Big unions

Example:

- Consider the family of sets

$$\mathcal{T} = \left\{ T \subseteq [5] \;\middle|\; \begin{array}{l} \text{the sum of the elements of} \\ T \text{ is less than or equal } 2 \end{array} \right\}$$

$$= \{ \emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\} \}$$

- The *big union* of the family $\mathcal{T}$ is the set $\bigcup \mathcal{T}$ given by the union of the sets in $\mathcal{T}$:

$$n \in \bigcup \mathcal{T} \iff \exists T \in \mathcal{T}. n \in T \quad .$$

Hence, $\bigcup \mathcal{T} = \{ 0, 1, 2 \}$.

Big intersections

Example:

- Consider the family of sets

$$\mathcal{S} = \left\{ S \subseteq [5] \;\middle|\; \text{the sum of the elements of } S \text{ is } 6 \right\}$$

$$= \{\{2,4\},\{0,2,4\},\{1,2,3\}\}$$

- The *big intersection* of the family $\mathcal{S}$ is the set $\bigcap \mathcal{S}$ given by the intersection of the sets in $\mathcal{S}$:

$$n \in \bigcap \mathcal{S} \iff \forall S \in \mathcal{S}. \, n \in S \quad .$$

Hence, $\bigcap \mathcal{S} = \{2\}$.

closure property

Theorem 114 *Let*

$$\mathcal{F} = \left\{ S \subseteq \mathbb{R} \;\middle|\; (0 \in S) \wedge \left(\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S\right) \right\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$*. Hence,* $\bigcap \mathcal{F} = \mathbb{N}$*.*

PROOF:

Because $0 \in \mathbb{N}$ and $\mathbb{N}$ is closed under successors.

Ex. $\mathbb{R} \in \mathcal{F}$

$\emptyset \notin \mathcal{F}$

$(i) \Rightarrow \bigcap \mathcal{F} \subseteq \mathbb{N}$ RTP $\forall x \in \bigcap \mathcal{F}.\ x \in \mathbb{N}$

Assume $x \in \bigcap \mathcal{F}$

Therefore $x \in S\ \forall S \in \mathcal{F}$

and since $\mathbb{N} \in \mathcal{F}$. $x \in \mathbb{N}$.

(ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$

iff $\mathbb{N} \subseteq S, \forall S \in \mathcal{F}$.

FACT: $A \subseteq \bigcap \mathcal{F}$

iff $\forall S \in \mathcal{F}$.

$A \subseteq S$

Let $S \in \mathcal{F}$.

$0 \in S$

$\wedge\ \forall x \in \mathbb{R}.\ x \in S \Rightarrow (x+1) \in S$

RTP: $\mathbb{N} \subseteq S$

iff

$\forall n \in \mathbb{N}.\ n \in S$

Base case: $0 \in S$ ✓

By induction: Inductive step: $n \in \mathbb{N}$

$n \in S \Rightarrow (n+1) \in S$

✓

$\boxtimes$

$\{1\} \times A = \{\langle 1, a\rangle \mid a \in A\}$

$\{2\} \times B = \{\langle 2, b\rangle \mid b \in B\}$

NB: $(\{1\} \times A) \cap (\{2\} \times B) = \emptyset$

Disjoint unions

Definition 116 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B \;=\; \big(\{1\} \times A\big) \cup \big(\{2\} \times B\big) \quad .$$

Thus,

$$\forall x.\, x \in (A \uplus B) \iff \big(\, \exists a \in A.\, x = (1, a)\,\big) \vee \big(\, \exists b \in B.\, x = (2, b)\,\big)\,.$$

datatype (α, β) dunion = one of α | two of β

Proposition 118 *For all finite sets* A *and* B,

$$A \cap B = \emptyset \implies \#(A \cup B) = \#A + \#B \quad .$$

PROOF IDEA:

$$A = \{\underbrace{a_1, \ldots, a_m}_{\# m}\} \qquad B = \{\underbrace{b_1, \ldots, b_n}_{\# n}\}$$

$$A \cup B = \{\underbrace{a_1 \ldots a_m \; b_1 \ldots b_n}_{\#(m+n)}\}$$

Corollary 119 *For all finite sets* A *and* B,

$$\#(A \uplus B) = \#A + \#B \quad .$$

$$\Big[\; \#(A \times B) = \#(A) \cdot \#(B)$$

$$\Big[\; \# P(X) = 2^{\# X}$$

$R = \{ \cdots, \underset{A}{a}, \underset{B}{b}), \cdots \}$

Relations

Definition 121 *A* <u>(binary) relation</u> R *from a set* A *to a set* B

$$R : A \rightarrowtail B \quad \textit{or} \quad R \in \mathrm{Rel}(A, B) \quad ,$$

is

$$R \subseteq A \times B \quad \textit{or} \quad R \in \mathcal{P}(A \times B) \quad .$$

Notation 122 *One typically writes* $a\,R\,b$ *for* $(a, b) \in R$.

programs

values.

$P \Downarrow V$

Informal examples:

- Computation.
- Typing.
- Program equivalence.
- Networks.
- Databases.

$P : \alpha$ Types

programs

e.g. relational DBs.

Examples:

- Empty relation.
  $\emptyset : A \twoheadrightarrow B$ $\qquad (a\ \emptyset\ b \iff \mathbf{false})$
- Full relation.
  $(A \times B) : A \twoheadrightarrow B$ $\qquad (a\ (A \times B)\ b \iff \mathbf{true})$
- Identity (or equality) relation.
  $\mathrm{id}_A = \{\ (a, a) \mid a \in A\ \} : A \twoheadrightarrow A$ $\qquad (a\ \mathrm{id}_A\ a' \iff a = a')$
- Integer square root.
  $R_2 = \{\ (m, n) \mid m = n^2\ \} : \mathbb{N} \twoheadrightarrow \mathbb{Z}$ $\qquad (m\ R_2\ n \iff m = n^2)$

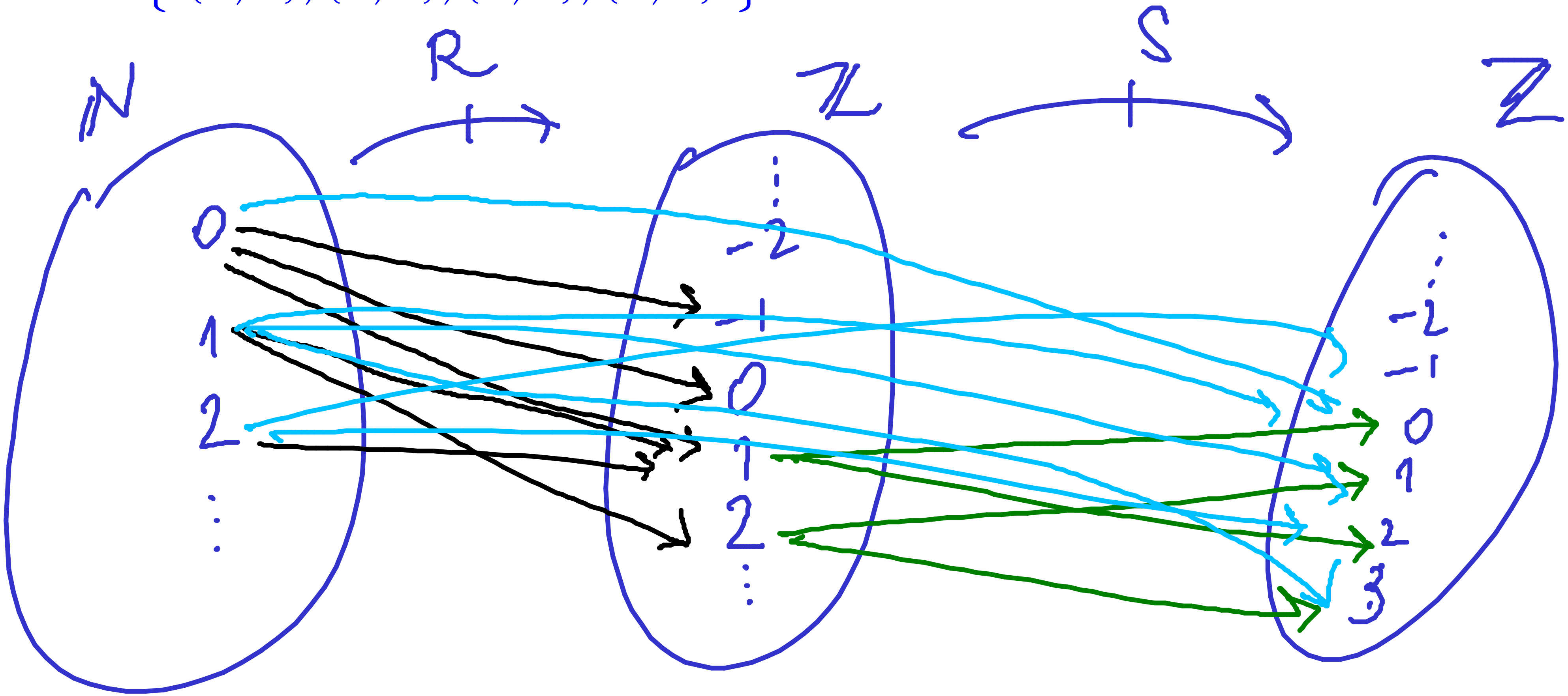
Ex: $4\ R_2\ 2$
$\wedge\ 4\ R_2\ (-2)$

Internal diagrams

Example:

$$R = \{ (0,0), (0,-1), (0,1), (1,2), (1,1), (2,1) \} : \mathbb{N} \rightarrow\!\!\!\!\!+\; \mathbb{Z}$$

$$S = \{ (1,0), (1,2), (2,1), (2,3) \} : \mathbb{Z} \rightarrow\!\!\!\!\!+\; \mathbb{Z}$$



Relational extensionality

$$R = S : A \twoheadrightarrow\!\!\!\!\!\!+\ B$$

iff

$$\forall a \in A. \forall b \in B.\ a\,R\,b \iff a\,S\,b$$

Relational composition

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \iff R \subseteq A \times B \iff R \in \mathcal{P}(A \times B)$$

$$\frac{B \overset{S}{\longrightarrow\!\!\!\!\!\!\!+\;\;} C}{A \overset{S \circ R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} C}$$

$$\underset{\substack{\uparrow \\ A}}{a}\ (S \circ R)\ \underset{\substack{\uparrow \\ C}}{c} \overset{\text{def}}{\iff} (\exists\, b \in B.\ a\,R\,b \wedge b\,S\,c)$$

$$A \xrightarrow{R} B \xrightarrow{id_B} B \qquad \left[\; b \; id_B \; b' \;\overset{}{\underset{def}{\iff}}\; b = b' \;\right]$$

$$A \xrightarrow{id_B \circ R} B$$

$$a\,(id_B \circ R)\,b \iff \exists\, b' \in B.\ a R b' \wedge b' \, id_B \, b$$

$$\iff \exists\, b' \in B.\ a R b' \wedge b' = b$$

$$\iff a R b$$

$$\Rightarrow \boxed{id_B \circ R = R}$$

Theorem 124 *Relational composition is associative and has the identity relation as neutral element.*

- *Associativity.*

  For all $R : A \rightarrowtriangle\!\!\!\!\!\!+ \, B$, $S : B \rightarrowtriangle\!\!\!\!\!\!+ \, C$, *and* $T : C \rightarrowtriangle\!\!\!\!\!\!+ \, D$,

$$(T \circ S) \circ R \;=\; T \circ (S \circ R)$$

$$\overset{?}{}\; T \circ S \circ R$$

- *Neutral element.*

  For all $R : A \rightarrowtriangle\!\!\!\!\!\!+ \, B$,

$$R \circ \mathrm{id}_A \;=\; R \;=\; \mathrm{id}_B \circ R \quad .$$

$$a\,((T\circ S)\circ R)\,d$$
$$\iff \exists b.\ aRb \wedge b(T\circ S)d$$
$$\iff \exists b.\ aRb \wedge \exists c.\ bSc \wedge cTd$$
$$\iff \exists b.\ \exists c.\ aRb \wedge bSc \wedge cTd$$

$$a\,(T\circ(S\circ R))\,d$$
$$\iff \exists c.\ a(S\circ R)c \wedge cTd$$
$$\iff \exists c.\ (\exists b.\ aRb \wedge bSc) \wedge cTd$$
$$\iff \exists c.\ \exists b.\ aRb \wedge bSc \wedge cTd.$$

$\boxtimes$

Definition 125

1. *For positive integers* m *and* n*, an* $(m \times n)$*-matrix* M *over a semiring* $(S, 0, \oplus, 1, \odot)$ *is given by entries* $M_{i,j} \in S$ *for all* $0 \le i < m$ *and* $0 \le j < n$*.*

$$(M+N)_{i,j} = M_{i,j} \oplus N_{i,j}$$

$(m\times n)$-matrices

$$\underset{m\times l}{(N \circledast M)_{i,j}} = \bigoplus_{k} \left(M_{i,k} \odot N_{k,j} \right)$$

M $(m \times n)$
N $(n \times l)$

Theorem 126 *Matrix multiplication is associative and has the identity matrix as neutral element.*