Ordered pairing

Notation:

$$(a, b) \quad \text{or} \quad \langle a, b \rangle$$

Fundamental property:

$$(a, b) = (x, y) \implies a = x \wedge b = y$$

A construction:

For every pair a and b, three applications of the pairing axiom provide the set

$$\langle a, b \rangle = \{ \{ a \} , \{ a, b \} \}$$

which defines an *ordered pairing* of a and b.

Proposition 109 (Fundamental property of ordered pairing)
For all a, b, x, y,

$$\langle a, b\rangle = \langle x, y\rangle \iff \big(a = x \wedge b = y\big) \quad .$$

YOUR PROOF:

$$\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\}$$

$$\iff a = x \wedge b = y$$

$\Longleftarrow$ easy $\Longrightarrow$

$$((\{a\} = \{x\} \vee \{a\} = \{x, y\})$$

$$\wedge (\{a, b\} = \{x\} \vee \{a, b\} = \{x, y\}))$$

Four cases —— Ex for you.

MY PROOF: *Let* a, b, x, y *be arbitrary.*

$(\Longleftarrow)$ *Vacuous.*

$(\Longrightarrow)$ *Assume* $\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$.

Then, $\{a\} = \{x\} \vee \cancel{\{a\}} = \{x,y\}$; *and, in either case,* $a = x$. *Hence,* $\{\{a\},\{a,b\}\} = \{\{a\},\{a,y\}\}$ *and, by Proposition 108.2 (on page 347),* $\{a,b\} = \{a,y\}$ *which, again by Proposition 108.2, implies* $b = y$.

Products

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{\, x \mid \exists\, a \in A, b \in B.\, x = (a, b)\,\}$$

where

$$\forall a_1, a_2 \in A, b_1, b_2 \in B.$$
$$(a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \wedge b_1 = b_2) \quad .$$

Thus,

$$\forall x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$

More generally, for a fixed natural number n and sets $A_1, \ldots, A_n$, we have

$$\begin{aligned} \prod_{i=1}^{n} A_i &= A_1 \times \cdots \times A_n \\ &= \{\, x \mid \exists\, a_1 \in A_1, \ldots, a_n \in A_n.\ x = (a_1, \ldots, a_n)\,\} \end{aligned}$$

where

$(((a_1, a_2), a_3) \ldots ?$

$$\forall\, a_1, a_1' \in A_1, \ldots, a_n, a_n' \in A_n.$$
$$(a_1, \ldots, a_n) = (a_1', \ldots, a_n') \iff (a_1 = a_1' \wedge \cdots \wedge a_n = a_n') \quad .$$

NB Cunningly enough, the definition is such that $\prod_{i=1}^{0} A_i = \{\, ()\,\}$.

Notation 110 *For a natural number* n *and a set* A*, one typically writes* A^n *for* $\prod_{i=1}^{n} A$.

Pattern-matching notation

Example: The subset of ordered pairs from a set A with equal components is formally

$$\{x \in A \times A \mid \exists a_1 \in A.\, \exists a_2 \in A.\, x = (a_1, a_2) \wedge a_1 = a_2\}$$

but often abbreviated using *pattern-matching notation* as

$$\{(a_1, a_2) \in A \times A \mid a_1 = a_2\} .$$

Notation: For a property $P(a, b)$ with a ranging over a set A and b ranging over a set B,

$$\{(a, b) \in A \times B \mid P(a, b)\}$$

abbreviates

$$\{x \in A \times B \mid \exists a \in A.\, \exists b \in B.\, x = (a, b) \wedge P(a, b)\} .$$

Proposition 111 *For all finite sets* A *and* B,

$$\#(A \times B) = \#A \cdot \#B \quad .$$

PROOF IDEA [a] :

$$A = \{a_1 \cdots a_k\}$$

$$B = \{b_1 \cdots b_j\}$$

$$A \times B = \left\{ \begin{array}{ccc} (a_1, b_1) & \cdots & (a_1, b_j) \\ \vdots & & \vdots \\ (a_k, b_1) & \cdots & (a_k, b_j) \end{array} \right\}$$

[a] See Theorem 162.2 on page 439.

Sets and logic

$\mathcal{P}(\mathbb{U})$	$\{\,\textbf{false}\,,\ \textbf{true}\,\}$
$\emptyset$	**false**
$\mathbb{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{c}$	$\neg(\cdot)$
$\bigcup$	$\exists$
$\bigcap$	$\forall$

Example:

$\sum \phi = 0$

- Consider the family of sets

$$\mathcal{T} = \left\{ T \subseteq [5] \;\middle|\; \begin{array}{l} \text{the sum of the elements of} \\ T \text{ is less than or equal } 2 \end{array} \right\}$$

$$= \{ \emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\} \}$$

- The *big union* of the family $\mathcal{T}$ is the set $\bigcup \mathcal{T}$ given by the union of the sets in $\mathcal{T}$:

$$n \in \bigcup \mathcal{T} \iff \exists T \in \mathcal{T}.\, n \in T \quad .$$

Hence, $\bigcup \mathcal{T} = \{0, 1, 2\}$.

Definition 112 *Let U be a set. For a collection of sets $\mathcal{F} \in \mathcal{P}(\mathcal{P}(U))$, we let the* big union *(relative to U) be defined as*

$$\bigcup \mathcal{F} = \{ x \in U \mid \exists A \in \mathcal{F}. x \in A \} \in \mathcal{P}(U) \quad .$$

Btw To get some intuition behind this definition, it might be useful to compare the construction with the ML function

```
flatten : 'a list list -> 'a list
```

associated with the ML `list` datatype constructor.

Examples:

1. For $A, A_1, A_2 \in \mathcal{P}(U)$,

$$\bigcup \emptyset = \emptyset$$

$$\bigcup \{A\} = A$$

$$\bigcup \{A_1, A_2\} = A_1 \cup A_2$$

$$\bigcup \{A, A_1, A_2\} = A \cup A_1 \cup A_2$$

2. For $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(U)))$, let us introduce the notation

$$\left\{ \bigcup \mathcal{A} \in \mathcal{P}(U) \;\middle|\; \mathcal{A} \in \mathcal{F} \right\}$$

map $\bigcup$ $\mathcal{F}$

for the set

$$\left\{ X \in \mathcal{P}(U) \;\middle|\; \exists \mathcal{A} \in \mathcal{F}. X = \bigcup \mathcal{A} \right\} \in \mathcal{P}(\mathcal{P}(U))$$

noticing that this is justified by the fact that, for all $x \in U$,

$$\begin{aligned}
& x \in \bigcup \{ X \in \mathcal{P}(U) \mid \exists \mathcal{A} \in \mathcal{F}. X = \bigcup \mathcal{A} \} \\
& \quad \iff \exists X \in \mathcal{P}(U). \exists \mathcal{A} \in \mathcal{F}. X = \bigcup \mathcal{A} \wedge x \in X \\
& \quad \iff \exists \mathcal{A} \in \mathcal{F}. x \in \bigcup \mathcal{A}
\end{aligned}$$

$$\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(U)))$$

$$\bigcup\left(\bigcup \mathcal{F}\right) = ?$$

$$\mathcal{F} = \{ \ldots \mathcal{A}, \ldots \mathcal{B} \ldots \}$$

$$\mathcal{A} = \{A_1, A_2 \ldots A_k\}$$

$$\mathcal{B} = \{B_1, B_2 \ldots B_j\}$$

$$\bigcup\left(\bigcup \mathcal{F}\right) =$$

$$\bigcup \{ \ldots A_1, A_2, \ldots, A_k, \ldots$$
$$B_1, B_2, \ldots B_j \ldots \}$$

$$= \{ \ldots A_1 \cup A_2 \cup \ldots \cup A_k \cup \ldots$$
$$\ldots \cup B_1 \cup B_2 \cup \ldots \cup B_j \cup \ldots \}$$

$$= \{ \ldots \cup \left(\underline{\bigcup \mathcal{A}}\right) \cup \ldots \cup \underline{\left(\bigcup \mathcal{B}\right)} \ldots \}$$

$$= \underline{\bigcup}\left(\mathrm{map} \bigcup \mathcal{F}\right)$$

We then have the following *associativity law*:

Proposition 113 *For all* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{U})))$,

$$\bigcup \left(\bigcup \mathcal{F} \right) = \bigcup \left\{ \bigcup \mathcal{A} \in \mathcal{P}(\mathcal{U}) \;\middle|\; \mathcal{A} \in \mathcal{F} \right\} \in \mathcal{P}(\mathcal{U}) \quad .$$

Btw In trying to understand this statement, ponder about the following analogous identity for the ML `list` datatype constructor: for all `F : 'a list list list`,

```
flatten ( flatten F )
  = flatten ( map flatten F ) : 'a list
```

The above two identities are the *associativity law* of a mathematical structure known as a *monad*, which has become a fundamental tool in functional programming.

MY PROOF: For $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(U)))$ and $x \in U$, one calculates that:

$$
\begin{aligned}
& x \in \bigcup \left(\bigcup \mathcal{F}\right) \\
& \quad \iff \exists X \in \bigcup \mathcal{F}.\ x \in X \\
& \quad \iff \exists \mathcal{A} \in \mathcal{F}.\ \exists X \in \mathcal{A}.\ x \in X \\
& \quad \iff \exists \mathcal{A} \in \mathcal{F}.\ x \in \bigcup \mathcal{A} \\
& \quad \iff x \in \bigcup \left\{ \bigcup \mathcal{A} \in \mathcal{P}(U) \mid \mathcal{A} \in \mathcal{F} \right\}
\end{aligned}
$$

Big intersections

Example:

- Consider the family of sets

$$\mathcal{S} = \left\{ S \subseteq [5] \;\middle|\; \begin{array}{l} \text{the sum of the elements of} \\ S \text{ is less than or equal } 6 \end{array} \right\}$$

$$= \{\{2,4\}, \{0,2,4\}, \{1,2,3\}\}$$

ϕ ? 2

- The *big intersection* of the family $\mathcal{S}$ is the set $\bigcap \mathcal{S}$ given by the intersection of the sets in $\mathcal{S}$:

$$n \in \bigcap \mathcal{S} \iff \forall S \in \mathcal{S}. n \in S \quad .$$

Hence, $\bigcup \mathcal{S} = \{2\}$.

Definition 114 *Let* U *be a set. For a collection of sets* $\mathcal{F} \subseteq \mathcal{P}(U)$*, we let the* big intersection *(relative to* U*) be defined as*

$$\bigcap \mathcal{F} = \{ x \in U \mid \forall A \in \mathcal{F}.\, x \in A \} \quad .$$

Examples: For $A, A_1, A_2 \in \mathcal{P}(U)$,

$$\bigcap \emptyset = U$$
$$\bigcap \{A\} = A$$
$$\bigcap \{A_1, A_2\} = A_1 \cap A_2$$
$$\bigcap \{A, A_1, A_2\} = A \cap A_1 \cap A_2$$

Theorem 115 *Let*

$$\mathcal{F} = \left\{ S \subseteq \mathbb{R} \,\middle|\, (0 \in S) \wedge (\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S) \right\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$. *Hence,* $\bigcap \mathcal{F} = \mathbb{N}$.

NB This result is typically interpreted as stating that:

$\mathbb{N}$ is the least set of numbers containing 0 and closed under successors.

? prop. 116 ?

Marcello will Explain ☺

Proposition 116 *Let* $\mathcal{U}$ *be a set and let* $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$ *be a family of subsets of* $\mathcal{U}$*.*

1. *For all* $S \in \mathcal{P}(\mathcal{U})$,

$$S = \bigcup \mathcal{F}$$

iff

$$\big[\forall A \in \mathcal{F}. A \subseteq S\big] = Q(S)$$

$$\wedge \ \big[\forall X \in \mathcal{P}(\mathcal{U}). (\underbrace{\forall A \in \mathcal{F}. A \subseteq X}_{Q(X)}) \Rightarrow \underbrace{S \subseteq X}_{\text{smallest for } Q}\big]$$

2. *For all* $T \in \mathcal{P}(\mathcal{U})$,

$$T = \bigcap \mathcal{F}$$

iff

$$\big[\forall A \in \mathcal{F}. T \subseteq A\big] \ Q'(T)$$

$$\wedge \ \big[\forall Y \in \mathcal{P}(\mathcal{U}). (\underbrace{\forall A \in \mathcal{F}. Y \subseteq A}_{Q'(Y)}) \Rightarrow \underbrace{Y \subseteq T}_{\text{biggest for } Q'}\big]$$

Union axiom

$\boxed{\text{Every collection of sets has a union.}}$

The set whose existence is postulated by the union axiom for a collection $\mathcal{F}$ is typically denoted

$$\bigcup \mathcal{F}$$

and, in the case $\mathcal{F} = \{A, B\}$, abbreviated to

$$A \cup B \quad .$$

Thus,

$$x \in \bigcup \mathcal{F} \iff \exists X \in \mathcal{F}. x \in X \quad ,$$

and hence

$$x \in (A \cup B) \iff (x \in A) \vee (x \in B) \quad .$$

Using the separation and union axioms, for every collection $\mathcal{F}$, consider the set

$$\left\{ x \in \bigcup \mathcal{F} \mid \forall X \in \mathcal{F}.\, x \in X \right\} \quad .$$

For *non-empty* $\mathcal{F}$ this set is denoted

$$\bigcap \mathcal{F}$$

because, in this case,

$$\forall x.\ x \in \bigcap \mathcal{F} \iff (\forall X \in \mathcal{F}.\, x \in X) \quad .$$

In particular, for $\mathcal{F} = \{A, B\}$, this is abbreviated to

$$A \cap B$$

with defining property

$$\forall x.\ x \in (A \cap B) \iff (x \in A) \land (x \in B) \quad .$$