Slides

for Part IA CST 2022/23

Discrete Mathematics

<www.cl.cam.ac.uk/teaching/2223/DiscMath>

Prof Marcelo Fiore

Marcelo.Fiore@cl.cam.ac.uk

What are we up to?

- Learn to read and write, and also work with, mathematical arguments.
- Doing some basic discrete mathematics.
- Getting a taste of computer science applications.

Lecture plan

I. Proofs.

II. Numbers.

III. Sets.

IV. Regular languages and finite automata.

Proofs

Objectives

- To develop techniques for analysing and understanding mathematical statements.
- To be able to present logical arguments that establish mathematical statements in the form of clear proofs.
- To prove Fermat's Little Theorem, a basic result in the theory of numbers that has many applications in computer science.

Proofs in practice

We are interested in examining the following statement:

The product of two odd integers is odd.

This seems innocuous enough, but it is in fact full of baggage.
For instance, it presupposes that you know:

- what a statement is;
- what the integers $(\ldots, -1, 0, 1, \ldots)$ are, and that amongst them there is a class of odd ones $(\ldots, -3, -1, 1, 3, \ldots)$;
- what the product of two integers is, and that this is in turn an integer.

More precisely put, we may write:

If m and n are odd integers then so is $m \cdot n$.

which further presupposes that you know:

- what variables are;
- what

  if ... then ...

  statements are, and how one goes about proving them;
- that the symbol "$\cdot$" is commonly used to denote the product operation.

Even more precisely, we should write

For all integers m and n, if m and n are odd then so is $m \cdot n$.

which now additionally presupposes that you know:

- what

  for all . . .

  statements are, and how one goes about proving them.

Thus, in trying to understand and then prove the above statement, we are assuming quite a lot of *mathematical jargon* that one needs to learn and practice with to make it a useful, and in fact very powerful, tool.

Some mathematical jargon

Statement

A sentence that is either true or false — but not both.

Example 1

'$e^{i\pi} + 1 = 0$'

Non-example

'This statement is false'

Predicate

A statement whose truth depends on the value of one or more variables.

Example 2

1. ‘$e^{ix} = \cos x + i \sin x$’

2. *‘the function f is differentiable’*

Theorem

A very important true statement.

Proposition

A less important but nonetheless interesting true statement.

Lemma

A true statement used in proving other true statements.

Corollary

A true statement that is a simple deduction from a theorem or proposition.

Example 3

1. *Fermat's Last Theorem*
2. *The Pumping Lemma*

Proof

Logical explanation of why a statement is true; a method for establishing truth.

Logic

The study of methods and principles used to distinguish good (correct) from bad (incorrect) reasoning.

Example 5

1. *Classical predicate logic*
2. *Hoare logic*
3. *Temporal logic*

Axiom

A basic assumption about a mathematical situation.

Axioms can be considered facts that do not need to be proved (just to get us going in a subject) or they can be used in definitions.

Example 6

1. Euclidean Geometry

2. Riemannian Geometry

3. Hyperbolic Geometry

Definition

An explanation of the mathematical meaning of a word (or phrase).

The word (or phrase) is generally defined in terms of properties.

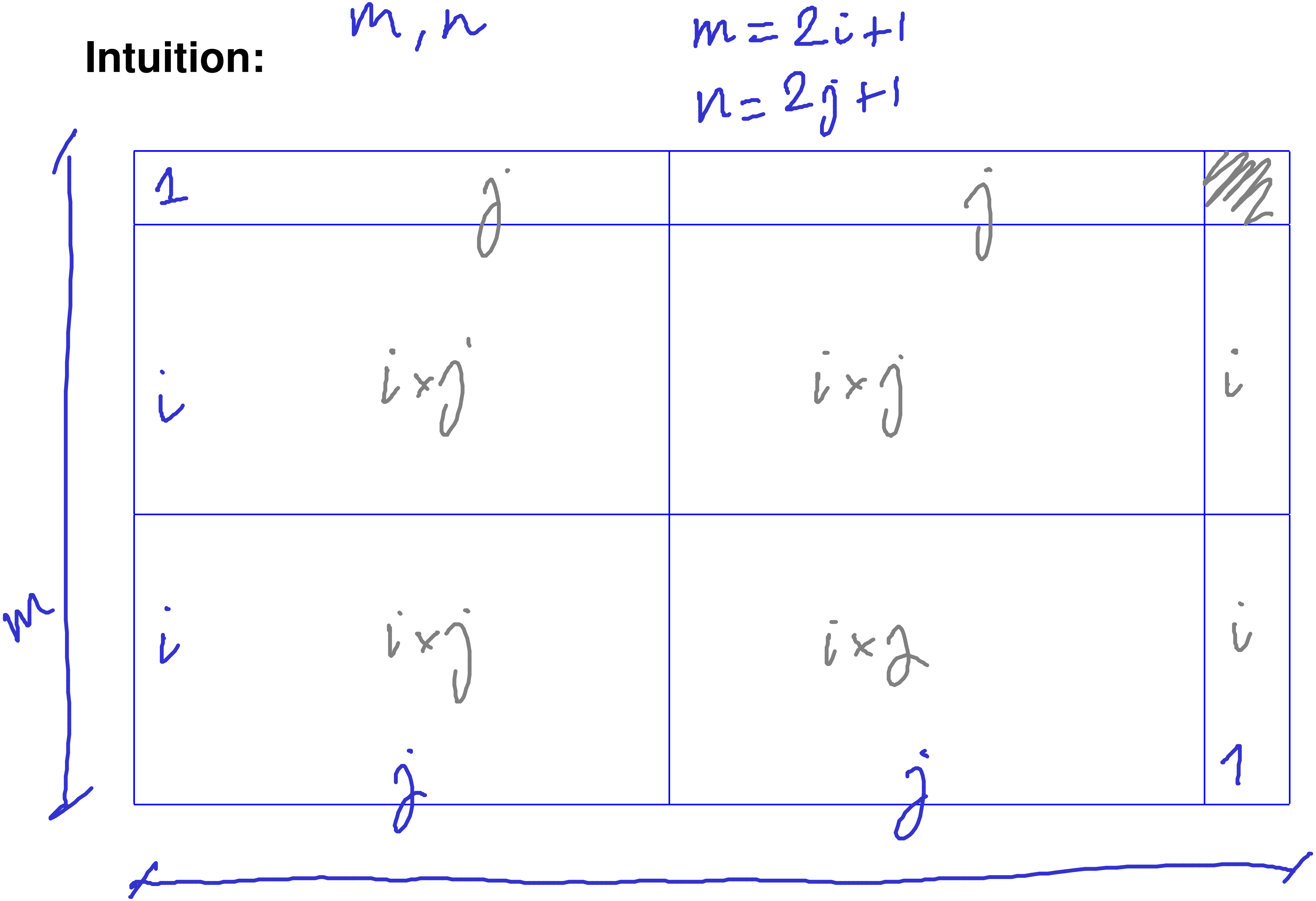
Warning: It is vitally important that you can recall definitions precisely. A common problem is not to be able to advance in some problem because the definition of a word is unknown.

Definition, theorem, intuition, proof in practice

Definition 7 *An integer is said to be* odd *whenever it is of the form* $2 \cdot i + 1$ *for some (necessarily unique) integer* i.

Proposition 8 *For all integers* m *and* n*, if* m *and* n *are odd then so is* $m \cdot n$.

Intuition:



PROOF OF Proposition 8:

Let m and n be odd integers

Then $m = 2i+1$ for an integer i

and $n = 2j+1$ for an integer j.

We need show $m \cdot n = 2k+1$ for some integer k.

Consider $m \cdot n = (2i+1) \cdot (2j+1)$

$$= \cdots = 4ij + 2i + 2j + 1$$

$$= 2(2ij + i + j) + 1$$

So we may take $k = 2ij + i + j$.

☒

Simple and composite statements

A statement is *simple* (or *atomic*) when it cannot be broken into other statements, and it is *composite* when it is built by using several (simple or composite statements) connected by *logical* expressions (e.g., if. . . then. . . ; . . . implies . . . ; . . . if and only if . . . ; . . . and. . . ; either . . . or . . . ; it is not the case that . . . ; for all . . . ; there exists . . . ; etc.)

Examples:

'2 is a prime number'

'for all integers m and n, if $m \cdot n$ is even then either n or m are even'

Proof Structure

Assumptions	Goals
statements that may be used for deduction	statements to be established

Implication

Theorems can usually be written in the form

if a collection of *assumptions* holds,
then so does some *conclusion*

or, in other words,

a collection of *assumptions* **implies** some *conclusion*

or, in symbols,

a collection of *hypotheses* $\Longrightarrow$ some *conclusion*

NB Identifying precisely what the assumptions and conclusions are is the first goal in dealing with a theorem.

Implications

- How to *prove* them as goals.
- How to *use* them as assumptions.

How to prove implication goals

The main proof strategy for implication:

To prove a goal of the form

$$P \implies Q$$

assume that P is true and prove Q.

NB *Assuming* is not *asserting*! Assuming a statement amounts to the same thing as adding it to your list of hypotheses.

Proof pattern:

In order to prove that

$$P \implies Q$$

1. Write: Assume P.
2. Show that Q logically follows.

Scratch work:

Before using the strategy

Assumptions	Goal
$P \implies Q$
$\vdots$

After using the strategy

Assumptions	Goal
Q
$\vdots$
P

Proposition 8 *If m and n are odd integers, then so is $m \cdot n$.*

PROOF:

Assume m and n are odd integers.

RTP: $m \cdot n$ is an odd integer.

So $m = 2i+1$ for an integer i

and $n = 2j+1$ for an integer j.

Then $m \cdot n = 2(2ij + i + j) + 1$ and hence it is odd.

$\boxtimes$

Definition 9 *A real number is:*

- rational *if it is of the form* m/n *for a pair of integers* m *and* n*; otherwise it is* irrational*.*
- positive *if it is greater than* 0*, and* negative *if it is smaller than* 0*.*
- nonnegative *if it is greater than or equal* 0*, and* nonpositive *if it is smaller than or equal* 0*.*
- natural *if it is a nonnegative integer.*

Proposition 10 *Let* x *be a positive real number. If* $\sqrt{x}$ *is rational then so is* x.

PROOF: Let x be a positive real number.

RTP: ($\sqrt{x}$ rational) $\Rightarrow$ (x rational)

Assume $\sqrt{x}$ is rational; that is, $\sqrt{x} = m/n$ for integers m and n.

RTP: x is rational; that is, $x = p/q$ for integers p and q.

Indeed, take $p = m^2$ and $q = n^2$. $\boxtimes$

How to use implication assumptions

Logical Deduction by Modus Ponens

A main rule of *logical deduction* is that of *Modus Ponens*:

> From the statements P and $P \implies Q$,
> the statement Q follows.

or, in other words,

> If P and $P \implies Q$ hold then so does Q.

or, in symbols,

$$\frac{P \qquad P \implies Q}{Q}$$

The use of implications:

To use an assumption of the form $P \implies Q$, aim at establishing P.
Once this is done, by Modus Ponens, one can conclude Q and so further assume it.

Theorem 11 *Let* P_1, P_2, *and* P_3 *be statements. If* $P_1 \implies P_2$ *and* $P_2 \implies P_3$ *then* $P_1 \implies P_3$.

PROOF:

<u>RTP</u> $(P_1 \Rightarrow P_2 \text{ and } P_2 \Rightarrow P_3) \Longrightarrow (P_1 \Rightarrow P_3)$

<u>Assume</u>: ① $P_1 \Rightarrow P_2$ and ② $P_2 \Rightarrow P_3$

<u>RTP</u> : $P_1 \Rightarrow P_3$

<u>Assume</u>: ③ P_1

<u>RTP</u> : P_3

By MP on ① and ③, we have ④ P_2.

By MP on ② and ④, we have P_3 as required. ☒