Complexity Theory

Lecture 9

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http://www.cl.cam.ac.uk/teaching/2223/Complexity

Primality

In 2002, Agrawal, Kayal and Saxena showed that PRIME is in P.

If a is co-prime to p,

$$(x - a)^p \equiv (x^p - a) \pmod{p}$$

if, and only if, p is a prime.

Checking this equivalence would take to long. Instead, the equivalence is checked *modulo* a polynomial $x^r - 1$, for “suitable” r.

The existence of suitable small r relies on deep results in number theory.

Factors

Consider the language Factor

$$\{(x,k) \mid x \text{ has a factor } y \text{ with } 1 < y < k\}$$

Factor $\in$ NP $\cap$ co-NP

Certificate of membership—a factor of x less than k.

Certificate of disqualification—the prime factorisation of x.

Graph Isomorphism

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is there a *bijection*

$$\iota : V_1 \to V_2$$

such that for every $u, v \in V_1$,

$$(u, v) \in E_1 \quad \text{if, and only if,} \quad (\iota(u), \iota(v)) \in E_2.$$

Graph Isomorphism

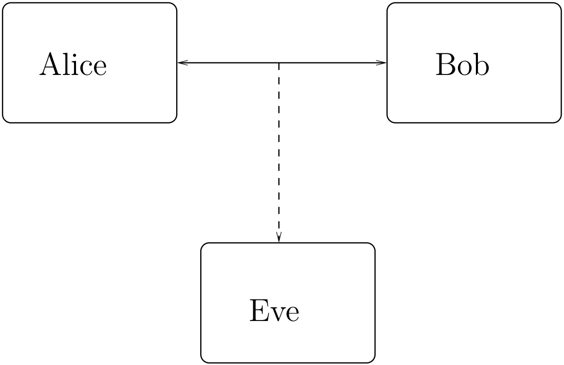
Graph Isomorphism is

- in NP
- not known to be in P
- not known to be in co-NP
- not known (or *expected*) to be NP-complete
- shown to be in *quasi-polynomial time*, i.e. in

$$\mathrm{TIME}(n^{(\log n)^k})$$

for a constant k.

Cryptography



Alice wishes to communicate with Bob without Eve eavesdropping.

Private Key

In a private key system, there are two secret keys
e – the encryption key
d – the decryption key
and two functions D and E such that:

for any x,

$$D(E(x, e), d) = x.$$

For instance, taking $d = e$ and both D and E as *exclusive or*, we have the *one time pad*:

$$(x \oplus e) \oplus e = x$$

One Time Pad

The one time pad is provably secure, in that the only way Eve can decode a message is by knowing the key.

If the original message x and the encrypted message y are known, then so is the key:

$$e = x \oplus y$$

Public Key

In public key cryptography, the encryption key e is public, and the decryption key d is private.
We still have,

for any x,

$$D(E(x, e), d) = x$$

If E is polynomial time computable (and it must be if communication is not to be painfully slow), then the following language is in NP:

$$\{(y, z) \mid y = E(x, e) \text{ for some } x \text{ with } x \leq_{\text{lex}} z\}$$

Thus, public key cryptography is not *provably secure* in the way that the one time pad is. It relies on the assumption that $\mathsf{P} \neq \mathsf{NP}$.

One Way Functions

A function f is called a *one way function* if it satisfies the following conditions:

1. f is one-to-one.
2. for each x, $|x|^{1/k} \leq |f(x)| \leq |x|^k$ for some k.
3. f is computable in polynomial time.
4. f^{-1} is *not* computable in polynomial time.

We cannot hope to prove the existence of one-way functions without at the same time proving $\mathsf{P} \neq \mathsf{NP}$.
It is strongly believed that the RSA function:

$$f(x, e, p, q) = (x^e \bmod pq, pq, e)$$

is a one-way function.

UP

Though one cannot hope to prove that the RSA function is one-way without separating P and NP, we might hope to make it as secure as a proof of NP-completeness.

Definition
A nondeterministic machine is *unambiguous* if, for any input x, there is at most one accepting computation of the machine.
UP is the class of languages accepted by unambiguous machines in polynomial time.

UP

Equivalently, UP is the class of languages of the form

$$\{x \mid \exists y R(x, y)\}$$

Where R is polynomial time computable, polynomially balanced, *and* for each x, there is *at most one* y such that $R(x, y)$.

UP One-way Functions

We have

$$P \subseteq UP \subseteq NP$$

It seems unlikely that there are any NP-complete problems in UP.

One-way functions exist *if, and only if,* $P \neq UP$.

One-Way Functions Imply P $\neq$ UP

Suppose f is a *one-way function*.

Define the language L_f by

$$L_f = \{(x, y) \mid \exists z(z \leq x \text{ and } f(z) = y)\}.$$

We can show that L_f is in UP but not in P.