Lecture 15

Presheaf categories

Let C be a small category. The functor category $\frac{\text{Set}^{C^{op}}}{\text{is called the category of presheaves on C}}$.

- objects are contravariant functors from C to Set
- morphisms are natural transformations

Much used in the semantics of various dependently-typed languages and logics.

Given a category C with a terminal object 1

A global element of an object $X \in obj \mathbb{C}$ is by definition a morphism $1 \rightarrow X$ in \mathbb{C}

E.g. in Set ...

E.g. in **Mon** ...

Given a category C with a terminal object 1

A global element of an object $X \in obj \mathbb{C}$ is by definition a morphism $1 \rightarrow X$ in \mathbb{C}

We say **C** is well-pointed if for all $f, g : X \rightarrow Y$ in **C** we have:

$$\left(\forall 1 \xrightarrow{x} X, f \circ x = g \circ x\right) \implies f = g$$

(Set is, Mon isn't.)

Idea:

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replace global elements of $X, 1 \xrightarrow{x} X$ by arbitrary morphisms $Y \xrightarrow{x} X$ (for any $Y \in obj C$) Some people use the notation $x \in_Y X$ and say "x is a generalised element of X at stage Y" Have to take into account "change of stage": $x \in_Y X \land Z \xrightarrow{f} Y \Rightarrow x \circ f \in_Z X$

(cf. Kripke's "possible world" semantics of intuitionistic and modal logics)

Yoneda functor

$$y: \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathsf{op}}}$$

(where C is a small category)

is the Curried version of the hom functor

 $\mathbf{C} \times \mathbf{C}^{\mathsf{op}} \cong \mathbf{C}^{\mathsf{op}} \times \mathbf{C} \xrightarrow{\operatorname{Hom}_{\mathbf{C}}} \mathbf{Set}$

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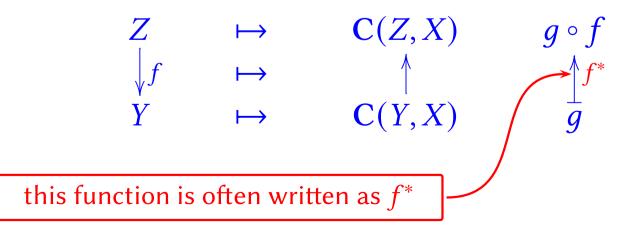
► For each C-object *X*, the object $yX \in Set^{C^{op}}$ is the functor $C(_,X) : C^{op} \rightarrow Set$ given by

$$\begin{array}{cccccccc} Z & \mapsto & \mathbf{C}(Z,X) & g \circ f \\ & \downarrow f & \mapsto & \uparrow & & \uparrow \\ Y & \mapsto & \mathbf{C}(Y,X) & & g \end{array}$$

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► For each C-morphism $Y \xrightarrow{f} X$, the morphism $yY \xrightarrow{yf} yX$ in Set^{C^{op}} is the natural transformation whose component at any given $Z \in C^{op}$ is the function

$$yY(Z) \xrightarrow{(yf)_Z} yX(Z)$$

$$\stackrel{\parallel}{\longrightarrow} C(Z,Y) \qquad C(Z,X)$$

$$g \longmapsto f \circ g$$

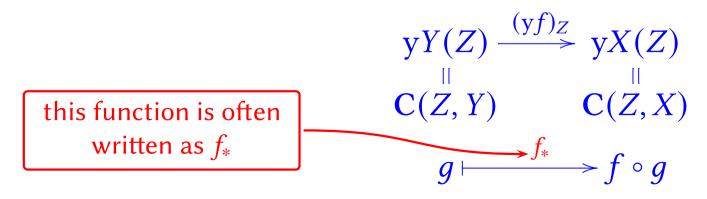
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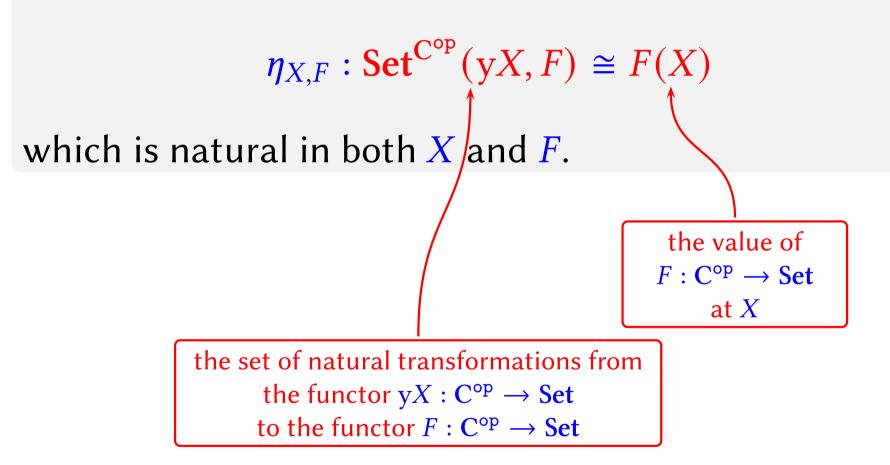


For each small category C, each object $X \in C$ and each presheaf $F \in Set^{C^{op}}$, there is a bijection of sets

 $\eta_{X,F}$: Set^{C^{op}} $(yX,F) \cong F(X)$

which is natural in both X and F.

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Definition of the function $\eta_{X,F}$: **Set**^{C^{op}}(yX,F) \rightarrow F(X):

for each θ : yX \rightarrow F in Set^{C^{op}} we have the function $C(X, X) = yX(X) \xrightarrow{\theta_X} F(X)$ and define

 $\eta_{X,F}(\theta) \triangleq \theta_X(\mathrm{id}_X)$

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Definition of the function $\eta_{X,F}^{-1} : F(X) \to \text{Set}^{C^{\text{op}}}(yX,F)$: for each $x \in F(X), Y \in \mathbb{C}$ and $f \in yX(Y) = \mathbb{C}(Y,X)$, we get a $F(X) \xrightarrow{F(f)} F(Y)$ in **Set** and hence $F(f)(x) \in F(Y)$;

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Define
$$\left(\eta_{X,F}^{-1}(x)\right)_{Y} : yX(Y) \to F(Y)$$
 by

$$\left(\eta_{X,F}^{-1}(x)\right)_{Y}(f) \triangleq F(f)(x)$$

check this gives a natural transformation $\eta_{X,F}^{-1}(x) : yX \to F$

Proof of
$$\eta_{X,F} \circ \eta_{X,F}^{-1} = id_{F(X)}$$

For any $x \in F(X)$ we have

$$\eta_{X,F}\left(\eta_{X,F}^{-1}(x)\right) \triangleq \left(\eta_{X,F}^{-1}(x)\right)_{X} (\operatorname{id}_{X})$$
$$\triangleq F(\operatorname{id}_{X})(x)$$
$$= \operatorname{id}_{F(X)}(x)$$
$$= x$$

by definition of $\eta_{X,F}$ by definition of $\eta_{X,F}^{-1}$ since *F* is a functor

Proof of
$$\eta_{X,F}^{-1} \circ \eta_{X,F} = \operatorname{id}_{\operatorname{Set}^{\operatorname{C^{op}}}(yX,F)}$$

For any $yX \xrightarrow{\theta} F$ in $\operatorname{Set}^{\operatorname{C^{op}}}$ and $Y \xrightarrow{f} X$ in \mathbb{C} , we have

$$\begin{pmatrix} \eta_{X,F}^{-1}(\eta_{X,F}(\theta)) \end{pmatrix}_{Y} f \triangleq \begin{pmatrix} \eta_{X,F}^{-1}(\theta_{X}(\mathrm{id}_{X}))) \end{pmatrix}_{Y} f \\ \triangleq F(f)(\theta_{X}(\mathrm{id}_{X})) \\ = \theta_{Y}(f) (\mathrm{id}_{X}) \\ \triangleq \theta_{Y}(\mathrm{id}_{X} \circ f) \\ = \theta_{Y}(f) \\ \hline \mathbf{naturality of } \theta \\ yX(Y) \xrightarrow{\theta_{Y}} F(Y) \\ \uparrow f^{*} \qquad \uparrow F(f) \\ yX(X) \xrightarrow{\theta_{X}} F(X) \\ \hline \end{pmatrix}$$

by definition of $\eta_{X,F}$ by definition of $\eta_{X,F}^{-1}$ by naturality of θ by definition of f^*

Proof of
$$\eta_{X,F}^{-1} \circ \eta_{X,F} = \operatorname{id}_{\operatorname{Set}^{\operatorname{Cop}}(YX,F)}$$

For any $yX \xrightarrow{\theta} F$ in Set^{C°P} and $Y \xrightarrow{f} X$ in **C**, we have

$$\begin{pmatrix} \eta_{X,F}^{-1}(\eta_{X,F}(\theta)) \end{pmatrix}_{Y} f \triangleq \begin{pmatrix} \eta_{X,F}^{-1}(\theta_{X}(\mathrm{id}_{X}))) \end{pmatrix}_{Y} f \\ \triangleq F(f)(\theta_{X}(\mathrm{id}_{X})) \\ = \theta_{Y}(f^{*}(id_{X})) \\ \triangleq \theta_{Y}(\mathrm{id}_{X} \circ f) \\ = \theta_{Y}(f)$$

by definition of $\eta_{X,F}$ by definition of $\eta_{X,F}^{-1}$ by naturality of θ by definition of f^*

so
$$\forall \theta, Y, \left(\eta_{X,F}^{-1}\left(\eta_{X,F}(\theta)\right)\right)_{Y} = \theta_{Y}$$

so $\forall \theta, \eta_{X,F}^{-1}\left(\eta_{X,F}(\theta)\right) = \theta$
so $\eta_{X,F}^{-1} \circ \eta_{X,F} = \text{id.}$

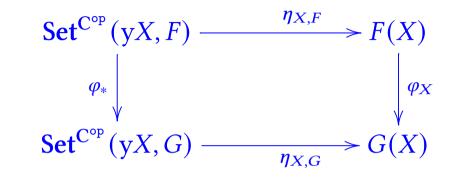
For each small category **C**, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{op}}$, there is a bijection of sets

$$\eta_{X,F}$$
: Set^{C^{op}}(yX, F) \cong F(X)

which is natural in both X and F.

Proof that $\eta_{X,F}$ is natural in *F*:

Given $F \xrightarrow{\varphi} G$ in Set^{C°P}, have to show that



commutes in Set. For all $yX \xrightarrow{\theta} F$ we have

$$\begin{aligned}
\varphi_X \left(\eta_{X,F}(\theta) \right) &\triangleq \varphi_X \left(\theta_X(\mathrm{id}_X) \right) \\
&\triangleq (\varphi \circ \theta)_X(\mathrm{id}_X) \\
&\triangleq \eta_{X,G}(\varphi \circ \theta) \\
&\triangleq \eta_{X,G}(\varphi_*(\theta))
\end{aligned}$$

Proof that $\eta_{X,F}$ is natural in X:

Given $Y \xrightarrow{f} X$ in **C**, have to show that

commutes in Set. For all $yX \xrightarrow{\theta} F$ we have $F(f)((\eta_{X,F}(\theta)) \triangleq F(f)(\theta_X(id_X)))$ $= \theta_Y(f^*(id_X)) \qquad \text{by naturality of } \theta$ $= \theta_Y(f)$ $= \theta_Y(f_*(id_Y))$ $\triangleq (\theta \circ yf)_Y(id_Y)$ $\triangleq \eta_{Y,F}(\theta \circ yf)$ $\triangleq \eta_{Y,F}((yf)^*(\theta))$ **Corollary** of the Yoneda Lemma:

the functor $y : \mathbb{C} \rightarrow Set^{\mathbb{C}^{op}}$ is full and faithful.

In general, a functor $F : \mathbb{C} \to \mathbb{D}$ is

Faithful if for all $X, Y \in \mathbb{C}$ the function

$\mathbf{C}(X,Y)$	\rightarrow	$\mathbf{D}(F(X),F(Y))$
f	\mapsto	F(f)

is injective:

 $\forall f, f' \in \mathbf{C}(X, Y), \ F(f) = F(f') \Rightarrow f = f'$

► full if the above functions are all surjective: $\forall g \in \mathbf{D}(F(X), F(Y)), \exists f \in \mathbf{C}(X, Y), F(f) = g$ **Corollary** of the Yoneda Lemma:

the functor $y : C \rightarrow Set^{C^{op}}$ is full and faithful.

Proof. From the proof of the Yoneda Lemma, for each $F \in Set^{C^{op}}$ we have a bijection

$$F(X) \xrightarrow{(\eta_{X,F})^{-1}} \operatorname{Set}^{\operatorname{Cop}}(\mathrm{y}X,F)$$

By definition of $(\eta_{X,F})^{-1}$, when F = yY the above function is equal to

$$yY(X) = C(X, Y) \rightarrow Set^{C^{op}}(yX, yY)$$

 $f \mapsto f_* = yf$

So, being a bijection, $f \mapsto yf$ is both injective and surjective; so y is both faithful and full.

Recall (for a small category **C**):

Yoneda functor $y : \mathbb{C} \to \text{Set}^{\mathbb{C}^{op}}$

Yoneda Lemma: there is a bijection $\operatorname{Set}^{\operatorname{C^{op}}}(\mathrm{y}X, F) \cong F(X)$ which is natural both in $F \in \operatorname{Set}^{\operatorname{C^{op}}}$ and $X \in \mathbb{C}$.

An application of the Yoneda Lemma:

Theorem. For each small category **C**, the category **Set**^{C°P} of presheaves is cartesian closed.

Proof sketch.

Terminal object in Set^{C^{op}} is the functor $1 : C^{op} \rightarrow Set$ given by

 $\begin{cases} 1(X) \triangleq \{0\} & \text{terminal object in Set} \\ 1(f) \triangleq id_{\{0\}} \end{cases}$

Proof sketch.

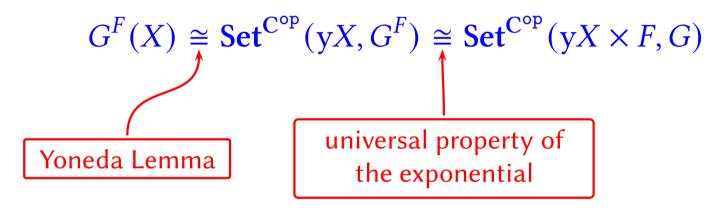
Product of $F, G \in \text{Set}^{\mathbb{C}^{op}}$ is the functor $F \times G : \mathbb{C}^{op} \to \text{Set}$ given by

 $\begin{cases} (F \times G)(X) \triangleq F(X) \times G(X) & \text{cartesian product of sets} \\ (F \times G)(f) \triangleq F(f) \times G(f) \end{cases}$

with projection morphisms $F \xleftarrow{\pi_1}{\leftarrow} F \times G \xrightarrow{\pi_2}{\rightarrow} G$ given by the natural transformations whose components at $X \in \mathbb{C}$ are the projection functions $F(X) \xleftarrow{\pi_1}{\leftarrow} F(X) \times G(X) \xrightarrow{\pi_2}{\rightarrow} G(X)$.

Proof sketch.

We can work out what the value of the exponential $G^F \in Set^{C^{op}}$ at $X \in C$ has to be using the Yoneda Lemma:



Proof sketch.

We can work out what the value of the exponential $G^F \in \text{Set}^{\mathbb{C}^{op}}$ at $X \in \mathbb{C}$ has to be using the Yoneda Lemma:

 $G^{F}(X) \cong \mathbf{Set}^{\mathbf{C}^{\mathsf{op}}}(\mathbf{y}X, G^{F}) \cong \mathbf{Set}^{\mathbf{C}^{\mathsf{op}}}(\mathbf{y}X \times F, G)$

We take the set $\operatorname{Set}^{C^{\operatorname{op}}}(YX \times F, G)$ to be the definition of the value of G^F at X...

Exponential objects in Set^{C°P}:

$$G^F(X) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathbf{y}X \times F, G)$$

Given
$$Y \xrightarrow{f} X$$
 in **C**, we have $yY \xrightarrow{yf} yX$ in $\operatorname{Set}^{\operatorname{C^{op}}}$ and hence
 $G^{F}(Y) \triangleq \operatorname{Set}^{\operatorname{C^{op}}}(yY \times F, G) \xrightarrow{} \operatorname{Set}^{\operatorname{C^{op}}}(yX \times F, G) \triangleq G^{F}(X)$
 $\theta \mapsto \theta \circ (yf \times \operatorname{id}_{F})$

We define

$$G^F(f) \triangleq (\mathbf{y}f \times \mathrm{id}_F)^*$$

Have to check that these definitions make G^F into a functor $C^{op} \rightarrow Set$.

Application morphisms in Set^{C^{op}}:

Given $F, G \in \mathbf{Set}^{\mathbf{C}^{\mathsf{op}}}$, the application morphism

 $app: G^F \times F \to G$

is the natural transformation whose component at $X \in \mathbb{C}$ is the function

 $(G^F \times F)(X) \triangleq G^F(X) \times F(X) \triangleq \operatorname{Set}^{\operatorname{C^{op}}}(yX \times F, G) \times F(X) \xrightarrow{\operatorname{app}_X} G(X)$

defined by

$$\texttt{app}_X(\theta, x) \triangleq \theta_X(\texttt{id}_X, x)$$

Have to check that this is natural in *X*.

Currying operation in Set^{C°P}**:**

$$\left(H \times F \xrightarrow{\theta} G\right) \mapsto \left(H \xrightarrow{\operatorname{cur} \theta} G^F\right)$$

Given $H \times F \xrightarrow{\theta} G$ in Set^{C^{op}}, the component of cur θ at $X \in \mathbf{C}$

$$H(X) \xrightarrow{(\operatorname{cur} \theta)_X} G^F(X) \triangleq \operatorname{Set}^{\operatorname{C^{op}}}(\mathsf{y}X \times F, G)$$

is the function mapping each $z \in H(X)$ to the natural transformation $yX \times F \to G$ whose component at $Y \in \mathbb{C}$ is the function

$$(\mathbf{y}X \times F)(Y) \triangleq \mathbf{C}(Y,X) \times F(Y) \to G(Y)$$

defined by

$$((\operatorname{cur} \theta)_X(z))_Y(f, y) \triangleq \theta_Y(H(f)(z), y)$$

Currying operation in Set^{C°P}**:**

$$\left(H \times F \xrightarrow{\theta} G\right) \mapsto \left(H \xrightarrow{\operatorname{cur} \theta} G^F\right)$$

$$((\operatorname{cur} \theta)_X(z))_Y(f, y) \triangleq \theta_Y(H(f)(z), y)$$

Have to check that this is natural in Y,

then that $(\operatorname{cur} \theta)_X$ is natural in *X*,

then that $\operatorname{cur} \theta$ is the unique morphism $H \xrightarrow{\varphi} G^F$ in $\operatorname{Set}^{C^{\operatorname{op}}}$ satisfying $\operatorname{app} \circ (\varphi \times \operatorname{id}_F) = \theta$.

So we can interpret simply typed lambda calculus in any presheaf category.

More than that, presheaf categories (usefully) model dependently-typed languages.