Lecture 15

Presheaf categories

Let $\mathbf{C}$ be a small category. The functor category $\boxed{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}$ is called the category of presheaves on $\mathbf{C}$.

- objects are contravariant functors from $\mathbf{C}$ to $\mathbf{Set}$
- morphisms are natural transformations

Much used in the semantics of various dependently-typed languages and logics.

Given a category $\mathbf{C}$ with a terminal object $\mathtt{1}$

A global element of an object $X \in \mathtt{obj}\,\mathbf{C}$ is by definition a morphism $\mathtt{1} \to X$ in $\mathbf{C}$

E.g. in **Set** …

E.g. in **Mon** …

Given a category $\mathbf{C}$ with a terminal object 1

A global element of an object $X \in \mathtt{obj}\,\mathbf{C}$ is by definition a morphism $1 \to X$ in $\mathbf{C}$

We say $\mathbf{C}$ is well-pointed if for all $f, g : X \to Y$ in $\mathbf{C}$ we have:

$$\left(\forall 1 \xrightarrow{x} X,\ f \circ x = g \circ x\right) \ \Rightarrow\ f = g$$

($\mathbf{Set}$ is, $\mathbf{Mon}$ isn't.)

Idea:

replace global elements of X, $\mathtt{1} \xrightarrow{x} X$

by arbitrary morphisms $Y \xrightarrow{x} X$ (for any $Y \in \mathtt{obj}\,\mathbf{C}$)

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Some people use the notation $\boxed{x \in_Y X}$ and say "x is a generalised element of X at stage Y"

Have to take into account "change of stage":

$$x \in_Y X \;\wedge\; Z \xrightarrow{f} Y \;\Rightarrow\; x \circ f \in_Z X$$

(cf. Kripke's "possible world" semantics of intuitionistic and modal logics)

Yoneda functor

$$\boxed{\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathrm{Hom}_{\mathbf{C}}} \mathbf{Set}$$

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- For each $\mathbf{C}$-object X, the object $\mathrm{y}X \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $\mathbf{C}(\_, X) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ given by

$$\begin{array}{ccccc}
Z & \mapsto & \mathbf{C}(Z, X) & & g \circ f \\
\big\downarrow f & \mapsto & \big\uparrow & & \big\uparrow \\
Y & \mapsto & \mathbf{C}(Y, X) & & g
\end{array}$$

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Z & \mapsto & \mathbf{C}(Z, X) & & g \circ f \\
\downarrow f & \mapsto & \uparrow & & \uparrow f^* \\
Y & \mapsto & \mathbf{C}(Y, X) & & g
\end{array}$$

this function is often written as f^*

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- For each $\mathbf{C}$-morphism $Y \xrightarrow{f} X$, the morphism $\mathrm{y}Y \xrightarrow{\mathrm{y}f} \mathrm{y}X$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the natural transformation whose component at any given $Z \in \mathbf{C}^{\mathrm{op}}$ is the function

$$\begin{array}{ccc} \mathrm{y}Y(Z) & \xrightarrow{(\mathrm{y}f)_Z} & \mathrm{y}X(Z) \\ \| & & \| \\ \mathbf{C}(Z,Y) & & \mathbf{C}(Z,X) \\ & & \\ g & \longmapsto & f \circ g \end{array}$$

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this function is often written as f_*

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \cong F(X)$$

which is natural in both X and F.

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the value of
$F : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$
at X

the set of natural transformations from
the functor $\mathrm{y}X : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$
to the functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$

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which is natural in both X and F.

Definition of the function $\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \to F(X)$:

for each $\theta : \mathrm{y}X \to F$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ we have the function $\mathbf{C}(X, X) = \mathrm{y}X(X) \xrightarrow{\theta_X} F(X)$ and define

$$\boxed{\eta_{X,F}(\theta) \triangleq \theta_X(\mathtt{id}_X)}$$

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Definition of the function $\eta_{X,F}^{-1} : F(X) \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F)$:

for each $x \in F(X)$, $Y \in \mathbf{C}$ and $f \in \mathrm{y}X(Y) = \mathbf{C}(Y, X)$,

we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathbf{Set}$ and hence $F(f)(x) \in F(Y)$;

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Define $\left(\eta_{X,F}^{-1}(x)\right)_Y : \mathrm{y}X(Y) \to F(Y)$ by

$$\boxed{\left(\eta_{X,F}^{-1}(x)\right)_Y (f) \triangleq F(f)(x)}$$

check this gives a natural transformation $\eta_{X,F}^{-1}(x) : \mathrm{y}X \to F$

Proof of $\boxed{\eta_{X,F} \circ \eta^{-1}_{X,F} = \mathtt{id}_{F(X)}}$

For any $x \in F(X)$ we have

$$
\begin{aligned}
\eta_{X,F}\left(\eta^{-1}_{X,F}(x)\right) &\triangleq \left(\eta^{-1}_{X,F}(x)\right)_X (\mathtt{id}_X) && \text{by definition of } \eta_{X,F} \\
&\triangleq F(\mathtt{id}_X)(x) && \text{by definition of } \eta^{-1}_{X,F} \\
&= \mathtt{id}_{F(X)}(x) && \text{since } F \text{ is a functor} \\
&= x
\end{aligned}
$$

Proof of $\boxed{\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathtt{id}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X,F)}}$

For any $\mathrm{y}X \xrightarrow{\theta} F$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and $Y \xrightarrow{f} X$ in $\mathbf{C}$, we have

$$
\begin{aligned}
\left(\eta^{-1}_{X,F}\,(\eta_{X,F}(\theta))\right)_Y f &\triangleq \left(\eta^{-1}_{X,F}\,(\theta_X(\mathtt{id}_X))\right)_Y f && \text{by definition of } \eta_{X,F} \\
&\triangleq F(f)(\theta_X(\mathtt{id}_X)) && \text{by definition of } \eta^{-1}_{X,F} \\
&= \theta_Y(f^*(id_X)) && \text{by naturality of } \theta \\
&\triangleq \theta_Y(\mathtt{id}_X \circ f) && \text{by definition of } f^* \\
&= \theta_Y(f)
\end{aligned}
$$

naturality of θ

$$
\begin{tikzcd}
\mathrm{y}X(Y) \arrow[r, "\theta_Y"] & F(Y) \\
\mathrm{y}X(X) \arrow[u, "f^*"] \arrow[r, "\theta_X"'] & F(X) \arrow[u, "F(f)"']
\end{tikzcd}
$$

Proof of $\boxed{\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathtt{id}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X,F)}}$

For any $\mathrm{y}X \xrightarrow{\theta} F$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and $Y \xrightarrow{f} X$ in $\mathbf{C}$, we have

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&\triangleq F(f)(\theta_X(\mathtt{id}_X)) && \text{by definition of } \eta^{-1}_{X,F} \\
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$$

so $\forall \theta, Y,\ \left(\eta^{-1}_{X,F}\left(\eta_{X,F}(\theta)\right)\right)_Y = \theta_Y$

so $\forall \theta,\ \eta^{-1}_{X,F}\left(\eta_{X,F}(\theta)\right) = \theta$

so $\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathtt{id}$.

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \cong F(X)$$

which is natural in both X and F.

Proof that $\eta_{X,F}$ is natural in F:

Given $F \xrightarrow{\varphi} G$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, have to show that

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \arrow[r, "\eta_{X,F}"] \arrow[d, "\varphi_*"'] & F(X) \arrow[d, "\varphi_X"] \\
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, G) \arrow[r, "\eta_{X,G}"'] & G(X)
\end{tikzcd}$$

commutes in $\mathbf{Set}$. For all $\mathrm{y}X \xrightarrow{\theta} F$ we have

$$\begin{aligned}
\varphi_X\left(\eta_{X,F}(\theta)\right) &\triangleq \varphi_X\left(\theta_X(\mathtt{id}_X)\right) \\
&\triangleq (\varphi \circ \theta)_X(\mathtt{id}_X) \\
&\triangleq \eta_{X,G}(\varphi \circ \theta) \\
&\triangleq \eta_{X,G}(\varphi_*(\theta))
\end{aligned}$$

Proof that $\eta_{X,F}$ is natural in X:

Given $Y \xrightarrow{f} X$ in $\mathbf{C}$, have to show that

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \arrow[rr, "\eta_{X,F}"] \arrow[d, "(\mathrm{y}f)^*"'] & & F(X) \arrow[d, "F(f)"] \\
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}Y, F) \arrow[rr, "\eta_{Y,F}"'] & & F(Y)
\end{tikzcd}$$

commutes in $\mathbf{Set}$. For all $\mathrm{y}X \xrightarrow{\theta} F$ we have

$$\begin{aligned}
F(f)((\eta_{X,F}(\theta)) &\triangleq F(f)(\theta_X(\mathtt{id}_X)) \\
&= \theta_Y(f^*(\mathtt{id}_X)) && \text{by naturality of } \theta \\
&= \theta_Y(f) \\
&= \theta_Y(f_*(\mathtt{id}_Y)) \\
&\triangleq (\theta \circ \mathrm{y}f)_Y(\mathtt{id}_Y) \\
&\triangleq \eta_{Y,F}(\theta \circ \mathrm{y}f) \\
&\triangleq \eta_{Y,F}((\mathrm{y}f)^*(\theta))
\end{aligned}$$

Corollary of the Yoneda Lemma:

the functor $\mathrm{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is full and faithful.

In general, a functor $F : \mathbf{C} \to \mathbf{D}$ is

- faithful if for all $X, Y \in \mathbf{C}$ the function

$$\begin{array}{ccc} \mathbf{C}(X,Y) & \to & \mathbf{D}(F(X),F(Y)) \\ f & \mapsto & F(f) \end{array}$$

  is injective:

$$\forall f, f' \in \mathbf{C}(X,Y),\ F(f) = F(f') \Rightarrow f = f'$$

- full if the above functions are all surjective:

$$\forall g \in \mathbf{D}(F(X),F(Y)), \exists f \in \mathbf{C}(X,Y),\ F(f) = g$$

Corollary of the Yoneda Lemma:

the functor $\mathrm{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is full and faithful.

Proof. From the proof of the Yoneda Lemma, for each $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ we have a bijection

$$F(X) \xrightarrow{(\eta_{X,F})^{-1}} \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F)$$

By definition of $(\eta_{X,F})^{-1}$, when $F = \mathrm{y}Y$ the above function is equal to

$$\begin{array}{rcl} \mathrm{y}Y(X) = \mathbf{C}(X,Y) & \to & \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, \mathrm{y}Y) \\ f & \mapsto & f_* = \mathrm{y}f \end{array}$$

So, being a bijection, $f \mapsto \mathrm{y}f$ is both injective and surjective; so y is both faithful and full. $\square$

Recall (for a small category $\mathbf{C}$):

Yoneda functor $\mathrm{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

Yoneda Lemma: there is a bijection $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \cong F(X)$ which is natural both in $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and $X \in \mathbf{C}$.

An application of the Yoneda Lemma:

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{op}}$ of presheaves is cartesian closed.

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Proof sketch.

Terminal object in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $1 : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ given by

$$\begin{cases} 1(X) \triangleq \{0\} & \text{terminal object in } \mathbf{Set} \\ 1(f) \triangleq \mathtt{id}_{\{0\}} \end{cases}$$

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

Product of $F, G \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $F \times G : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ given by

$$\begin{cases} (F \times G)(X) \triangleq F(X) \times G(X) & \text{cartesian product of sets} \\ (F \times G)(f) \triangleq F(f) \times G(f) \end{cases}$$

with projection morphisms $F \xleftarrow{\pi_1} F \times G \xrightarrow{\pi_2} G$ given by the natural transformations whose components at $X \in \mathbf{C}$ are the projection functions $F(X) \xleftarrow{\pi_1} F(X) \times G(X) \xrightarrow{\pi_2} G(X)$.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

We can work out what the value of the exponential $G^F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ at $X \in \mathbf{C}$ has to be using the Yoneda Lemma:

$$G^F(X) \underset{\text{Yoneda Lemma}}{\cong} \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, G^F) \underset{\text{universal property of the exponential}}{\cong} \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G)$$

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

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$$G^F(X) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, G^F) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G)$$

We take the set $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G)$ to be the definition of the value of G^F at X…

Exponential objects in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$:

$$\boxed{G^F(X) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G)}$$

Given $Y \xrightarrow{f} X$ in $\mathbf{C}$, we have $\mathrm{y}Y \xrightarrow{\mathrm{y}f} \mathrm{y}X$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and hence

$$\begin{array}{rcl} G^F(Y) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}Y \times F, G) & \longrightarrow & \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G) \triangleq G^F(X) \\ \theta & \longmapsto & \theta \circ (\mathrm{y}f \times \mathtt{id}_F) \end{array}$$

We define

$$\boxed{G^F(f) \triangleq (\mathrm{y}f \times \mathtt{id}_F)^*}$$

Have to check that these definitions make G^F ino a functor $\mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

Application morphisms in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$:

Given $F, G \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, the application morphism

$$\mathtt{app} : G^F \times F \longrightarrow G$$

is the natural transformation whose component at $X \in \mathbf{C}$ is the function

$$(G^F \times F)(X) \triangleq G^F(X) \times F(X) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G) \times F(X) \xrightarrow{\mathtt{app}_X} G(X)$$

defined by

$$\boxed{\mathtt{app}_X(\theta, x) \triangleq \theta_X(\mathtt{id}_X, x)}$$

Have to check that this is natural in X.

Currying operation in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$:

$$\left(H \times F \xrightarrow{\theta} G\right) \mapsto \left(H \xrightarrow{\mathtt{cur}\,\theta} G^F\right)$$

Given $H \times F \xrightarrow{\theta} G$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, the component of $\mathtt{cur}\,\theta$ at $X \in \mathbf{C}$

$$H(X) \xrightarrow{(\mathtt{cur}\,\theta)_X} G^F(X) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G)$$

is the function mapping each $z \in H(X)$ to the natural transformation $\mathrm{y}X \times F \to G$ whose component at $Y \in \mathbf{C}$ is the function

$$(\mathrm{y}X \times F)(Y) \triangleq \mathbf{C}(Y, X) \times F(Y) \to G(Y)$$

defined by

$$\boxed{((\mathtt{cur}\,\theta)_X(z))_Y(f, y) \triangleq \theta_Y(H(f)(z), y)}$$

Currying operation in $\mathbf{Set}^{\mathbf{C}^{op}}$:

$$\left(H \times F \xrightarrow{\theta} G\right) \mapsto \left(H \xrightarrow{\mathtt{cur}\,\theta} G^F\right)$$

$$\boxed{((\mathtt{cur}\,\theta)_X(z))_Y(f, y) \triangleq \theta_Y(H(f)(z), y)}$$

Have to check that this is natural in Y,

then that $(\mathtt{cur}\,\theta)_X$ is natural in X,

then that $\mathtt{cur}\,\theta$ is the unique morphism $H \xrightarrow{\varphi} G^F$ in $\mathbf{Set}^{\mathbf{C}^{op}}$ satisfying $\mathtt{app} \circ (\varphi \times \mathtt{id}_F) = \theta$.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

So we can interpret simply typed lambda calculus in any presheaf category.

More than that, presheaf categories (usefully) model dependently-typed languages.