University of Cambridge
2022/23 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 6

1. Recall (from Lecture 2) that a pre-ordered set $(P, \leq_P)$ determines a category $\mathbf{C}_P$ whose objects are the elements of P and whose morphism sets $\mathbf{C}_P(x, x')$ contain at most one element and do so iff $x \leq_P x'$. Note that given two pre-ordered sets $(P, \leq_P)$ and $(Q, \leq_Q)$, a functor $F : \mathbf{C}_P \to \mathbf{C}_Q$ is the same thing as a monotone function from $(P, \leq_P)$ to $(Q, \leq_Q)$.

   (a) Given two such functors $F, G : \mathbf{C}_P \to \mathbf{C}_Q$, how many natural transformations are there from F to G?

   (b) Given monotone functions $F : \mathbf{C}_P \to \mathbf{C}_Q$ and $G : \mathbf{C}_Q \to \mathbf{C}_P$, give a property of F and G which ensures that, regarding them as functors, G is right adjoint to F.

2. Recall that **Preord** denotes the category of pre-ordered sets and monotone functions. For each set X, let $(\mathrm{Pow}\, X, \subseteq) \in \mathrm{obj}\, \mathbf{Preord}$ be the set of all subsets of X partially ordered by inclusion (given $A, A' \in \mathrm{Pow}\, X$, $A \subseteq A'$ means $\forall x \in A,\ x \in A'$). Given a function $f : X \to Y$, let $f^{-1} : \mathrm{Pow}\, Y \to \mathrm{Pow}\, X$ be the function that maps each subset $B \subseteq Y$ to the subset $f^{-1}B \subseteq X$ defined by $f^{-1}B \triangleq \{x \in X \mid f(x) \in B\}$.

   (a) Show that f^{-1} is a monotone function and hence gives a morphism $(\mathrm{Pow}\, Y, \subseteq) \to (\mathrm{Pow}\, X, \subseteq)$ in **Preord**.

   (b) Regarding f^{-1} as a functor as in question (1), show that it has both left and right adjoints, given on objects by the following 'generalized quantifiers'

   $$\exists_f A \triangleq \{y \in Y \mid \exists x \in X,\ f(x) = y \ \wedge\ x \in A\}$$
   $$\forall_f A \triangleq \{y \in Y \mid \forall x \in X,\ f(x) = y \ \Rightarrow\ x \in A\}$$

   (for all $A \in \mathrm{Pow}\, X$). [Hint: use your answer to question 1b.]

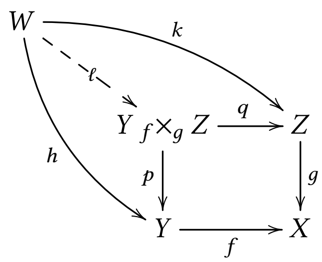
3. A category $\mathbf{C}$ *has pullbacks* if for every pair of $\mathbf{C}$-morphisms with a common codomain, $Y \xrightarrow{f} X \xleftarrow{g} Z$, there is an object $Y\,{}_f{\times}_g\,Z$ and morphisms p, q making the following diagram commute in $\mathbf{C}$ (that is, $f \circ p = g \circ q$)

   $$\begin{array}{ccc} Y\,{}_f{\times}_g\,Z & \xrightarrow{q} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array} \tag{1}$$

   and with the following universal property:

   For all $Y \xleftarrow{h} W \xrightarrow{k} Z$ in $\mathbf{C}$ with $f \circ h = g \circ k$, there is a unique morphism $\ell \in \mathbf{C}(W, Y\,{}_f{\times}_g\,Z)$

satisfying $p \circ \ell = h$ and $q \circ \ell = k$



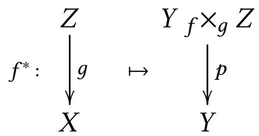
(2)

(a) For each $X \in \operatorname{obj} \mathbf{C}$, the *slice category* $\mathbf{C}/X$ is defined as follows: its objects are given by pairs (A, p) where $A \in \operatorname{obj} \mathbf{C}$ and $p \in \mathbf{C}(A, X)$; given two such objects (A, p) and (B, q), a morphism $f : (A, p) \to (B, q)$ in $\mathbf{C}/X$ is by definition a $\mathbf{C}$-morphism $f \in \mathbf{C}(A, B)$ such that $q \circ f = p$; composition and identities in $\mathbf{C}/X$ are given by those in $\mathbf{C}$.

Show that $\mathbf{C}$ has pullbacks iff for all $X \in \operatorname{obj} \mathbf{C}$ the slice category $\mathbf{C}/X$ has binary products.

(b) Show that if $\mathbf{C}$ has a terminal object and pullbacks, then it has binary products.

(c) Suppose $\mathbf{C}$ has pullbacks. Given $f \in \mathbf{C}(Y, X)$, show that the mapping



is the object part of a functor $f^* : \mathbf{C}/X \to \mathbf{C}/Y$ between slice categories.

(d) Show that the functor f^* in part (c) always has a left adjoint $f_! : \mathbf{C}/Y \to \mathbf{C}/X$, which on objects sends $(W, h) \in \operatorname{obj}(\mathbf{C}/Y)$ to $f_!(W, h) \triangleq (W, f \circ h) \in \operatorname{obj}(\mathbf{C}/X)$.

4. Suppose (T, η, μ) is a monad on a category $\mathbf{C}$ (see Lecture 16). Thus $T : \mathbf{C} \to \mathbf{C}$ is a functor and $\eta : \mathrm{id}_{\mathbf{C}} \to T$ and $\mu : T \circ T \to T$ are natural transformations satisfying $\mu \circ T\eta = \mathrm{id}_T = \mu \circ \eta_T$ and $\mu \circ \mu_T = \mu \circ T\mu$ (see Exercise Sheet 5, question 5 for the notation being used in those equations). The *Kleisli category* $\mathbf{C}_T$ of the monad is defined as follows. It has the same objects as $\mathbf{C}$; we will write FX for the object of $\mathbf{C}_T$ corresponding to an object $X \in \operatorname{obj} \mathbf{C}$. Given $X, Y \in \operatorname{obj} \mathbf{C}$, the set of morphisms in the Kleisli category from FX to FY is defined to be $\mathbf{C}_T(FX, FY) \triangleq \mathbf{C}(X, TY)$.

(a) Complete the definition of $\mathbf{C}_T$ by giving the definition of identity morphisms and composition satisfying the usual associativity and unity properties.

(b) Show that the mapping $X \in \operatorname{obj} \mathbf{C} \mapsto FX \in \operatorname{obj} \mathbf{C}_T$ extends to a functor $F : \mathbf{C} \to \mathbf{C}_T$.

(c) Show that the functor F has a right adjoint $G : \mathbf{C}_T \to \mathbf{C}$.

(d) Show that the monad associated with the adjunction $F \dashv G$ (see Lecture 16) is (T, η, μ).