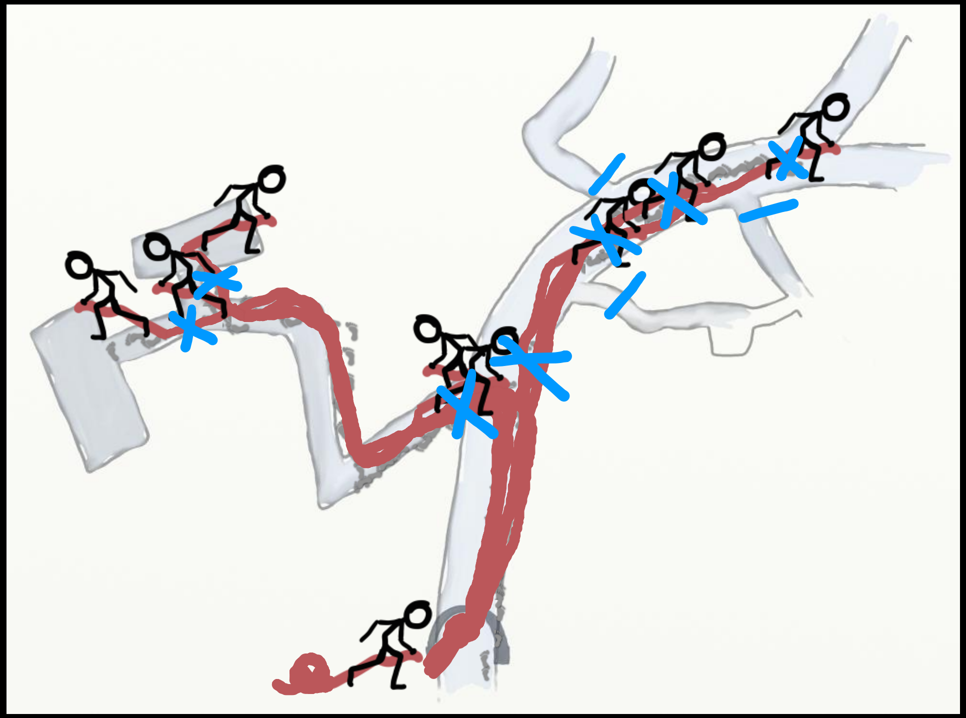
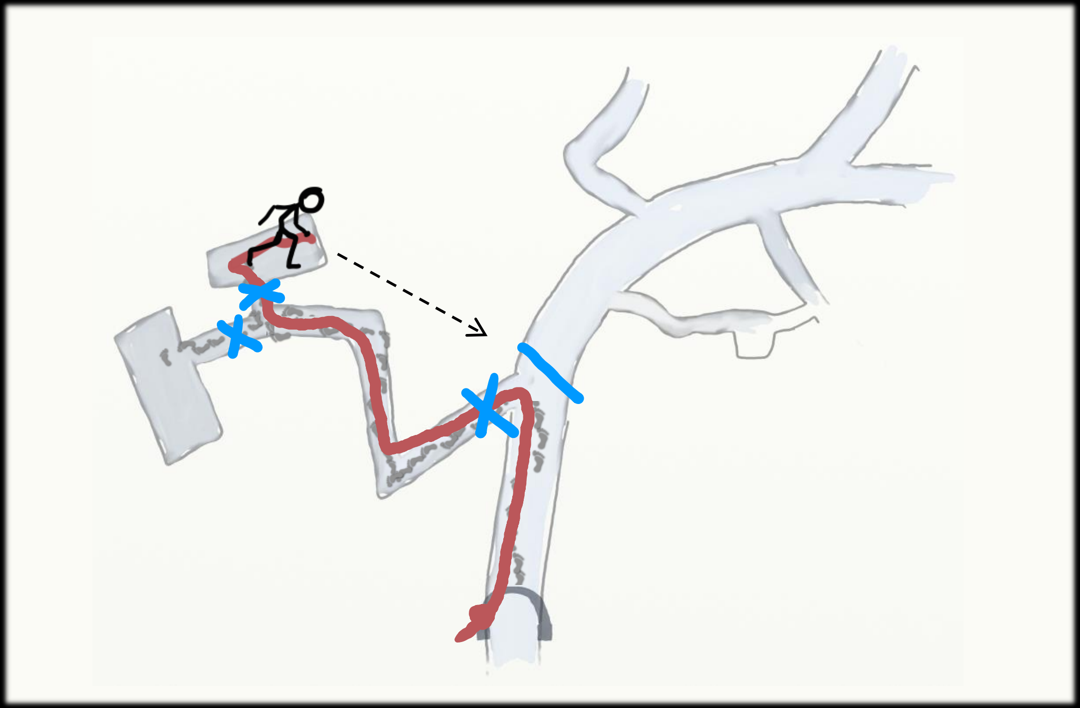
SECTION 5.1

Depth-first search

Ariadne's thread ...



but why not just teleport?



Analysis of running time for recursive dfs

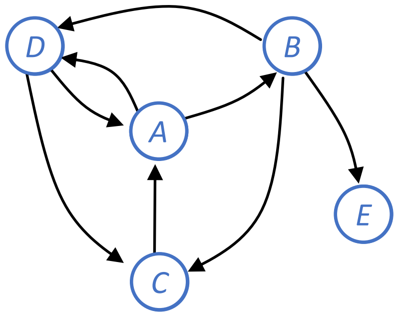
```
# visit all vertices reachable from s
def dfs_recurse(g, s):
    for v in g.vertices:
        v.visited = False
    visit(s)

def visit(v):
    v.visited = True
    for w in v.neighbours:
        if not w.visited:
            visit(w)
```

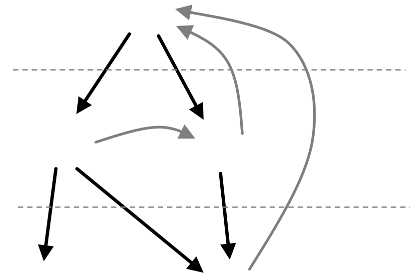
Analysis of running time for stack-based dfs

```python
# visit all vertices reachable from s
def dfs(g, s):
    for v in g.vertices:
        v.seen = False
    toexplore = Stack([s])
    s.seen = True

    while not toexplore.is_empty():
        v = toexplore.popright()
        for w in v.neighbours:
            if not w.seen:
                toexplore.pushright(w)
                w.seen = True
```

SECTION 5.2

Breadth-first search / finding shortest path



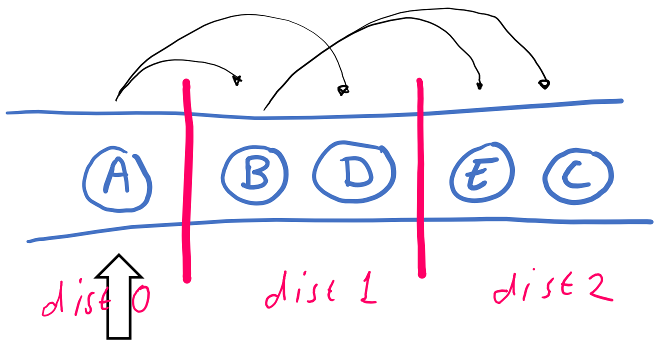


distance from A = 0

distance from A = 1

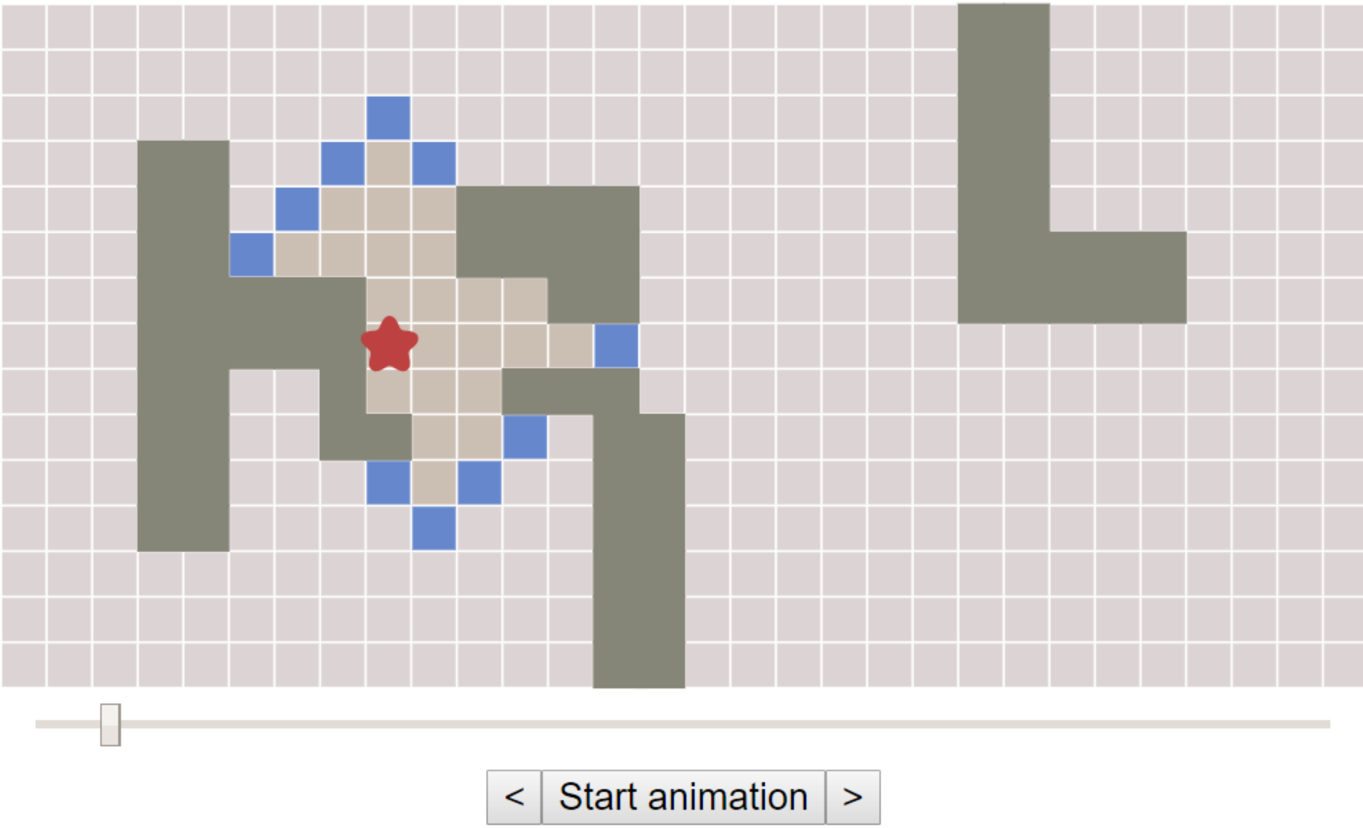
distance from A = 2

```
# Visit all the vertices in g reachable from start vertex s
def bfs(g, s):
    for v in g.vertices:
        v.seen = False
    toexplore = Queue([s])
    s.seen = True

    while not toexplore.is_empty():
        v = toexplore.popleft()
        for w in v.neighbours:
            if not w.seen:
                toexplore.pushright(w)
                w.seen = True
```



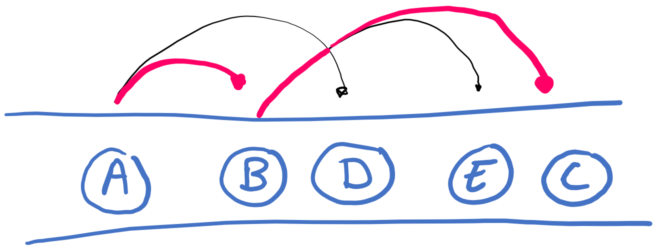
Breadth First Search

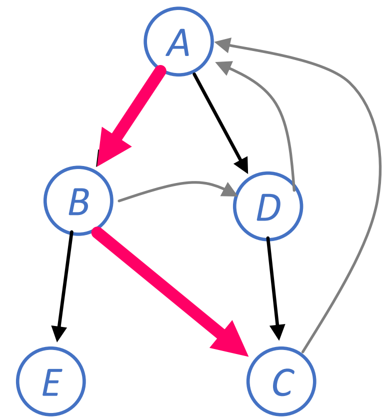
The key idea for all of these algorithms is that we keep track of an expanding ring called the ***frontier***. On a grid, this process is sometimes called "flood fill", but the same technique works for non-grids. **Start the animation** to see how the frontier expands:



https://www.redblobgames.com/pathfinding/a-star/introduction.html#breadth-first-search

```
# Find a path from s to t, if one exists
def bfs_path(g, s, t):
    for v in g.vertices:
        (v.seen, v.come_from) = (False, None)
...
    while not toexplore.is_empty():
        v = toexplore.popleft()
        for w in v.neighbours:
            if not w.seen:
                toexplore.pushright(w)
                (w.seen, w.come_from) = (True, v)
...
    if t.come_from has not been set:
        there is no path from s to t
    else:
        reconstruct the path from s to t,
        working backwards
```





Analysis of running time for bfs

```
# Visit all the vertices in g reachable from start vertex s
def bfs(g, s):
    for v in g.vertices:
        v.seen = False
    toexplore = Queue([s])
    s.seen = True

    while not toexplore.is_empty():
        v = toexplore.popleft()
        for w in v.neighbours:
            if not w.seen:
                toexplore.pushright(w)
                w.seen = True
```

Analysis of running time for stack-based dfs

```
# visit all vertices reachable from s
def dfs(g, s):
    for v in g.vertices:             } O(V)
        v.seen = False
    to_explore = Stack([s])          } O(1)
    s.seen = True

    while not to_explore.is_empty(): } at most once per vertex, so O(V)
        v = toexplore.popright()
        for w in v.neighbours:
            if not w.seen:           } run for every edge out of every
                toexplore.pushright(w)     vertex we visit, so O(E)
        w.seen = True
```

total $O(V+E)$

Department of Computer Science

cl.cam.ac.uk/teaching/2223/Algorithm2/materials.html

Schedule

This is the planned lecture schedule. It will be updated as and when actual lectures deviate from schedule. Links are to prerecorded videos. Slides will be uploaded the night before a lecture, and re-uploaded after the lecture with annotations made during the lecture.

5. Graphs and path finding

Lecture 13
- 5, 5.1 Graphs (14:27) code — graphs
- 5.2 Depth-first search (11:37)
- 5.3 Breadth-first search (6:43)
- Optional tick: bfs-all from ex4.q6

Lecture 14
- 5.4 Dikstra's algorithm (15:25) plus proof (24:01)

Lecture 15
- 5.5 Algorithms and proofs (9:29)
- 5.6 Bellman-Ford (12:13)
- Optional challenge: chatgpt-bfs
- Optional tick: bf-cycle from ex4.q19

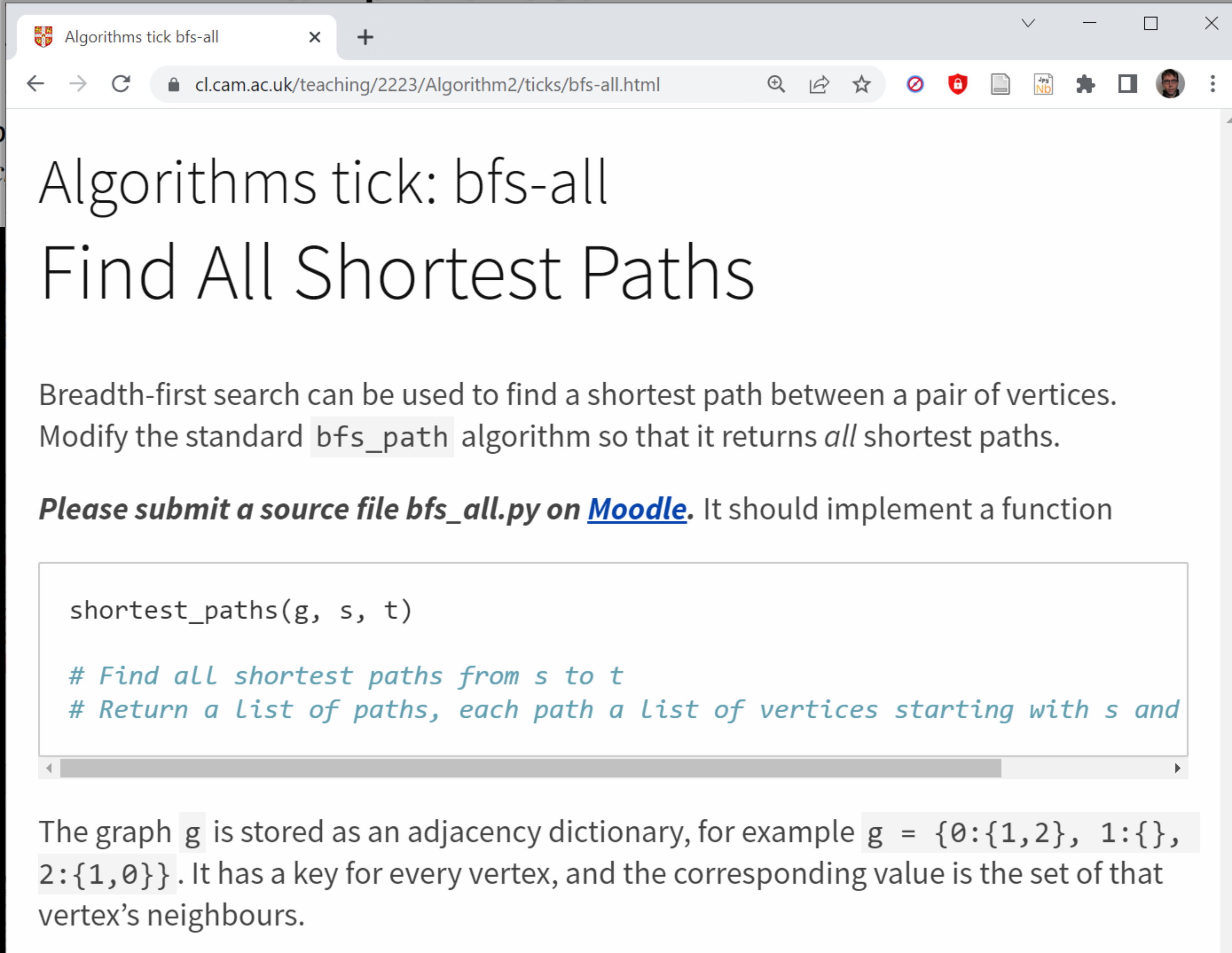
Lecture 16
- 5.7 Dynamic programming (13:06)
- 5.8 Johnson's algorithm (13:43)
- Example sheet 4 [pdf]

6. Graphs and subgraphs

Lecture 17
- 6.1 Flow networks (9:31) code — subgraphs
- 6.2 Ford-Fulkerson algorithm (21:55)

Example sheet 4

Question 6. Modify b ... rse *website, for you to chec*



Algorithms tick bfs-all

cl.cam.ac.uk/teaching/2223/Algorithm2/ticks/bfs-all.html

Algorithms tick: bfs-all

Find All Shortest Paths

Breadth-first search can be used to find a shortest path between a pair of vertices. Modify the standard `bfs_path` algorithm so that it returns *all* shortest paths.

Please submit a source file bfs_all.py on Moodle. It should implement a function

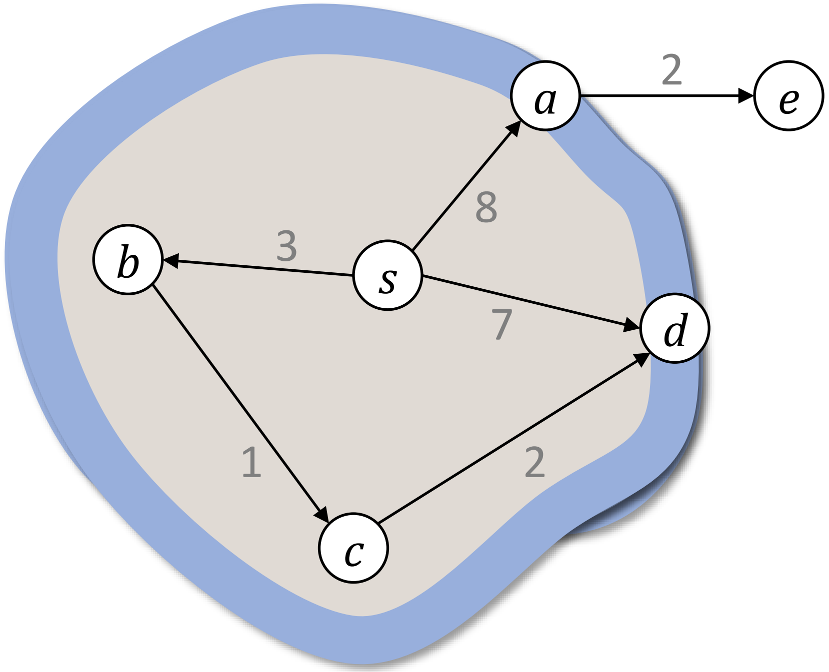
```
shortest_paths(g, s, t)

# Find all shortest paths from s to t
# Return a list of paths, each path a list of vertices starting with s and
```

The graph `g` is stored as an adjacency dictionary, for example `g = {0:{1,2}, 1:{}, 2:{1,0}}`. It has a key for every vertex, and the corresponding value is the set of that vertex's neighbours.

SECTION 5.3

Dijkstra’s algorithm

In a graph where the edges have costs (e.g. travel time), we can find shortest paths by using a similar “grow the frontier” algorithm to bfs.




```
def dijkstra(g, s):
    for v in g.vertices:
        v.distance = ∞
    s.distance = 0
    toexplore = PriorityQueue([s], sortkey = λv: v.distance)

    while not toexplore.is_empty():
        v = toexplore.popmin()
        # Assert: v.distance is distance(s to v)
        # Assert: v is never put back into toexplore
        for (w,edgecost) in v.neighbours:
            dist_w = v.distance + edgecost
            if dist_w < w.distance:
                w.distance = dist_w
                if w in toexplore:
                    toexplore.decreasekey(w)
                else:
                    toexplore.push(w)
```

popped	toexplore
$\{\}$	$[s]$
$\{s\}$	$[b, a, d]$
$\{s, b\}$	$[c, a, d]$
$\{s, b, c\}$	$[d, a]$

```
1   def dijkstra(g, s):
2       for v in g.vertices:
3           v.distance = ∞
4       s.distance = 0
5       toexplore = PriorityQueue([s], sortkey = λv: v.distance)
6
7       while not toexplore.is_empty():
8           v = toexplore.popmin()
9           # Assert: v.distance is distance(s to v)
10          # Assert: v is never put back into toexplore
11          for (w,edgecost) in v.neighbours:
12              dist_w = v.distance + edgecost
13              if dist_w < w.distance:
14                  w.distance = dist_w
15                  if w in toexplore:
16                      toexplore.decreasekey(w)
17                  else:
18                      toexplore.push(w)
```

$$\text{Total} = O(V) + O(V) \times \underbrace{C_{\text{popmin}}}_{O(\log n)} + O(E) \times \underbrace{C_{\text{push/dec.key}}}_{O(1)}$$

where n = # items stored

$n \le V$ by line 10

$$= O(E + V \log V)$$

Right from the beginning, and all through the course, we stress that the programmer's task is not just to write down a program, but that his main task is to give a formal proof that the program he proposes meets the equally formal functional specification.

Programming is one of the most difficult branches of applied mathematics; the poorer mathematicians had better remain pure mathematicians.

Edsger Dijkstra (1930—2002)
On the cruelty of really teaching computer science, 1988

Problem statement

Given a directed graph in which each edge is labelled with a cost ≥ 0, and a start vertex s, compute the distance from s to every other vertex, where …

$\mathrm{cost}(u \to v)$ is the *cost* associated with edge $u \to v$

$\mathrm{cost}(u \to \cdots \to v)$ is the sum of edge costs along the path $u \to \cdots \to v$

$$\mathrm{distance}(u \text{ to } v) = \begin{cases} \text{min cost of any path } u \to \cdots \to v, & \text{if one exists} \\ 0, & \text{if } u = v \\ \infty, & \text{otherwise} \end{cases}$$

The "breakpoint" proof strategy

1. Decide on a property we want to be true at all times
2. Assume it's true up to time $T - 1$
3. Show that it must therefore be true at time T

Assertion line 10.

A vertex v, once popped, is never put back into the priority queue

Proof.

1. A vertex w is only pushed into the priority queue when we discover a path shorter than $w.\texttt{distance}$
2. Once v is popped, $v.\texttt{distance} = \text{distance}(s \text{ to } v)$, so there can be no shorter path

Hence v is never pushed back.

```
8  v = toexplore.popmin()
9  # Assert: v.distance is distance(s to v)
10 # Assert: v is never put back into toexplore
11 for (w,edgecost) in v.neighbours:
12     dist_w = v.distance + edgecost
13     if dist_w < w.distance:
14         w.distance = dist_w
15         if w in toexplore:
16             toexplore.decreasekey(w)
17         else:
18             toexplore.push(w)
```

Theorem.

i. The algorithm terminates

ii. When it does, for every vertex v, $v.\mathtt{distance} = \mathrm{distance}(s \text{ to } v)$

iii. The two assertions never fail