Normal-order reduction strategy

Informal Description

Always reduce the leftmost outermost redex.

Example

$(\lambda x.x(\mathbf{I}z))\mathbf{I} \longrightarrow \mathbf{I}(\mathbf{I}z) \longrightarrow \mathbf{I}z \longrightarrow z$

Fact

If a term has a normal form, then the normal order reduction strategy will find it.

Call-by-need reduction

Informal description

Call-by-need is similar to call-by-name in that it only evaluates an argument when necessary, and different from call by name in that it never evaluates a reduction more than once.

Examples

$(\lambda x.y)(Iz) \longrightarrow y[x = Iz] = y;$

$(\lambda x.y)\Omega \longrightarrow y[x = \Omega] = y;$

$(\lambda x.f(x)(x))(Iz) \longrightarrow f(x)(x)[x = Iz] \longrightarrow f(x)(x)[x = z] = f(z)(z).$

Call-by-name reduction

Informal description

To compute the value of $(\lambda x.M)N$, first substitute N for x in M.

Examples

$(\lambda x.y)(Iz) \longrightarrow y$, where $I \equiv \lambda z.z;$

$(\lambda x.y)\Omega \longrightarrow y$, where $\Omega \equiv (\lambda x.xx)(\lambda x.xx);$

$(\lambda x.f(x)(x))(Iz) \longrightarrow f(Iz)(Iz) \twoheadrightarrow f(z)(z).$

Call-by-value reduction

Informal description

To compute the value of $(\lambda x.M)N$, first compute the value of N.

Examples

$(\lambda x.y)(Iz) \longrightarrow (\lambda x.y)z \longrightarrow y;$

$(\lambda x.y)\Omega \longrightarrow (\lambda x.y)\Omega \longrightarrow \ldots;$

$(\lambda x.f(x)(x))(Iz) \longrightarrow (\lambda x.f(x)(x))(z) \longrightarrow f(z)(z).$

Ordered Pairs

Pairing operator

$$\textbf{pair} \equiv \lambda xyf.fxy$$

Destructors for pairing

$$\begin{aligned} \textbf{fst} &\equiv \lambda p.p\ \textbf{true} \\ \textbf{snd} &\equiv \lambda p.p\ \textbf{false} \end{aligned}$$

Facts

$$\begin{aligned} \textbf{fst}\ (\textbf{pair}\ MN) &\twoheadrightarrow M \\ \textbf{snd}\ (\textbf{pair}\ MN) &\twoheadrightarrow N \end{aligned}$$

Church's numerals

$$\underline{n} \equiv \lambda fx.f^n(x)$$

For example

$$\begin{aligned} \underline{0} &\equiv \lambda fx.x \\ \underline{1} &\equiv \lambda fx.f(x) \\ \underline{2} &\equiv \lambda fx.f(f(x)) \\ \underline{3} &\equiv \lambda fx.f(f(f(x))) \\ &\vdots \end{aligned}$$

The Booleans

Booleans

$$\textbf{true} \equiv \lambda xy.x \qquad \textbf{false} \equiv \lambda xy.y$$

Conditional operator

$$\textbf{if} \equiv \lambda pxy.pxy$$

Logical operators

$$\begin{aligned} \textbf{and} &\equiv \lambda pq.\,\textbf{if}\ p\ q\ \textbf{false} \\ \textbf{not} &\equiv \lambda p.\,\textbf{if}\ p\ \textbf{false}\ \textbf{true} \end{aligned}$$

Tuples

Tuples can be encoded using nested pairing:

$$(M_1, \ldots, M_n) \equiv \textbf{pair}\ M_1\ (M_2, \ldots, M_n), \quad n \geq 2$$

General projections

$$\begin{aligned} \textbf{proj}_i^n &\equiv \lambda p.\ \textbf{fst}\ (\textbf{snd}^{i-1}(p)), \quad 1 \leq i < n \\ \textbf{proj}_n^n &\equiv \lambda p.\ \textbf{snd}^{n-1}(p) \end{aligned}$$

Fact

$\textbf{proj}_i^n(M_1, \ldots M_n) \twoheadrightarrow M_i$ for each $1 \leq i \leq n$.

Predecessor

Auxilary function

$$\textbf{prefn} \equiv \lambda fp.\textbf{pair}\ (f(\textbf{fst}\ p))(\textbf{fst}\ p)$$

Predecessor

$$\textbf{pre} \equiv \lambda nfx.\textbf{snd}\ (n(\textbf{prefn}\ f)(\textbf{pair}\ x\ x))$$

Subtraction

$$\textbf{sub} \equiv \lambda mn.n\ \textbf{pre}\ m$$

Basic Operations

$$\begin{aligned} \textbf{suc} &\equiv \lambda nfx.f(nfx) \\ \textbf{add} &\equiv \lambda mnfx.mf(nfx) \\ \textbf{iszero} &\equiv \lambda n.n(\lambda x.\textbf{false})\textbf{true} \end{aligned}$$

Facts

$$\begin{aligned} \textbf{suc}\ (\underline{n}) &\twoheadrightarrow \underline{n+1} \\ \textbf{add}\ \underline{m}\ \underline{n} &\twoheadrightarrow \underline{m+n} \\ \textbf{iszero}(\underline{0}) &\twoheadrightarrow \textbf{true} \\ \textbf{iszero}(\underline{n+1}) &\twoheadrightarrow \textbf{false} \end{aligned}$$