Quantum LOCAL Model & Cycle Coloring

Longcheng Li

Cambridge

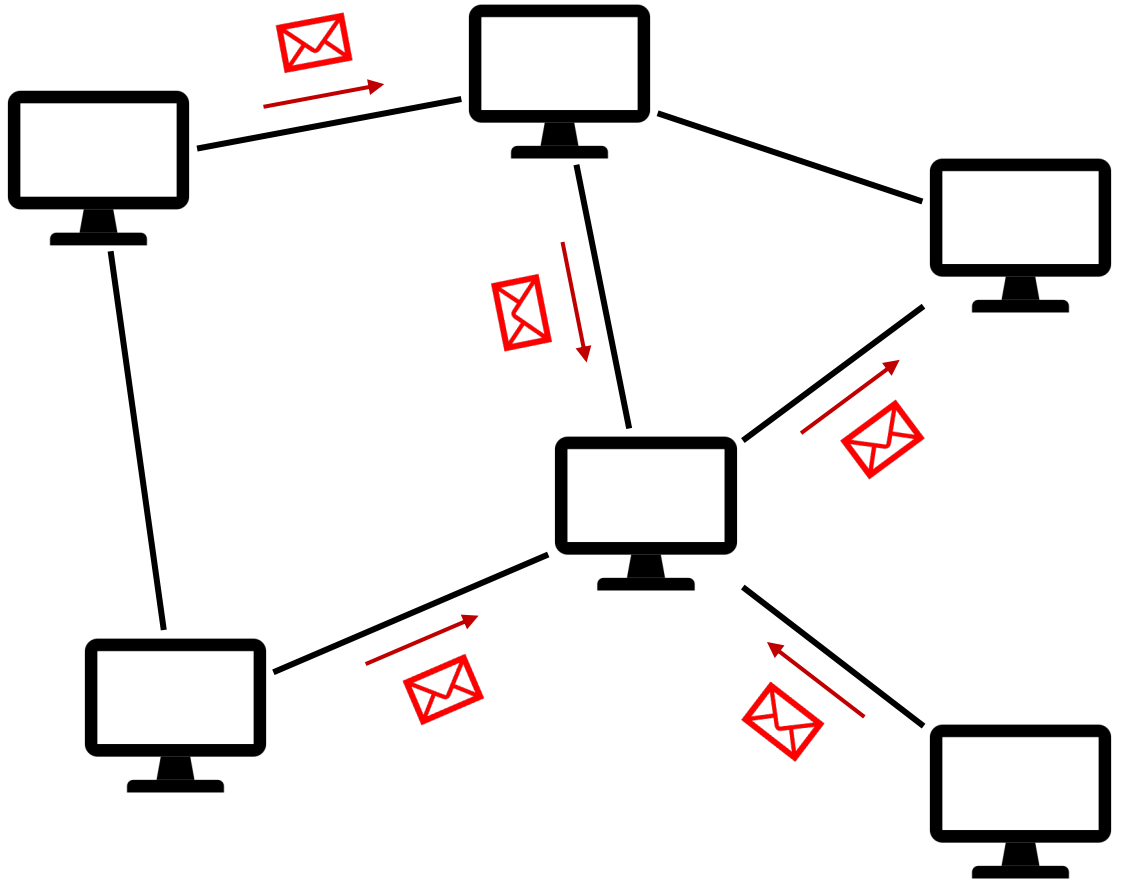
Contents

1. Classical LOCAL model & distributed cycle coloring
2. Quantum LOCAL model & advantages from non-local games
3. New results: Quantum cycle coloring

Distributed computing

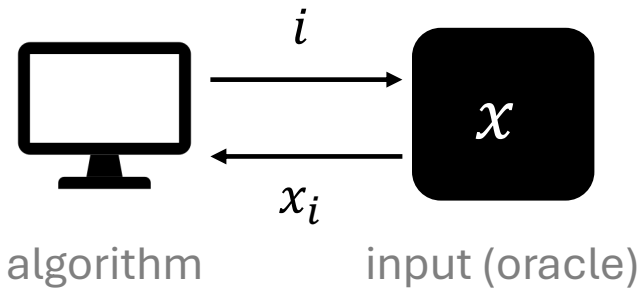
- A network of **computers**, each with its own local input
- Linked by **communication links**
- Cooperate by **sending messages** to solve a global task

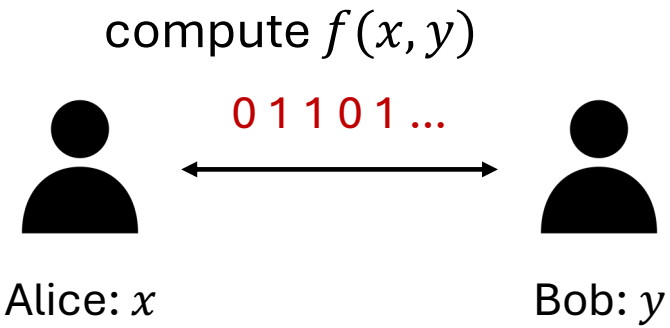
e.g. the Internet, data centers, sensor networks, blockchains

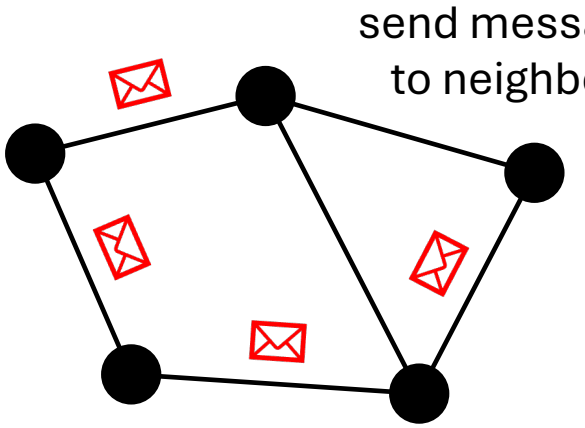


To make things easier

Complexity theorists like to isolate one specific cost:







Query complexity

queries

Communication complexity

bits communicated

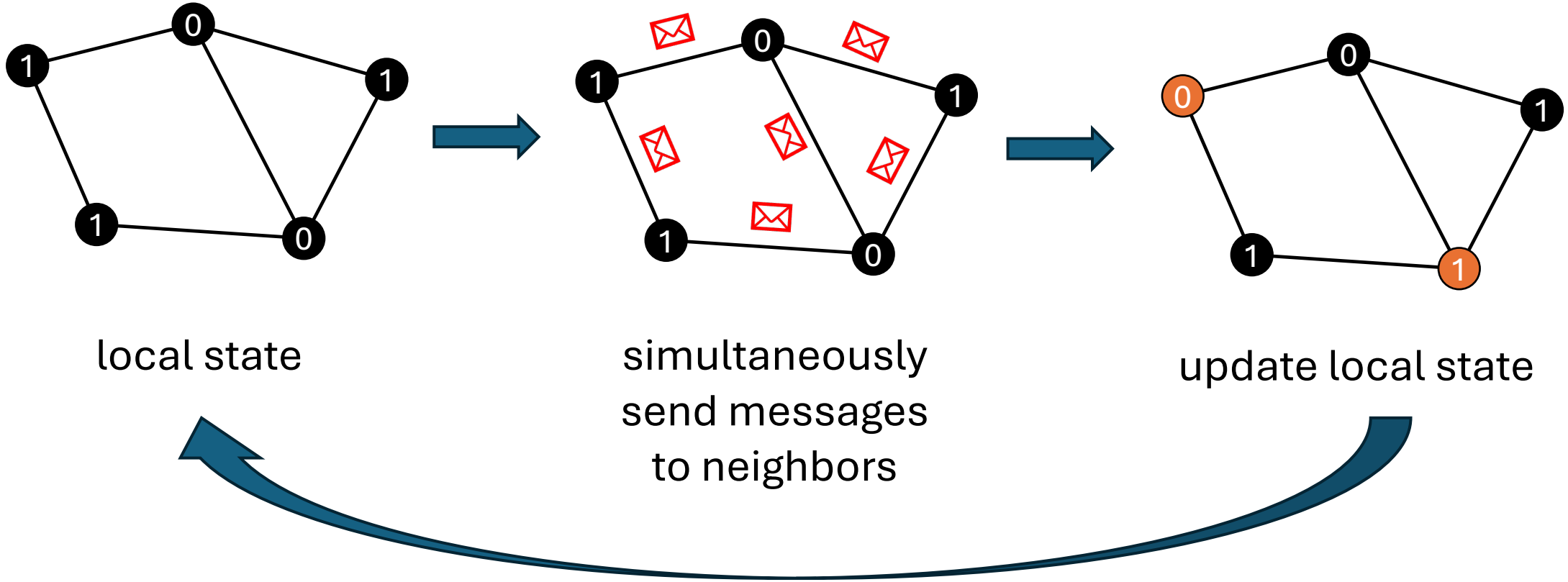
Distributed computing

LOCAL model: # rounds

Everything else (e.g. local computation) is free.

What happens in each round

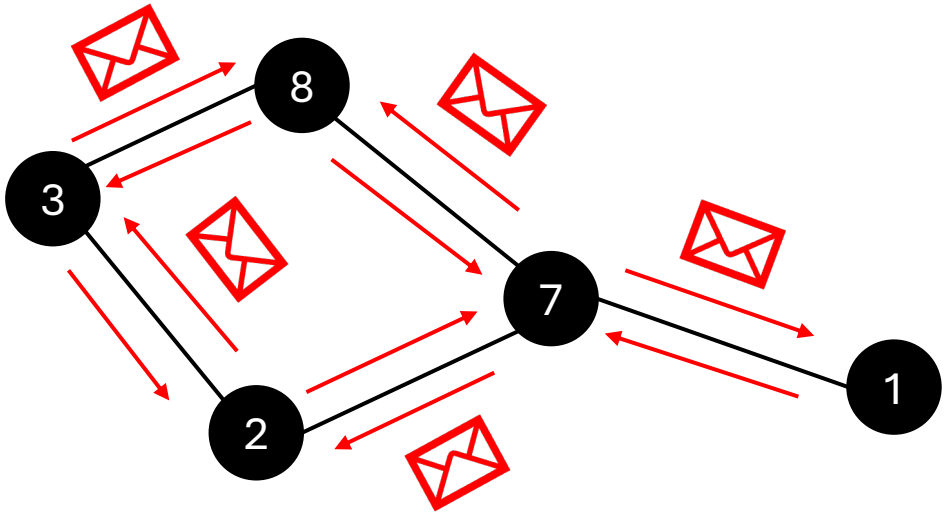
In one round:



LOCAL model

Given a graph $G = (V, E)$

- V are n processors
- E are communication links



- Start with unique IDs $\in [\mathrm{poly}(n)]$ (MAC address)
- Unbounded local computation and unbounded message length
- Locality/Round complexity: min #communication rounds

* $[m] := \{1,2,3, \dots, m\}$

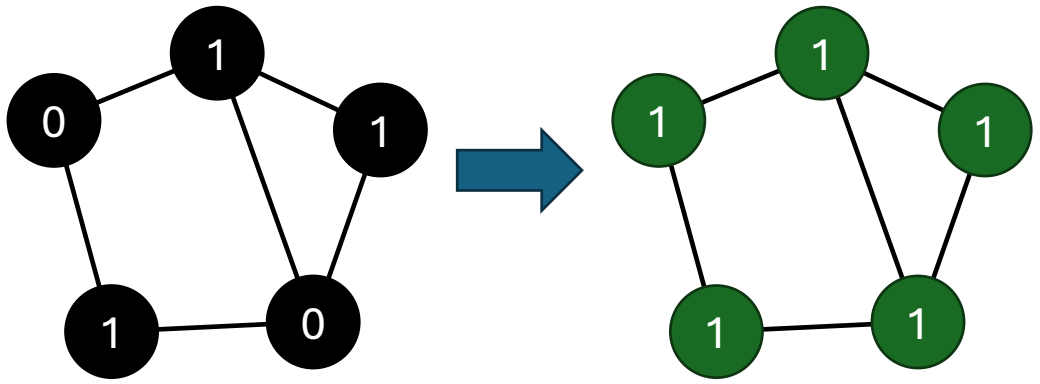
Examples

Compute a global function:

- Consensus: agree on a single bit

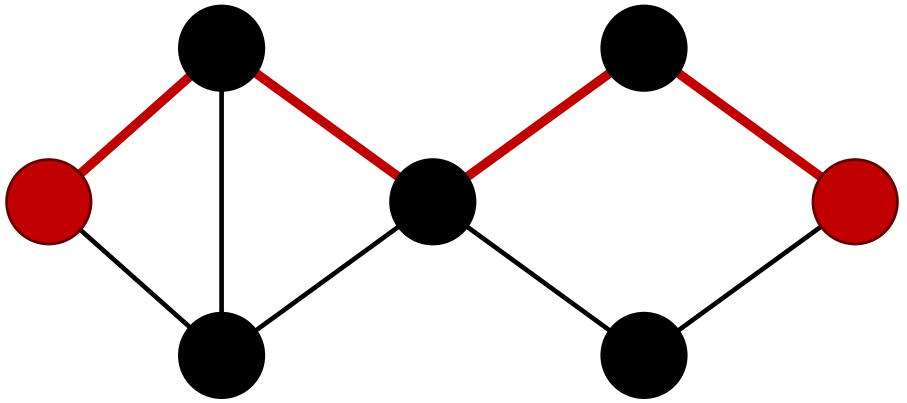
Learn network graph properties:

- Diameter of the network?



inputs

outputs: all equal



diameter = 4

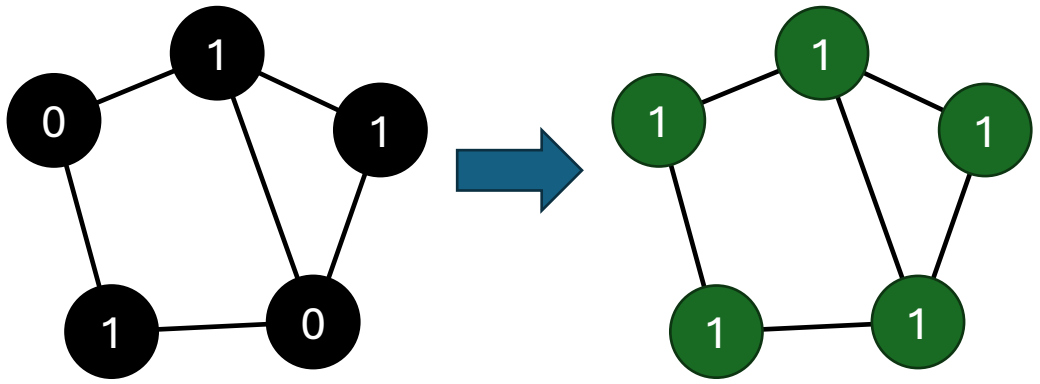
Examples

Compute a global function:

- Consensus: agree on a single bit

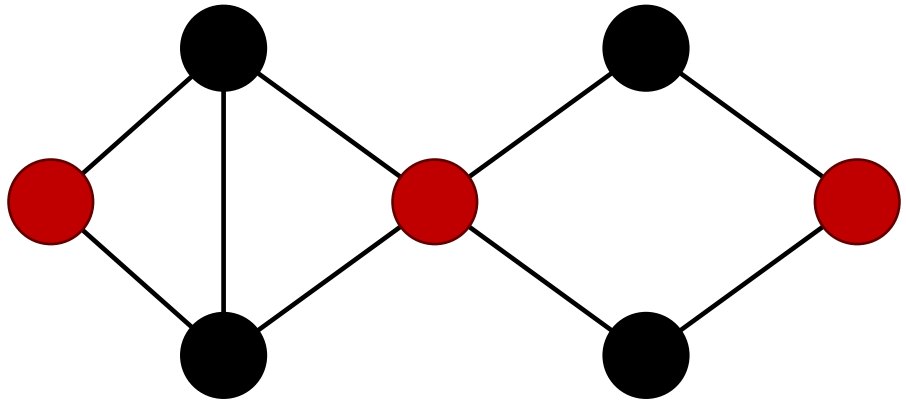
Learn network graph properties:

- Diameter of the network?
- Maximal independent set



inputs

outputs: all equal



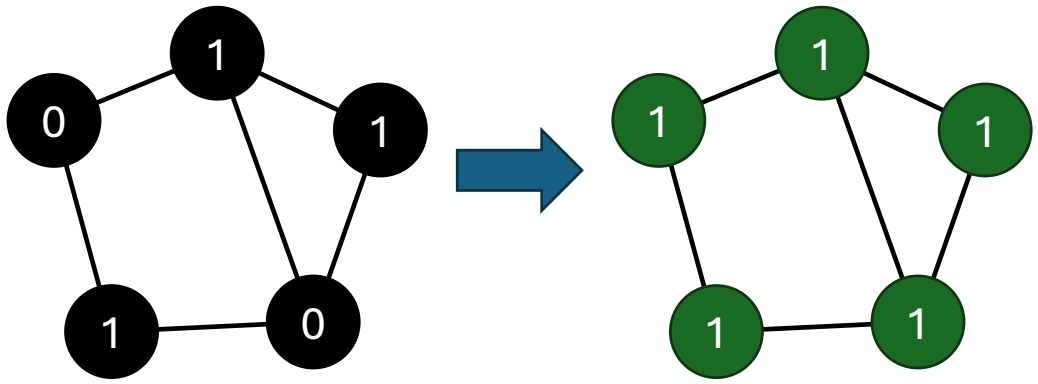
independent and maximal

Examples

"Distributed graph problems"

Compute a global function:

- Consensus: agree on a single bit

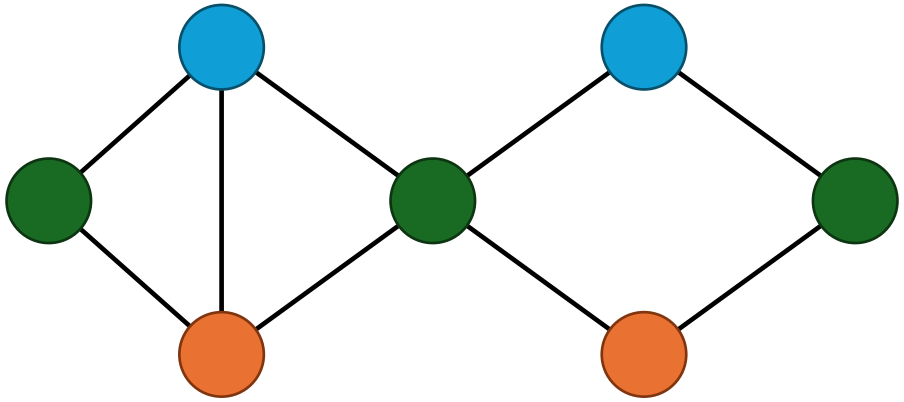


inputs

outputs: all equal

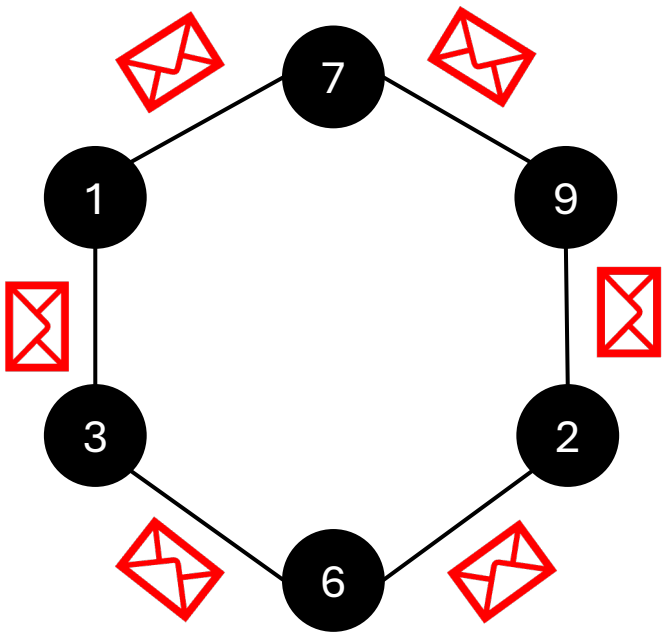
Learn network graph properties:

- Diameter of the network?
- Maximal independent set
- Coloring



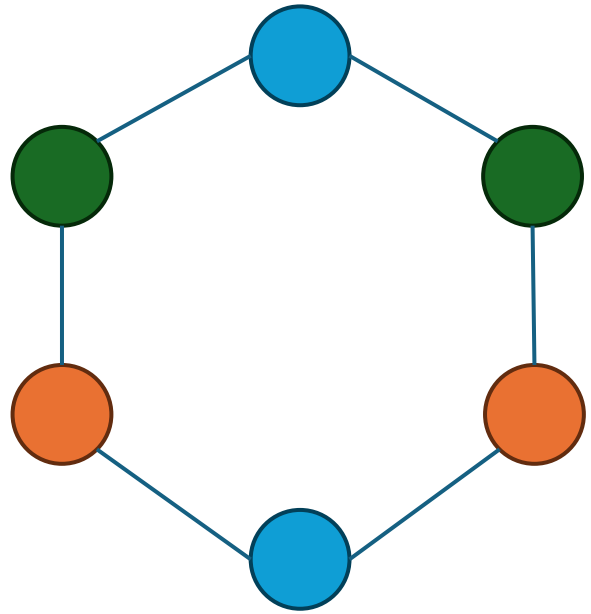
proper 3-coloring:
neighbors have different colors

Coloring a cycle [Cole–Vishkin'86]



n-cycle

After some rounds of communication

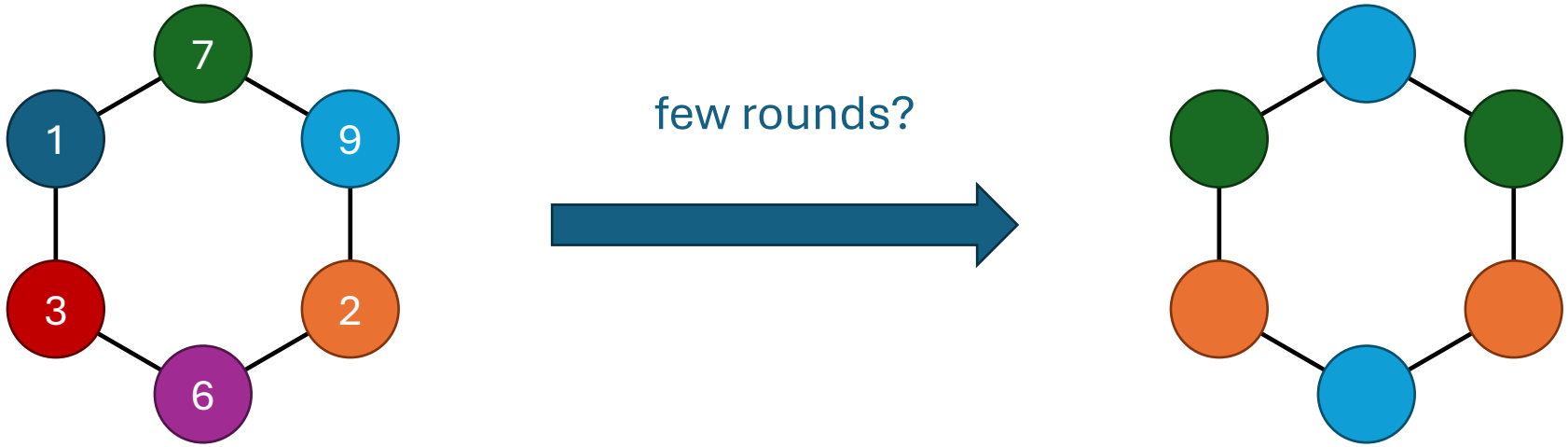


a proper 3-coloring

Reducing number of colors

- Unique IDs = a proper n^c-coloring
- Question:

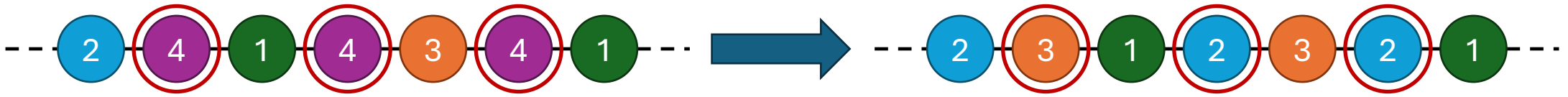
Starting from a q-coloring, how can we reduce #colors with communication?



IDs: an n^c-coloring

a 3-coloring

$q \to (q-1)$ in one round $(q \geq 4)$



a q-coloring $(q = 4)$ a $(q-1)$-coloring

How?

For each node with color q:

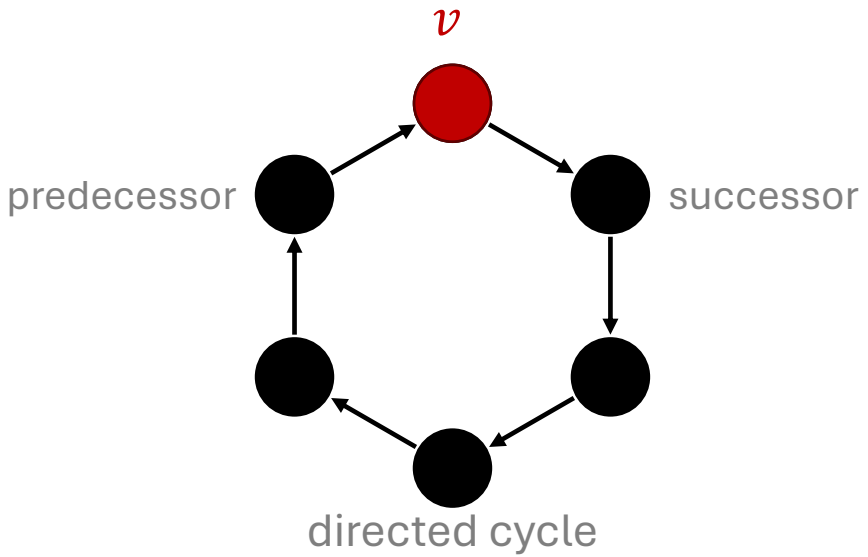
- pick a new color from $\{1,2,3\} \setminus \{\text{colors of 2 neighbors}\}$.

Repeating it gives a $\text{poly}(n)$-round algorithm for 3-coloring

Can we do faster?

$q \to 2\lceil \log q \rceil$ colors in one round

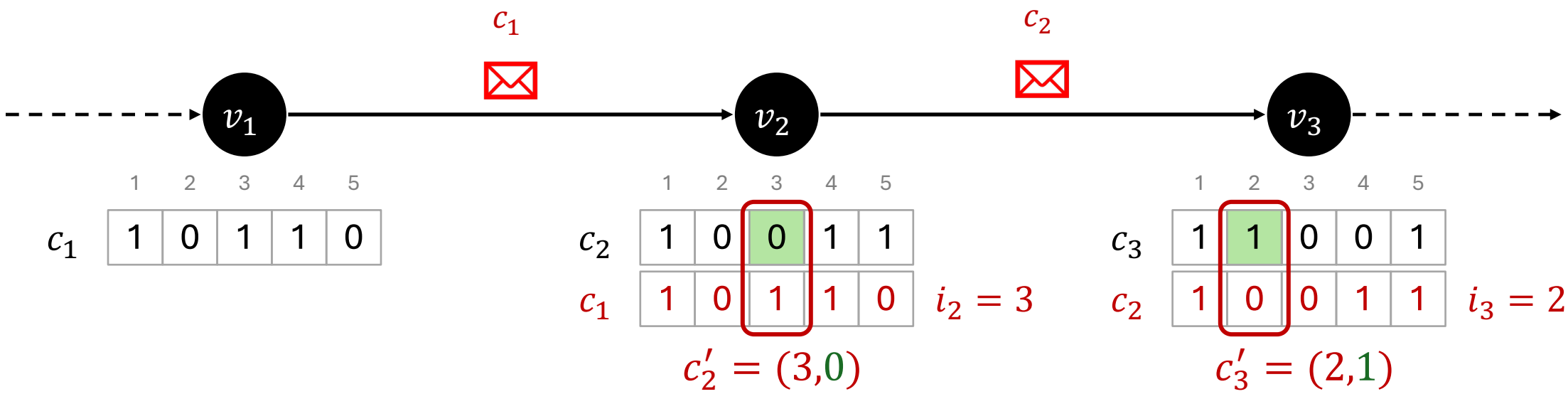
First, assume nodes know the *direction* of the cycle:



- Write colors in binary: $c_v \in \{0,1\}^L$ with $L = \lceil \log q \rceil$ bits

  e.g. $q = 32, L = 5$: $c_v = 10011$

$q \to 2\lceil \log q \rceil$ colors in one round



1. Every v_ℓ sends its color c_ℓ to $v_{\ell+1}$
2. Find the least index i where bits differ.
3. Set the new color $c'_\ell := (i, c_\ell[i])$

index bit

#new colors: $2L = 2\lceil \log q \rceil$

Why does it work?

- If indices differ, then done ✓
- Otherwise, bits must differ ✓

$\} \; c'_i \neq c'_{i+1}$

* c proper ⇒ $c_{\ell-1} \neq c_\ell$, so i_ℓ is well defined

Apply color reduction iteratively

IDs

$$n^c \longrightarrow O(\log n) \longrightarrow O(\log\log n) \longrightarrow \cdots \longrightarrow 6 \longrightarrow 5 \longrightarrow 4 \longrightarrow 3$$

$q \to 2\lceil \log q \rceil$ (Cole–Vishkin step)
$O(\log^* n)$ rounds

$q \to (q-1)$
3 rounds

It is super fast: $O(\log^* n)$ rounds in total

$\log^* n :=$ how many times we must apply log to reduce n to a constant

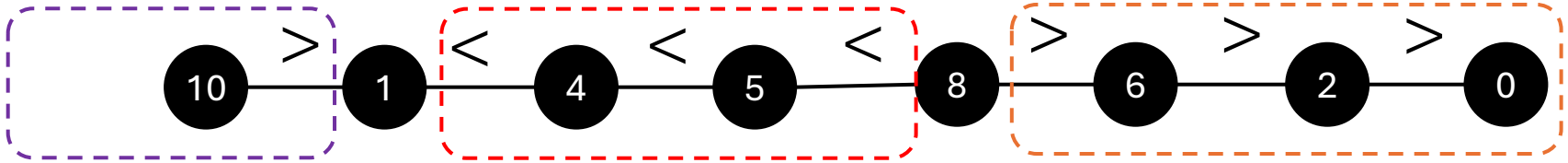
$2^{65536} \to 65536 \to 16 \to 4 \to 2 \to 1$: $\log^* n \leq 5$ for all $n \leq 2^{65536}$

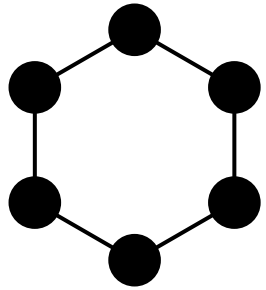
Cole–Vishkin Algorithm (1986)

Theorem: we can 3-color a directed cycle in $O(\log^* n)$ rounds.

1. What about undirected cycles?

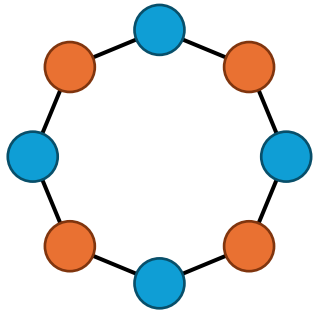
   Decompose cycle into directed paths:





2. What about 2-coloring (even) cycles?

   $\Theta(n)$ rounds: u, v get the same color iff $\mathrm{dist}(u, v)$ is even, a global property that no node can see in $o(n)$ rounds.



Is $O(\log^* n)$ optimal?

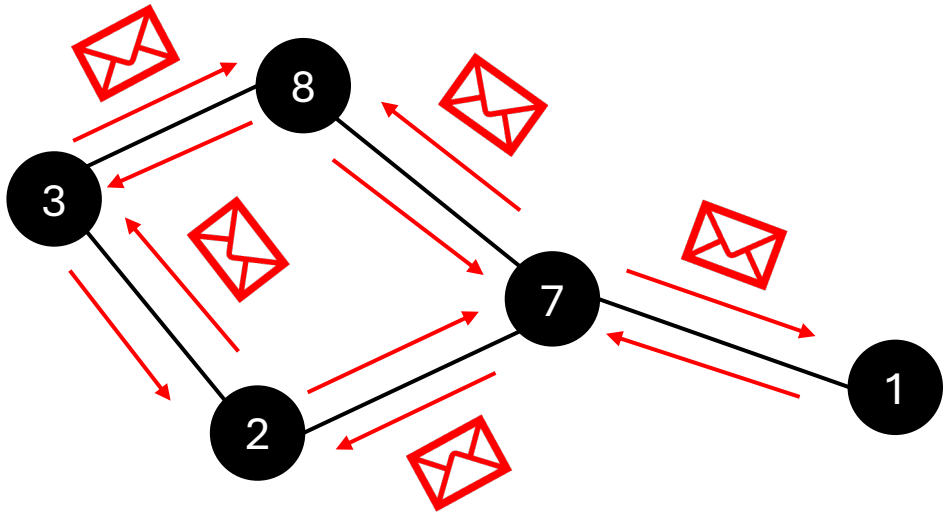
Theorem [Linial'92, Naor'91]: 3-coloring an n-cycle needs $\Omega(\log^* n)$ rounds, even for randomized algorithms.



Randomized LOCAL model

- Nodes have unbounded private randomness.
- Only require success with high probability i.e., $1 - 1/\mathrm{poly}(n)$.
- ~~Start with unique ID $\in [\mathrm{poly}(n)]$~~

independently sample ID from $[n^{10}]$
$\Rightarrow$ pairwise distinct w.p. $\geq 1 - n^{-8}$

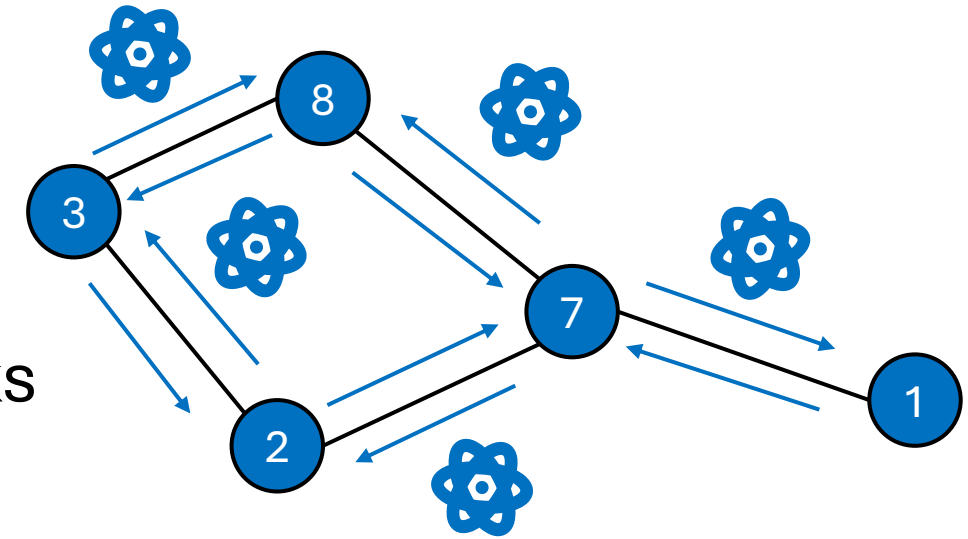


* In centralized computing, we can boost success probability by repeating, but it is not obvious in distributed computing.

Quantum LOCAL model

Given a graph $G = (V, E)$

- V are n quantum processors
- E are quantum communication links



- ~~Start with unique identifiers $\in [\mathrm{poly}(n)]$~~ (random $\subseteq$ quantum)
- Unbounded local computation and unbounded message length
- Locality/round complexity: min #communication rounds
- No prior-shared randomness or entanglement

Quantum advantages in LOCAL model

Can quantum algorithms solve problems in fewer rounds?

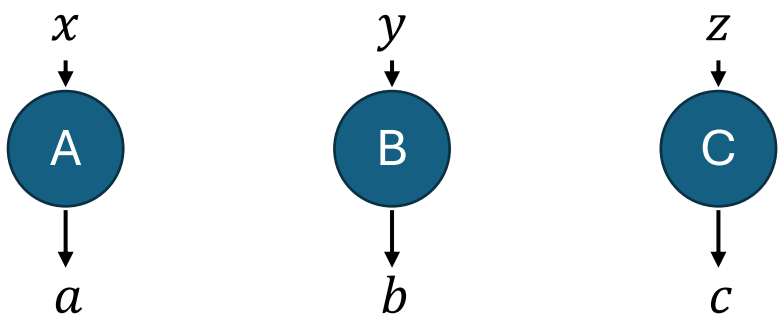


- Advantages from communication / query model (like Grover) **do not** work due to unlimited bandwidth.
- Yes, from non-local quantum games.

GHZ game

3 players input $x, y, z \in \{0,1\}$ with $x \oplus y \oplus z = 0$ and output a, b, c **w/o communication**.

They win iff $a \oplus b \oplus c = x \vee y \vee z$

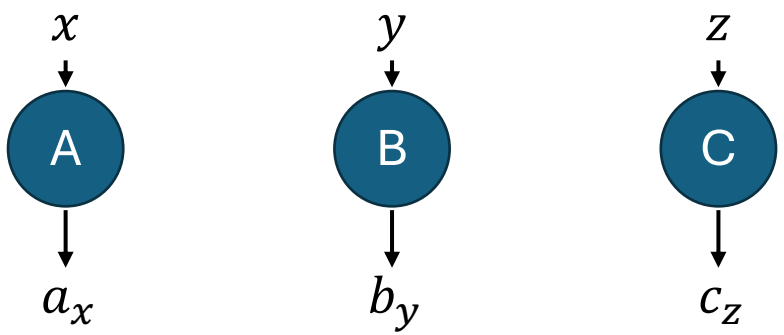


x y z	$x \vee y \vee z$	constraint
0 0 0	0	$a \oplus b \oplus c = 0$
0 1 1	1	$a \oplus b \oplus c = 1$
1 0 1	1	$a \oplus b \oplus c = 1$
1 1 0	1	$a \oplus b \oplus c = 1$

GHZ game: classical players

3 players input $x, y, z \in \{0,1\}$ with $x \oplus y \oplus z = 0$ and output a, b, c **w/o communication**.

They win iff $a \oplus b \oplus c = x \vee y \vee z$



x y z	$x \vee y \vee z$	constraint
0 0 0	0	$a_0 \oplus b_0 \oplus c_0 = 0$
0 1 1	1	$a_0 \oplus b_1 \oplus c_1 = 1$
1 0 1	1	$a_1 \oplus b_0 \oplus c_1 = 1$
1 1 0	1	$a_1 \oplus b_1 \oplus c_0 = 1$

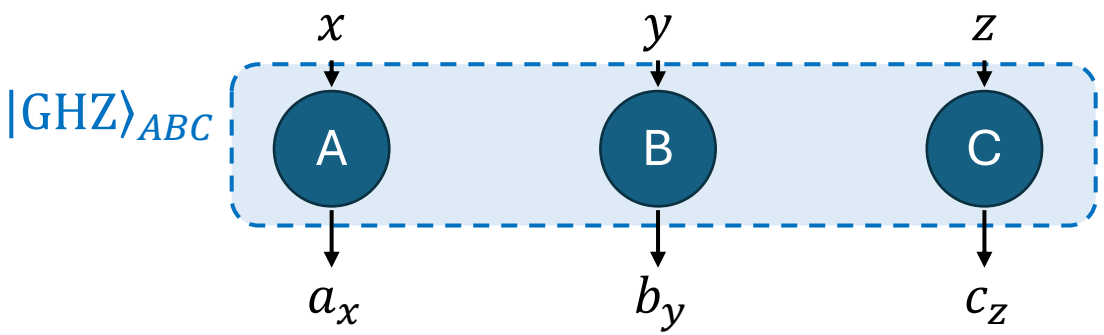
Total parity: $0 \neq 1$

- At most 3 of 4 constraints can be satisfied
- Classical: winning probability ≤ 3/4

GHZ game: quantum players

3 players input $x, y, z \in \{0,1\}$ with $x \oplus y \oplus z = 0$ and output a, b, c **w/o communication**.

They win iff $a \oplus b \oplus c = x \vee y \vee z$



$$|\mathrm{GHZ}\rangle_{ABC} := \frac{|0\rangle_A|0\rangle_B|0\rangle_C + |1\rangle_A|1\rangle_B|1\rangle_C}{\sqrt{2}}$$

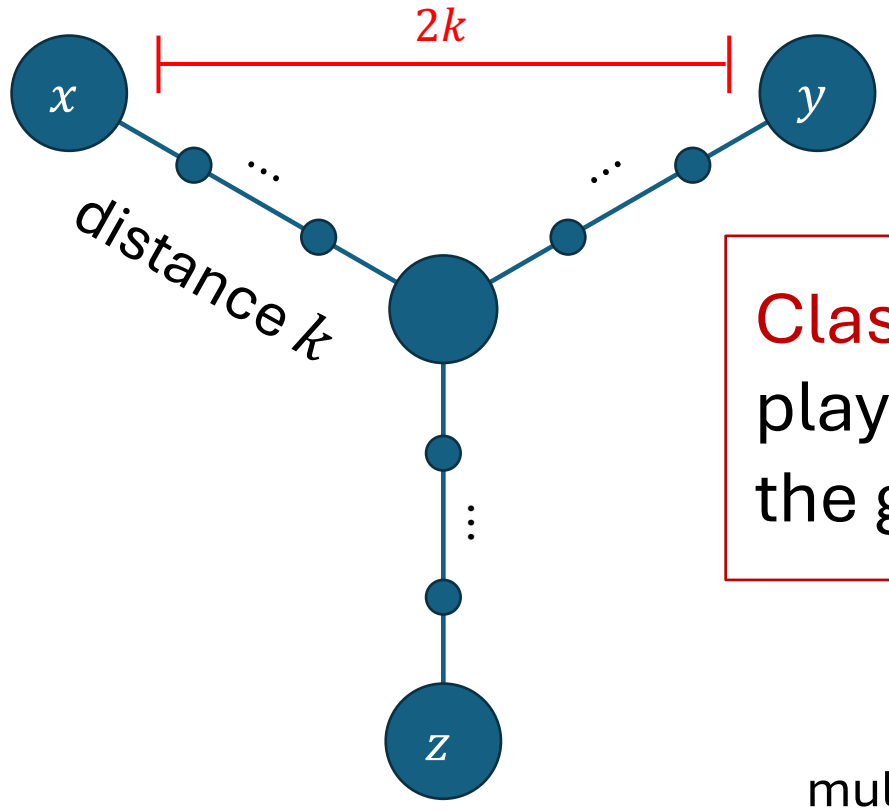
x y z	measure	outcome product	$a \oplus b \oplus c$
0 0 0	XXX	$+1$	0 ✓
0 1 1	XYY	-1	1 ✓
1 0 1	YXY	-1	1 ✓
1 1 0	YYX	-1	1 ✓

* $|\mathrm{GHZ}\rangle$ is a ± 1 eigenvector of each observable.

Quantum strategy: measure in X (Y) basis if input is 0 (1). ⇒ win with probability 1

GHZ game on a tripod [GKM'09]

Problem: Require 3 leaves to play GHZ game on a tripod graph



Classical: needs $2k$ rounds because players must communicate to win the game.

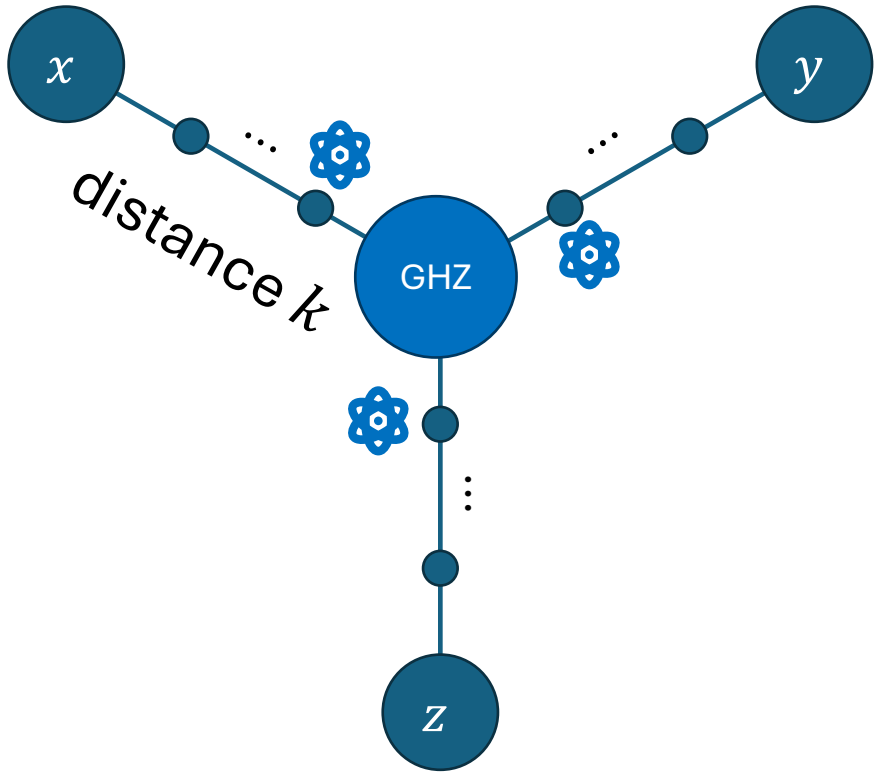
$< 2k$ rounds, winning prob $\leq 3/4$



multiple copies of tripods: global winning prob $\leq \epsilon$

GHZ game on a tripod [GKM'09]

Problem: Require 3 leaves to play GHZ game on a tripod graph

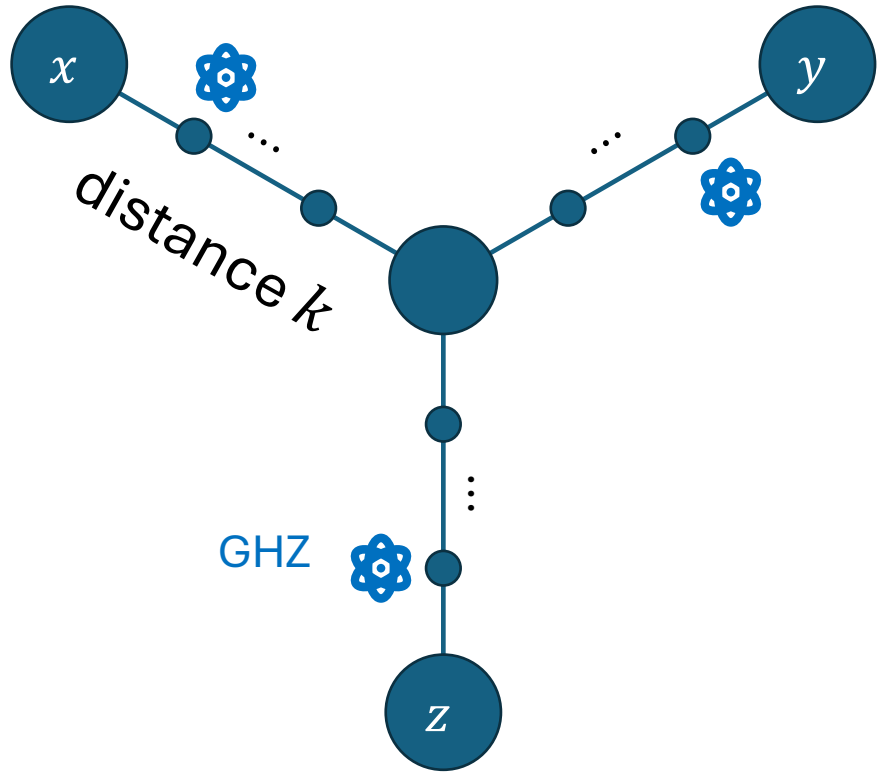


Quantum strategy:

1. Center prepares $|\mathrm{GHZ}\rangle$

GHZ game on a tripod [GKM'09]

Problem: Require 3 leaves to play GHZ game on a tripod graph

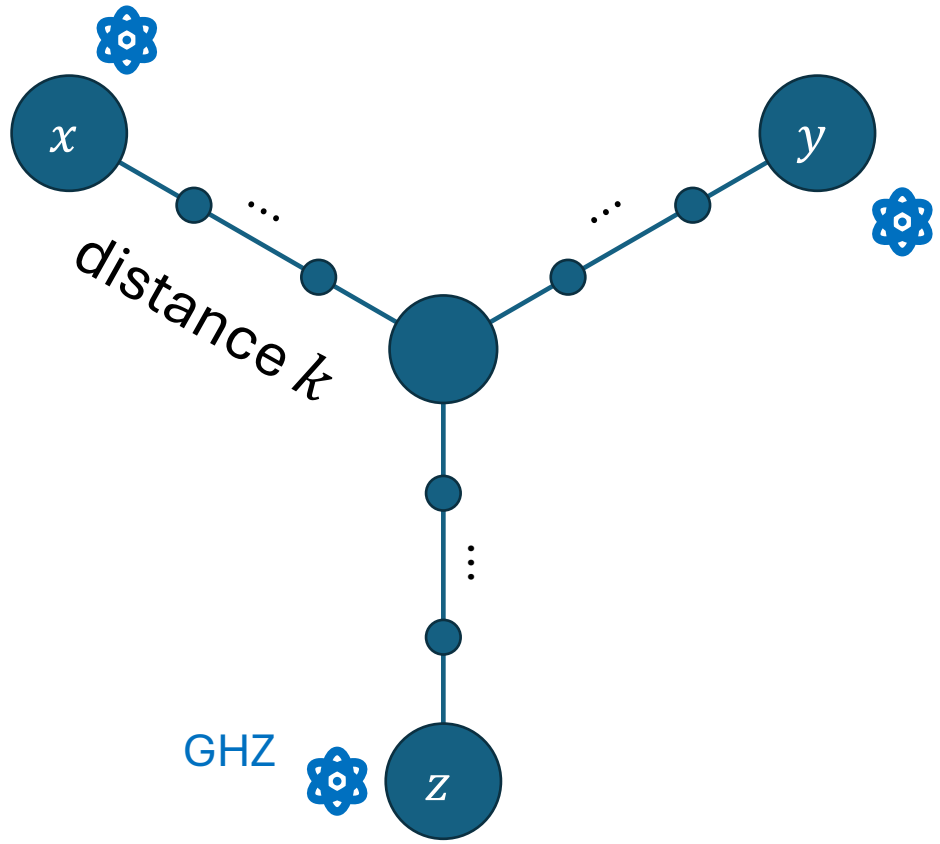


Quantum strategy:

1. Center prepares $|\mathrm{GHZ}\rangle$
2. Sends one qubit down each leg (k rounds)

GHZ game on a tripod [GKM'09]

Problem: Require 3 leaves to play GHZ game on a tripod graph



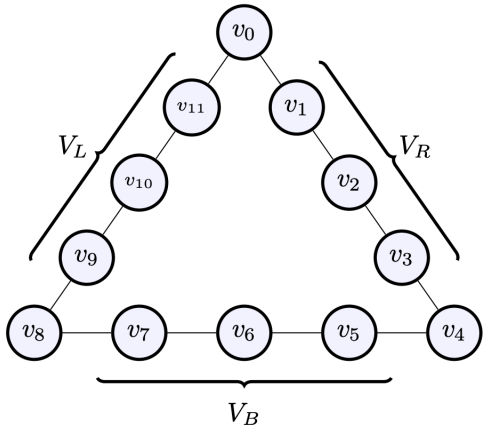
Quantum strategy:

1. Center prepares $|\mathrm{GHZ}\rangle$
2. Sends one qubit down each leg (k rounds)
3. Leaves play the GHZ game

k quantum rounds vs. $2k$ classical rounds

How large can it be?

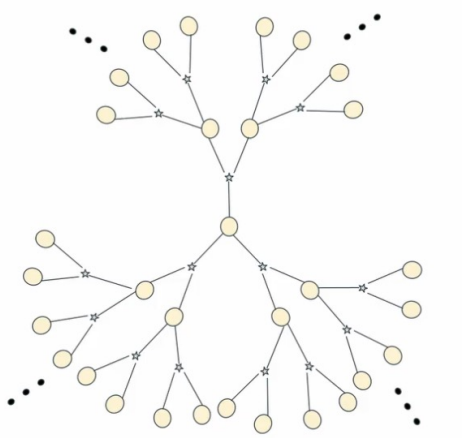
Le Gall, Nishimura, Rosmanis ’19



classical $\Omega(n)$
vs. quantum 2

global problem

Balliu et al. ’25

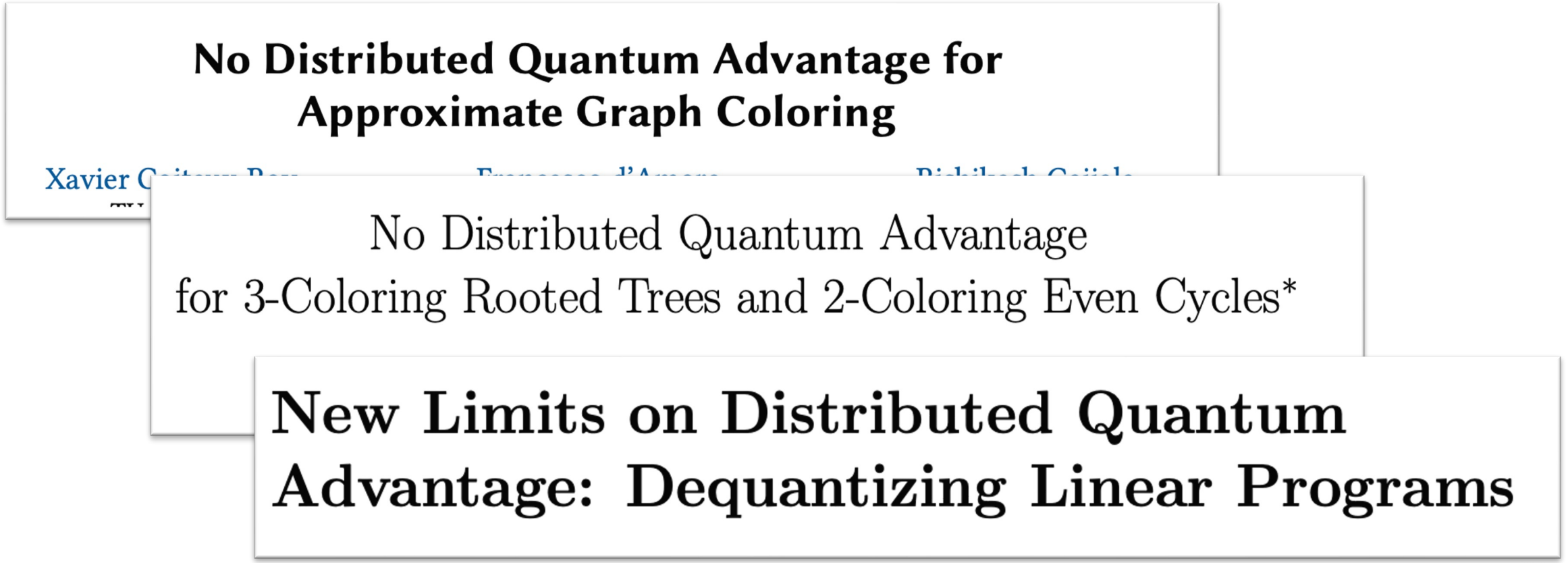


classical $\Omega(\log n \cdot \log^{0.99} \log n)$
vs. quantum $O(\log n)$

“locally verifiable problem”

This is somewhat cheating: can we speed up useful graph problems?

80% of quantum-LOCAL papers prove lower bounds

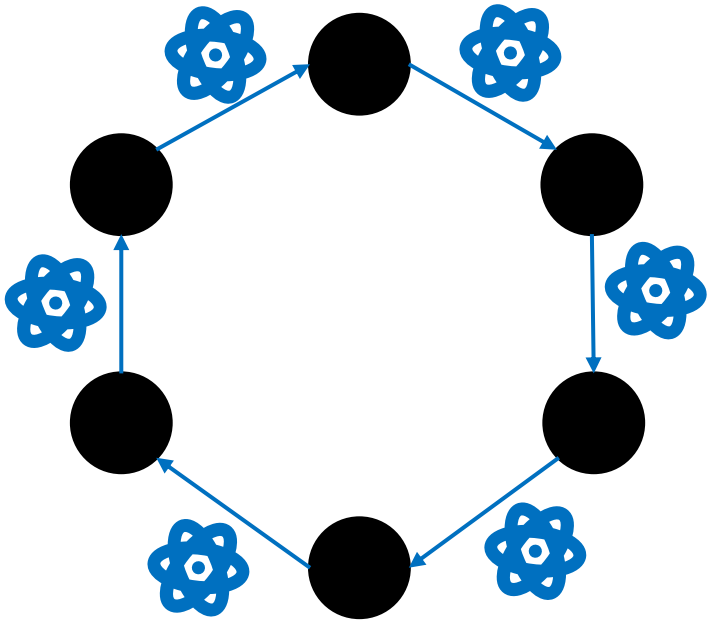


Most people believe that no useful graph problem admits quantum speedup in LOCAL model due to unlimited bandwidth.

Back to cycle coloring (quantum)

3-coloring a directed n-cycle:

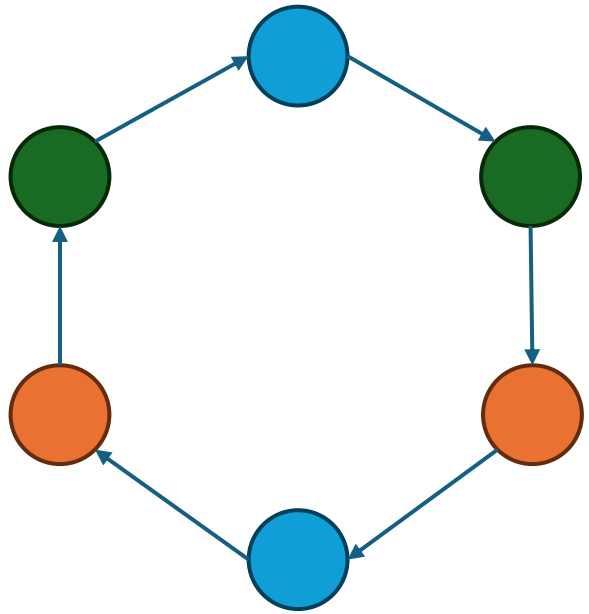
- Classical LOCAL: $\Theta(\log^* n)$ rounds
- Quantum LOCAL: ?



directed n-cycle

How many quantum rounds?

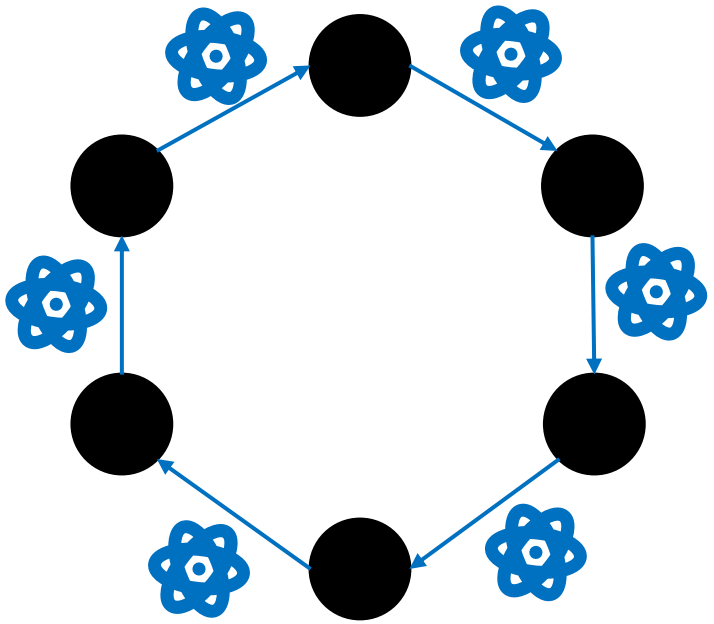
Spoiler: constant rounds!



output proper 3-coloring with probability $1 - 1/\mathrm{poly}(n)$

What was known before?

- 0-round is impossible
- **One-way** 1-round is impossible [LGR22]



One-way communication:
each node sends messages only to one side

New results

Instead of more rounds, the right lens is more colors:

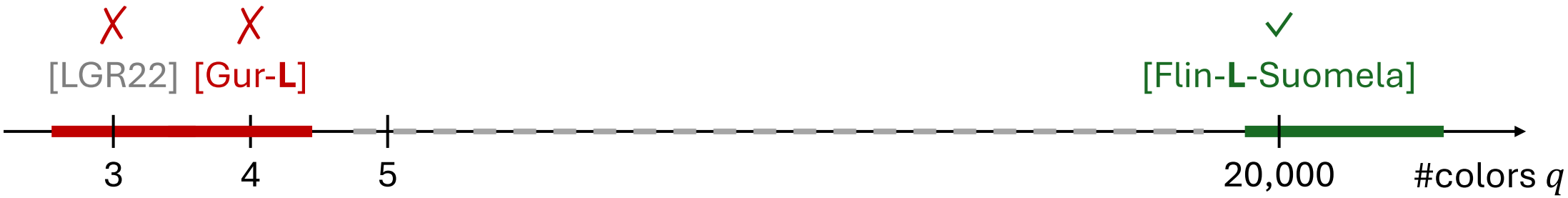


Figure: Whether one-way one-round ***q*-coloring** exists

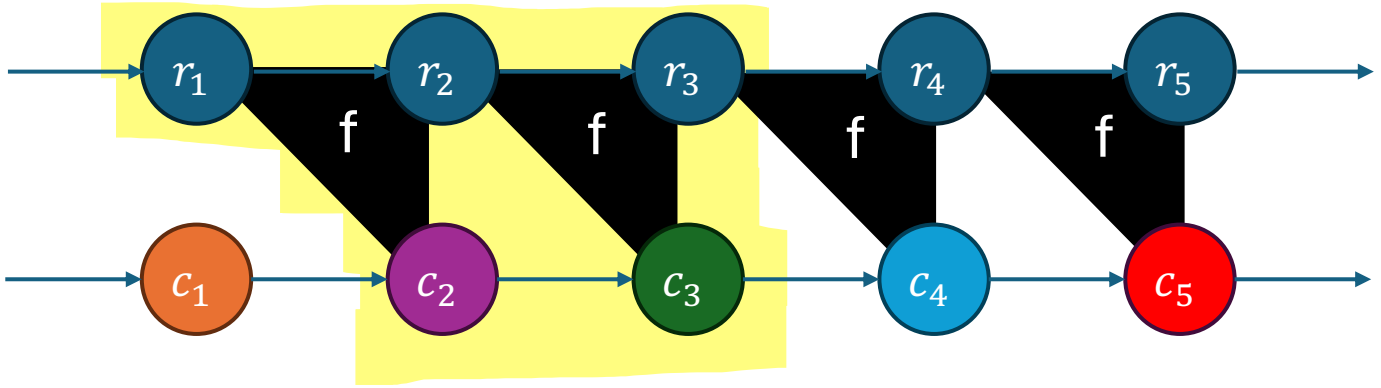
Key technique: An extremal combinatorial view from matrix spaces

Classical one-way one-round algorithm

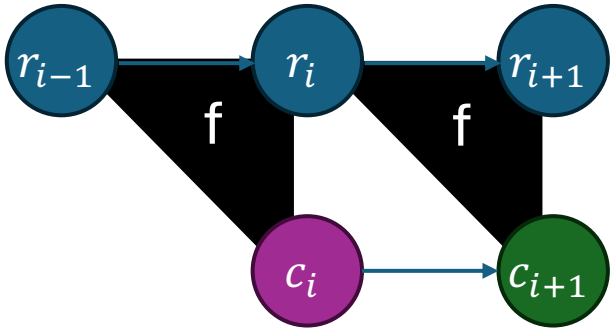
Algorithm for v_i

1. Sample a random seed $r_i \in [N]$ (N can be arbitrarily large)
2. Send r_i to next processor
3. Apply a function $f: [N]^2 \to [q]$ to get its color

$$c_i := f(r_{i-1}, r_i).$$



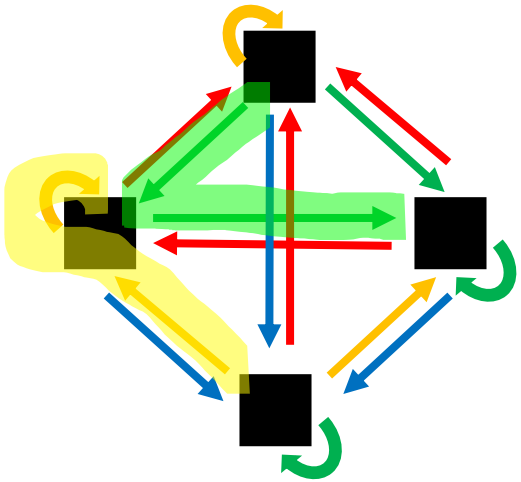
An extremal combinatorial view



Collision probability on one edge

$$\Pr[c_i = c_{i+1}] = \Pr[f(r_{i-1}, r_i) = f(r_i, r_{i+1})]$$

View $f: [N]^2 \to [q]$ as an edge q-coloring of complete directed graph



Monochromatic 2-walk: $x \to y \to z$ in a single color

A collision happens when

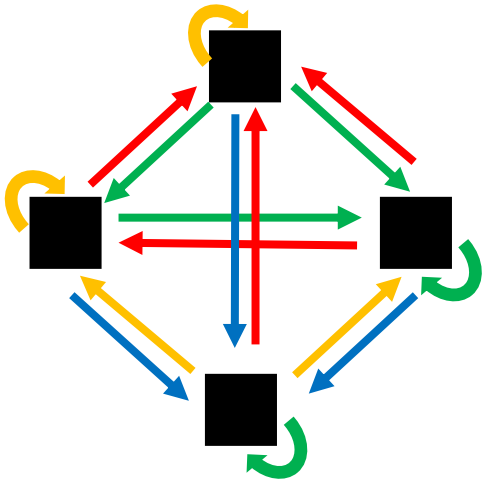
$$c_i = c_{i+1} \iff f(r_{i-1}, r_i) = f(r_i, r_{i+1}) \iff r_{i-1} \to r_i \to r_{i+1} \text{ is monochromatic}$$

$$\Pr[c_i = c_{i+1}] = \frac{\#\text{ monochromatic } 2\text{−walks}}{N^3}$$

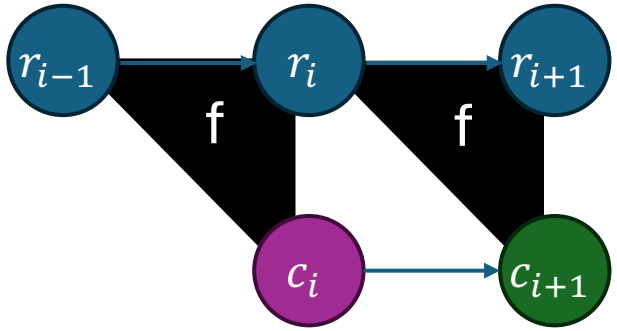
A Ramsey-type supersaturation*

* A coloring must contain many copies of monochromatic structure.

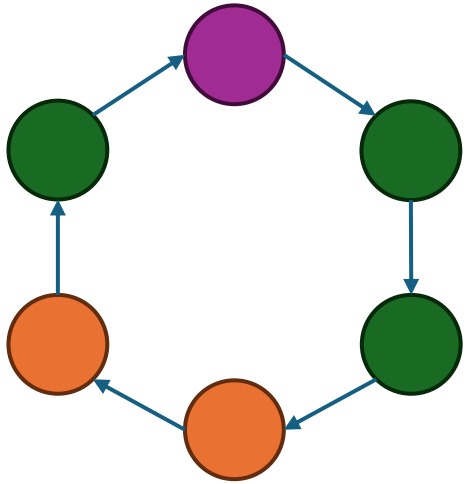
Lemma [HSW17]: Any q-coloring of $[N]^2$ contains $\Omega_q(N^3)$ monochromatic 2-walks.



Many monochromatic 2-walks



Each edge has large collision probability



Algorithm succeeds with small prob

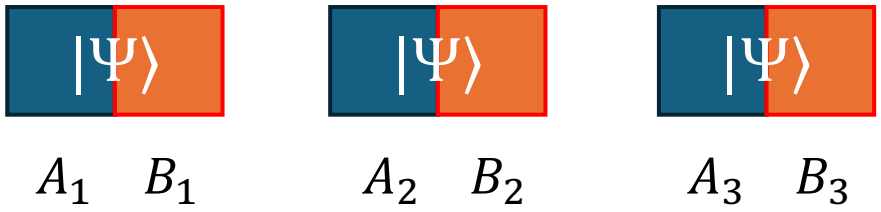
Quantum one-way one-round coloring

(Uniform) one-way one-round quantum algorithm:

- Each node prepares a uniform bipartite state

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{i \in [N]} |i\rangle_A |i\rangle_B$$

(N can be arbitrarily large)



Quantum one-way one-round coloring

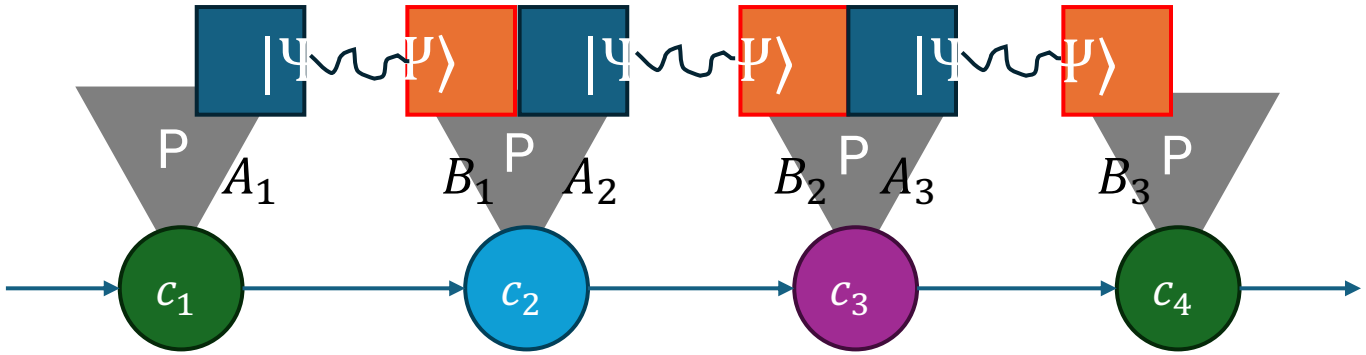
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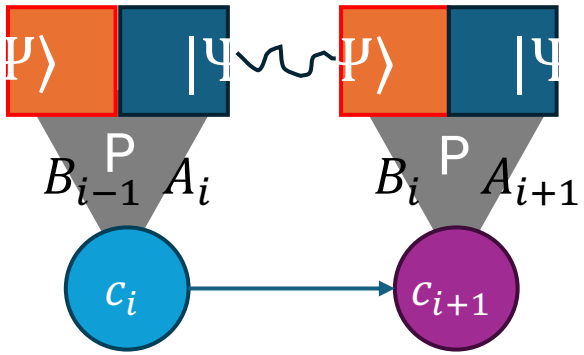
$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{i \in [N]} |i\rangle_A |i\rangle_B$$

(N can be arbitrarily large)

- Send register B to its successor
- Apply a projective measurement $\{P_a\}_{a \in [q]}$ on its (B, A)



Quantum: collision probability = energy



- Each P_a acts on $\mathbb{C}^N \otimes \mathbb{C}^N \cong \mathbb{C}^{N\times N}$ $\quad e_i \otimes e_j \leftrightarrow E_{i,j}$ (vectorization)
- $\{P_a\}_{a\in[q]}$ gives a decomposition

$$\mathbb{C}^{N\times N} = \oplus_{a\in[q]} W_a.$$

Definition: For $W \subseteq \mathbb{C}^{N\times N}$, with basis $\{M_k\}$ of W, define its *energy*

$$\mathcal{E}(W) \coloneqq \sum_{k,\ell} \|M_k M_\ell\|_F^2$$

"multiplicative energy of a matrix space"

Quantum collision probability:

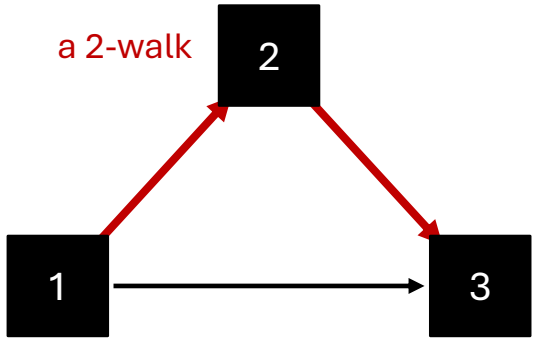
$$\Pr[c_i = c_{i+1}] = \frac{1}{N^3} \sum_{a=1}^{q} \mathcal{E}(W_a)$$

Energy generalizes #2-walks

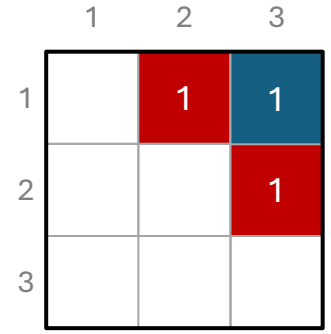
Interpret the classical picture using matrix spaces:

Directed graph $G = ([N], E)$

Matrix space $W = \mathrm{span}\{E_{x,y} : (x,y) \in E\}$



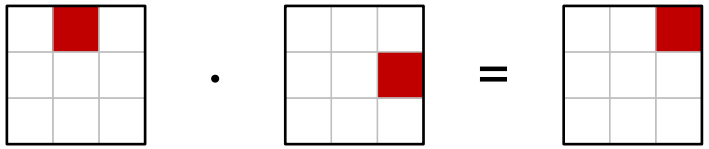
edge $(x,y) \leftrightarrow$ unit $E_{x,y}$



$W = \mathrm{span}\{E_{1,2}, E_{2,3}, E_{1,3}\}$

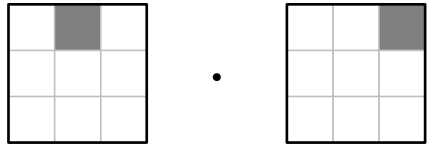
$(x,y), (w,z)$ form a 2-walk $\Leftrightarrow$ $E_{x,y}E_{w,z} = \delta_{y,w}E_{x,z}$ has norm 1

the walk 1→2→3



$E_{1,2}$ $\cdot$ $E_{2,3}$ $=$ $E_{1,3}$

not a walk



$E_{1,2}$ $\cdot$ $E_{1,3}$ $= 0$

$\mathcal{E}(W) =$ #2-walks in G (here: 1 two-walk)

Quantum supersaturation?

For every q-decomposition of $\mathbb{C}^{N\times N}$, do we have $\Omega_q(N^3)$ total energy?

Classical: q-coloring of $[N]^2$ $\Omega_q(N^3)$ monochromatic 2-walks

$q \leq 4$: holds*

[Gur-**L**]

collision probability $= \Omega(1)$ for every algorithm

⇒ no one-way one-round quantum 4-coloring

$q = 20{,}000$: fails

[Flin-**L**-Suomela]

collision probability $\to 0$ as $N \to \infty$

⇒ one-way one-round 20,000-coloring whp

⇒ 3-coloring cycles in 4 rounds

Classically it holds for every q; entanglement breaks it for large q.

* Impossibility needs to consider general quantum messages, formulated by a "weighted" energy.

Take away

- Cole–Vishkin 3-colors an n-cycle in $\Theta(\log^* n)$ classical rounds.
- Quantum algorithms 3-color an n-cycle in 4 rounds whp.

*Is it practical?

No, the best classical algorithm needs 3 rounds with IDs $\in [10^{100}]$