Quantum Complexity Theory

Hamiltonian Complexity



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Classical Cook-Levin Theorem

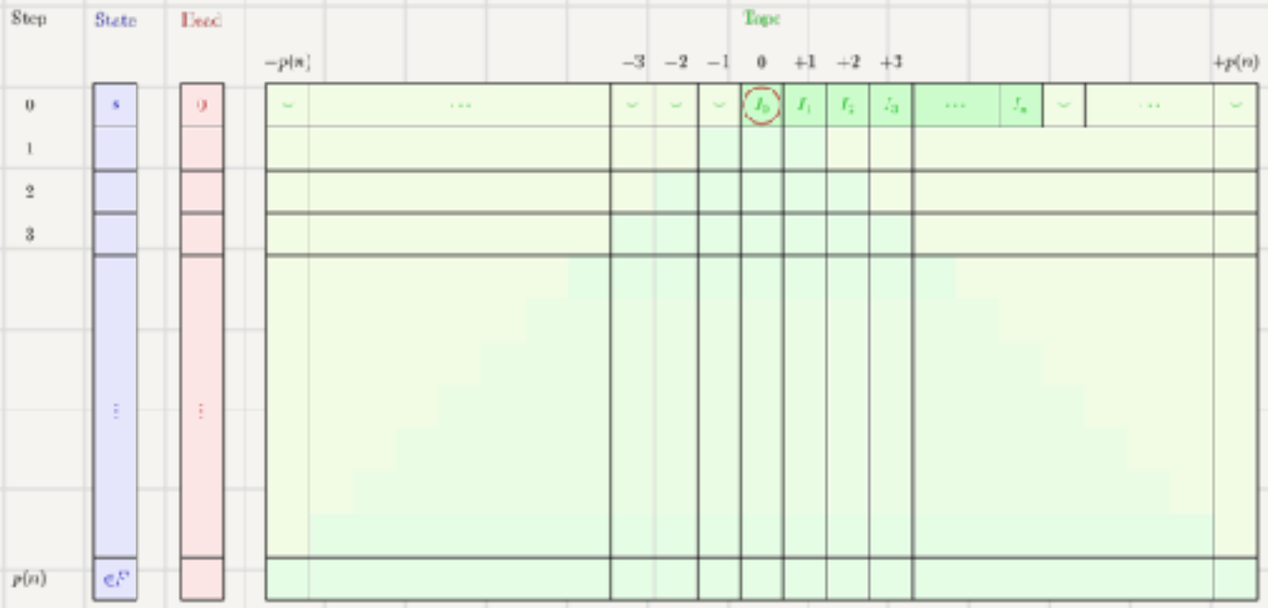
Theorem. k-SAT is NP-Complete, for $k > 2$.



Proof Sketch. Encode NP-computation as tableau

Local Constraints:

- Initialisation
- Computation
- Accepting output



Constraint Satisfaction Problems (CSP)

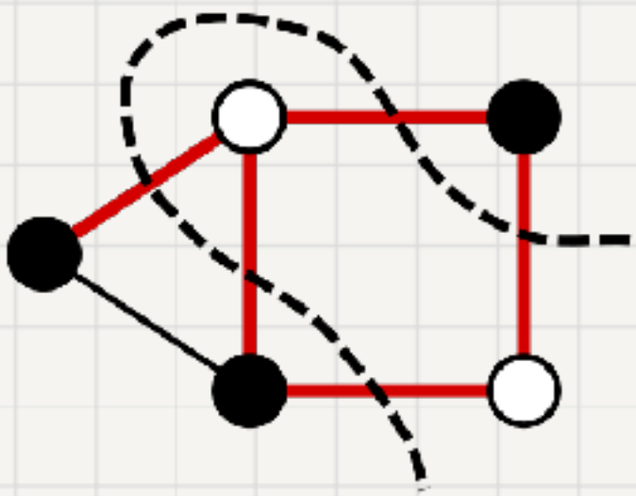
- Variables $x_1, x_2, \ldots, x_n$
- Constraints $C_1, C_2, \ldots, C_m$ over k variables

Goal: find an assignment $a_1, a_2, \ldots, a_n \in \Sigma$ satisfying

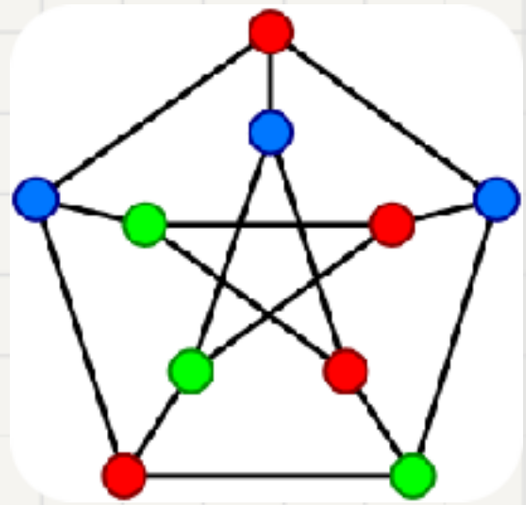
max number of constraints

$\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

3SAT



MaxCut



3COL

k-local Hamiltonians

Energy operators describing interactions of quantum systems

A k-local Hamiltonian on n qubits is a Hermitian operator

$$H = \sum_{i=1}^{m} H_i$$

Each H_i acts nontrivially on at most k qubits

k-local Hamiltonians

Since H is Hermitian, by the Spectral Theorem:

$$H = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

real eigenvalues

real number

Energy $\langle\varphi|H|\varphi\rangle = \sum_i \lambda_i \langle\varphi|\psi_i\rangle\langle\psi_i|\varphi\rangle = \sum_i \lambda_i |\langle\varphi|\psi_i\rangle|^2$

Ground state has minimum energy

Embedding k-SAT into local Hamiltonians

$$c = (x_1 \vee \neg x_2)$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

$|01\rangle$ is an unsatisfying assignment

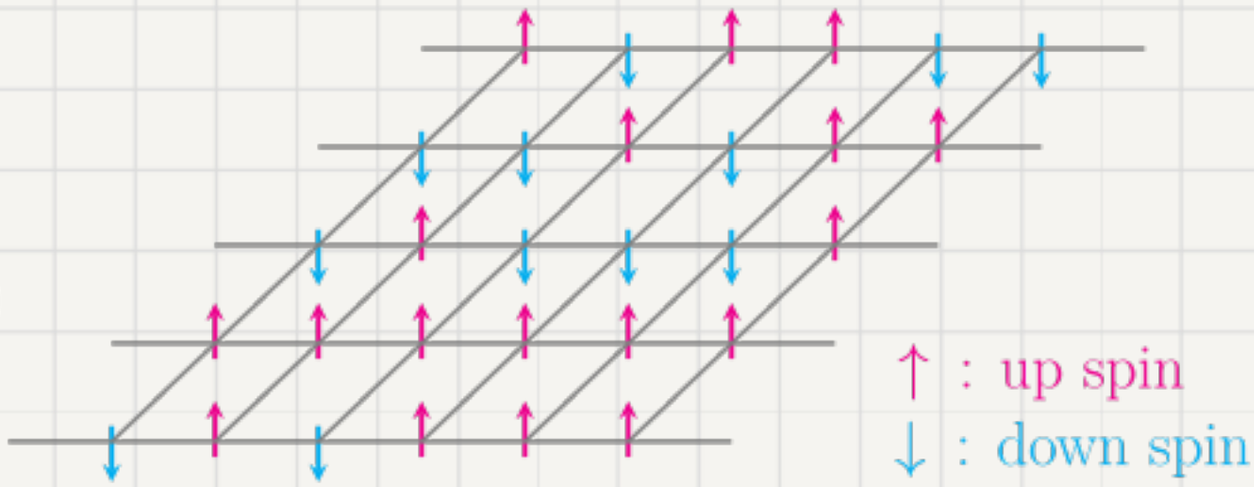
Energy measures the extent of constraint violations

Physics perspective

Ising model (spin)

Heisenberg model (ferromagnetic)

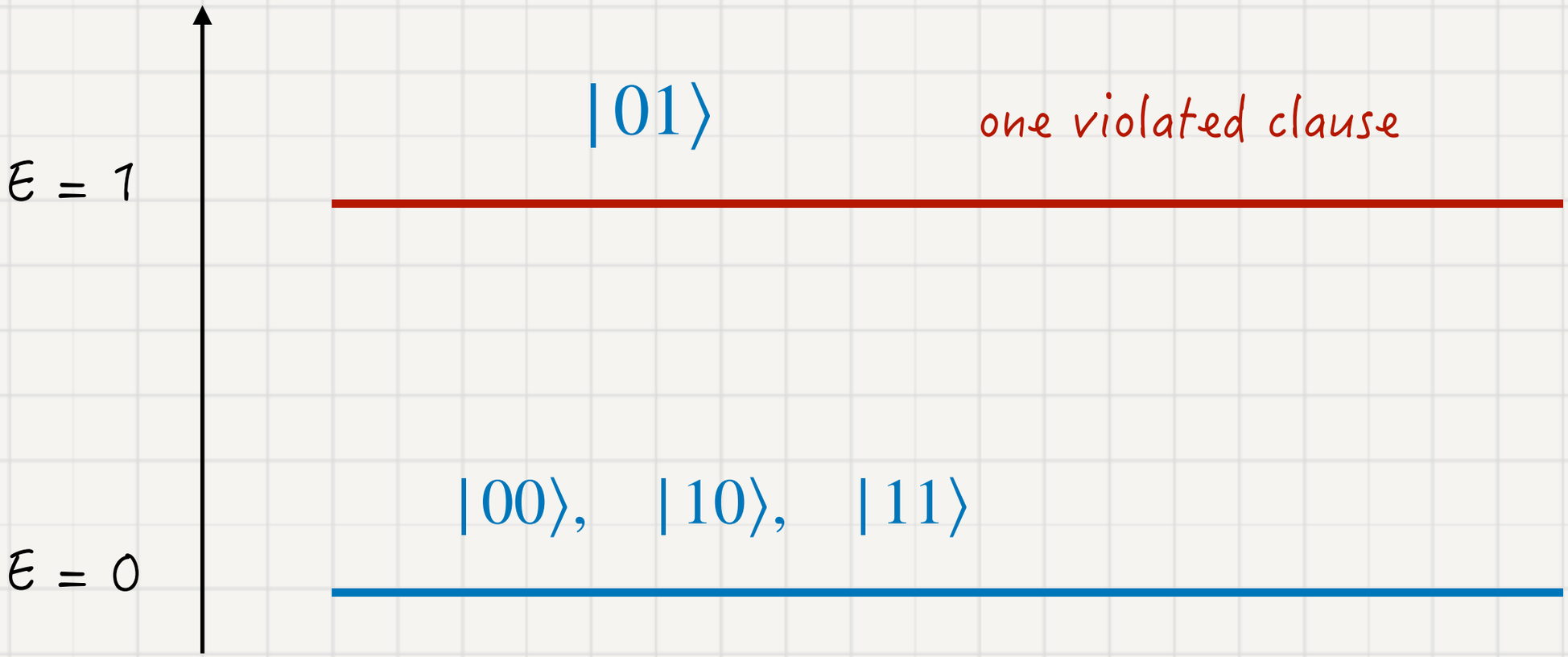
Bose-Hubbard model (bosons)



One clause, two energy levels

$c = (x_1 \vee \neg x_2)$

$H = |01\rangle\langle 01|$



A uniform superposition puts 1/4 of its weight on the bad assignment.

$$\langle ++|H|++\rangle = |\langle 01|++\rangle|^2 = \frac{1}{4}$$

Quote of the day

"...the exact application of these laws leads to equations much too complicated to be soluble."



Paul Dirac, 1929 — Quantum mechanics of many-electron systems

Simple local rules. A difficult global ground state.

Example 1: Max-Sat

$$\phi = c_1 \wedge c_2 \wedge c_3, \quad c_1 = (x_1 \vee x_2), c_2 = (\neg x_2 \vee x_3), c_3 = (x_3 \vee x_4)$$

$$H = H_{c_1} \otimes I_{3,4} + I_1 \otimes H_{c_2} \otimes I_4 + I_{1,2} \otimes H_{c_3}$$

$$\langle x | H | x \rangle = 0 \iff \phi(x) = 1$$

$\langle x | H | x \rangle$ counts the number of unsatisfied clauses

Example 2: Max-Cut / Ising Model

Partition edges of an undirected graph $G = (V, E)$ into two disjoint sets E_1, E_2 such that the maximum number of edges possible crosses between E_1 and E_2.

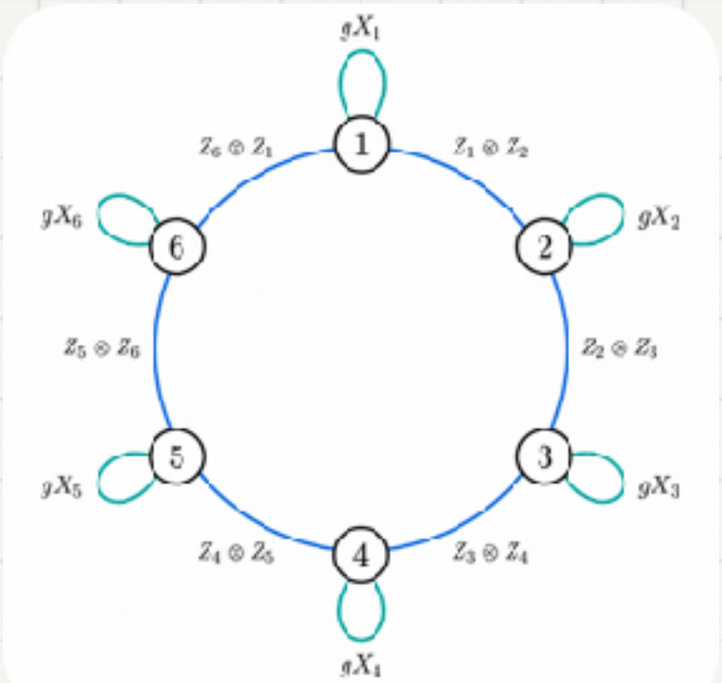
$$H_{ij} = I - Z_i \otimes Z_j \quad \forall (i,j) \in E$$

Recall $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

So $Z_i \otimes Z_j = |00\rangle\langle 00| - |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|$

Example 3: Quantum Ising Model

$$H(g) = -\sum_i Z_i \otimes Z_j - g \sum_i X_i$$



For $g = 0$:
MaxCut/Ising

Example 4: Quantum Heisenberg Model

Quantum magnetism in atomic systems

$$H(J_x, J_y, J_z, g) = J_x \sum_i X_i X_{i+1} + J_y \sum_i Y_i Y_{i+1} + J_z \sum_i Z_i Z_{i+1} + g \sum_i X_i$$

Local Hamiltonian Problem

$$H = \sum_{i=1}^{m} H_i, \quad \text{supp}(H_i) \leq k$$

Promise: efficiently computable $\alpha(n), \beta(n) \in \mathbb{R}$

satisfying $\alpha(n) - \beta(n) \geq 1/p(n)$

Output:

- If $\lambda_{\min}(H) \leq \alpha(n)$, accept
- If $\lambda_{\min}(H) \geq \beta(n)$, reject

H does not need to be diagonal, no geometric restrictions

A promise problem has a no-man's-land

Between the thresholds, either answer is allowed.



$$\beta - \alpha \geq \frac{1}{\text{poly}(n)}$$

The promise: no trick questions hiding in the gap.

Classical–Quantum Dictionary

Classical	Quantum
CSP	Hamiltonian
Variables	Qubits
Constraints	Hamiltonian terms
Solution quality	Energy
Optimal solution	Ground state
P	BQP
NP	QMA
Cook–Levin SAT formula	Feynman–Kitaev Hamiltonian