Quantum Complexity Theory

Problem set 2: Mixed states, observables, and channels **Solutions**

We use the ordered two-qubit basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ throughout.

1. Density matrices and the spectral theorem

(a) The orthonormal eigenvectors are $|\pm\rangle$, with eigenvalues $\lambda_\pm = (1 \pm t)/2$. Since ρ_t is Hermitian and has trace 1, it is a density matrix exactly when $-1 \le t \le 1$.

(b) Its spectral decomposition is $\rho_t = \frac{1+t}{2}|+\rangle\langle+| + \frac{1-t}{2}|-\rangle\langle-|$.

(c) Write $\rho = \sum_j \lambda_j |b_j\rangle\langle b_j|$, where $\lambda_j \ge 0$ and $\sum_j \lambda_j = 1$. Then

$$1 - \mathrm{Tr}(\rho^2) = \sum_j \lambda_j (1 - \lambda_j).$$

This is zero exactly when every λ_j is 0 or 1, hence exactly when ρ is a rank-one projector, i.e. pure. Here $\mathrm{Tr}(\rho_t^2) = (1 + t^2)/2$, so ρ_t is pure exactly for $t = \pm 1$.

2. Different ensembles and coherence

(a) The opposite off-diagonal entries cancel:

$$\frac{1}{2} \sum_{s \in \{+,-\}} |\psi_s\rangle\langle\psi_s| = \frac{1}{8}\left[\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} + \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\right] = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} = \tau.$$

The density matrices agree, so no measurement on the qubit distinguishes the preparations.

(b) Both τ and $|\psi_+\rangle\langle\psi_+|$ give $p(0) = 3/4$ and $p(1) = 1/4$.

(c) We have

$$H\tau H^\dagger = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad p_\tau(\pm) = \frac{1}{2}, \qquad p_{\psi_+}(\pm) = \frac{(\sqrt{3} \pm 1)^2}{8} = \frac{2 \pm \sqrt{3}}{4}.$$

This imbalance detects the off-diagonal coherence, invisible in the computational basis.

3. Projective measurements and forgotten outcomes

(a) Both are Hermitian and idempotent, with $M_{\mathrm{even}} M_{\mathrm{odd}} = 0$ and $M_{\mathrm{even}} + M_{\mathrm{odd}} = I_4$.

(b) Each outcome has probability 1/2. The conditional density matrices are

$$\rho_{\mathrm{even}} = |\Phi^+\rangle\langle\Phi^+|, \quad |\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}; \qquad \rho_{\mathrm{odd}} = |\Psi^+\rangle\langle\Psi^+|, \quad |\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}.$$

Both are entangled: they are pure, but each one-qubit reduced state is $I_2/2$.

(c) Forgetting the parity outcome gives

$$\rho' = \frac{1}{2}\rho_{\mathrm{even}} + \frac{1}{2}\rho_{\mathrm{odd}} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The $00 \leftrightarrow 11$ and $01 \leftrightarrow 10$ off-diagonal entries survive. Measuring both bits and forgetting both outcomes instead gives $I_4/4$, with no off-diagonal entries.

4. Reduced states and purification

(a) The cross terms vanish under either partial trace, giving $\rho_A = \rho_B = \mathrm{diag}(3/4, 1/4)$. The joint state is pure; the reduced states are mixed, with $\mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(\rho_B^2) = 5/8$.

(b) Because $\langle + \mid - \rangle = 0$, tracing out B from $|\Psi'\rangle\langle\Psi'|$ gives the same ρ_A. The required local unitary is H_B: $|\Psi'\rangle = (I_A \otimes H_B)|\Psi\rangle$.

(c) Take $|\Omega\rangle_{AB} = \sum_j \sqrt{\lambda_j}|b_j\rangle_A |j\rangle_B$ for orthonormal $|j\rangle_B$. It is normalized, and

$$\mathrm{Tr}_B |\Omega\rangle\langle\Omega| = \sum_{j,k} \sqrt{\lambda_j \lambda_k}|b_j\rangle\langle b_k|\langle k \mid j\rangle = \sum_j \lambda_j |b_j\rangle\langle b_j| = \rho_A.$$

5. Trace distance and distinguishing states

(a) The eigenvalues of $\rho - \sigma$ are $p_j - q_j$. Hence $D(\rho,\sigma) = \frac{1}{2}\sum_j |p_j - q_j|$.

(b) Here

$$\Delta := \rho - \sigma = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{Z - X}{2}, \qquad \Delta^2 = \frac{I}{2}.$$

Since $\operatorname{Tr}\Delta = 0$, its eigenvalues are $\pm 1/\sqrt{2}$. Thus

$$D(\rho,\sigma) = \frac{1}{\sqrt{2}}, \qquad p_{\text{guess}} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right).$$

(c) Measure $P_\pm = \frac{1}{2}\left(I \pm (Z - X)/\sqrt{2}\right)$, guessing ρ on $+$ and σ on $-$. These are the positive and negative spectral projectors of Δ, and give

$$p_{\text{success}} = \frac{1}{2}[\operatorname{Tr}(P_+\rho) + \operatorname{Tr}(P_-\sigma)] = \frac{1}{2}[1 + \operatorname{Tr}(P_+\Delta)] = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right).$$

After a computational-basis measurement, the best rule is to guess ρ on 0 and σ on 1. Its success probability is $\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}$.

6. Observables and correlations

(a) Since X, Z are Hermitian, $C^\dagger = C$. Using $XZ + ZX = 0$ gives

$$C^2 = \tfrac{1}{2}(X^2 + XZ + ZX + Z^2) = I.$$

Since also $\operatorname{Tr} C = 0$, its eigenvalues are ± 1, with spectral projectors $P_\pm = (I \pm C)/2$.

(b) For $\rho = |0\rangle\langle 0|$,

$$p(\pm 1) = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right), \qquad \mathbb{E}[C] = p(+1) - p(-1) = \frac{1}{\sqrt{2}} = \operatorname{Tr}(C\rho).$$

For $\rho = I/2$, both probabilities are $1/2$, so $\mathbb{E}[C] = 0 = \operatorname{Tr}(CI/2)$.

(c) Since $B_0 + B_1 = \sqrt{2}Z$ and $B_0 - B_1 = \sqrt{2}X$, the operator simplifies to

$$Z \otimes (B_0 + B_1) + X \otimes (B_0 - B_1) = \sqrt{2}(Z \otimes Z + X \otimes X).$$

Both $Z \otimes Z$ and $X \otimes X$ fix $|\Phi^+\rangle$, so the requested expectation is $\boxed{2\sqrt{2}}$.

7. Quantum channels and dephasing

(a) Choose $K_0 = \sqrt{1-p}\, I$ and $K_1 = \sqrt{p}\, Z$. Then $\sum_k K_k^\dagger K_k = (1-p)I + pI = I$, so

$$\operatorname{Tr}\mathcal{E}_p(\rho) = \operatorname{Tr}\left(\rho \sum_k K_k^\dagger K_k\right) = \operatorname{Tr}\rho.$$

For every auxiliary system R and every $\omega_{RA} \geq 0$,

$$(\mathrm{id}_R \otimes \mathcal{E}_p)(\omega) = \sum_k (I_R \otimes K_k)\omega(I_R \otimes K_k^\dagger) \geq 0,$$

because each summand is a positive semidefinite conjugation. This proves complete positivity.

(b) Direct multiplication gives

$$\mathcal{E}_p\begin{pmatrix} a & c \\ \bar{c} & 1-a \end{pmatrix} = \begin{pmatrix} a & (1-2p)c \\ (1-2p)\bar{c} & 1-a \end{pmatrix}.$$

Thus $\mathcal{E}_{1/2}(\rho) = \sum_{j=0}^{1} |j\rangle\langle j|\rho|j\rangle\langle j| = \operatorname{diag}(a, 1-a)$, exactly computational-basis measurement with the outcome discarded. In particular, $\mathcal{E}_{1/2}(|+\rangle\langle +|) = I/2$.

(c) Write $\rho = \sum_{x,y} \rho_{xy}|x\rangle\langle y|$. Since $\mathrm{CNOT}|x,0\rangle = |x,x\rangle$,

$$\rho \otimes |0\rangle\langle 0| \;\longmapsto\; \sum_{x,y} \rho_{xy}|xx\rangle\langle yy|,$$

$$\rho'_A = \sum_{x,y} \rho_{xy}|x\rangle\langle y|\langle y \mid x\rangle = \sum_x \rho_{xx}|x\rangle\langle x| = \mathcal{E}_{1/2}(\rho).$$