Quantum Complexity Theory

Problem set 2: Mixed states, observables, and channels

Show your working. Use the density-matrix formalism from Lecture 3. For two qubits, use the ordered basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, with system A first and system B second. You may use

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}.$$

1. Density matrices and the spectral theorem

For $t \in \mathbb{R}$, let $\rho_t = \frac{1}{2}\begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$.

(a) Find the eigenvalues of ρ_t and an orthonormal eigenbasis. For which values of t is ρ_t a density matrix?

(b) For these values of t, write ρ_t as a probabilistic mixture of two orthogonal pure states.

(c) Show that a density matrix ρ is pure if and only if $\mathrm{Tr}(\rho^2) = 1$. Compute $\mathrm{Tr}(\rho_t^2)$ and determine when ρ_t is pure. Use the spectral theorem.

2. Different ensembles and coherence

Let $|\psi_\pm\rangle = (\sqrt{3}|0\rangle \pm |1\rangle)/2$ and $\tau = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$.

(a) Compute the density matrix of an equal mixture of $|\psi_+\rangle$ and $|\psi_-\rangle$. Compare it with τ. Can any measurement on the supplied qubit distinguish these two preparation procedures?

(b) Compare the computational-basis measurement probabilities for τ and $|\psi_+\rangle\langle\psi_+|$.

(c) Compute $H\tau H^\dagger$. Find the Hadamard-basis measurement probabilities for τ and $|\psi_+\rangle\langle\psi_+|$. Explain how this measurement reveals coherence that is invisible in part (b).

3. Projective measurements and forgotten outcomes

Two qubits are prepared in $|{+}{+}\rangle = |+\rangle \otimes |+\rangle$. Consider the measurement

$$M_{\text{even}} = |00\rangle\langle 00| + |11\rangle\langle 11|, \qquad M_{\text{odd}} = |01\rangle\langle 01| + |10\rangle\langle 10|.$$

(a) Verify that $\{M_{\text{even}}, M_{\text{odd}}\}$ is a projective measurement.

(b) Find each outcome probability and the corresponding normalised post-measurement density matrix. Are the conditional states entangled?

(c) Discard the measurement outcome and write the resulting density matrix. Compare it with the state obtained by measuring both qubits in the computational basis and discarding both outcomes. Which off-diagonal entries survive in each case?

4. Reduced states and purification

Let $|\Psi\rangle_{AB} = (\sqrt{3}|00\rangle + |11\rangle)/2$.

(a) Compute $\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$ and $\rho_B = \mathrm{Tr}_A(|\Psi\rangle\langle\Psi|)$. Which of the joint and reduced states are pure?

(b) Check that $|\Psi'\rangle = (\sqrt{3}|0+\rangle + |1-\rangle)/2$ is another purification of ρ_A. Find a unitary acting only on B that maps $|\Psi\rangle$ to $|\Psi'\rangle$.

(c) Given a spectral decomposition $\rho_A = \sum_j \lambda_j |b_j\rangle\langle b_j|$, construct a purification and verify it by taking a partial trace.

5. Trace distance and distinguishing states

Recall that $D(\rho,\sigma)=\frac{1}{2}\|\rho-\sigma\|_1$. For a Hermitian matrix, the trace norm is the sum of the absolute values of its eigenvalues. For two states supplied with equal prior probabilities, the optimal guessing probability is $p_{\text{guess}}=(1+D(\rho,\sigma))/2$.

(a) Let $\rho=\operatorname{diag}(p_1,\dots,p_d)$ and $\sigma=\operatorname{diag}(q_1,\dots,q_d)$ be density matrices. Show that

$$D(\rho,\sigma)=\frac{1}{2}\sum_j |p_j-q_j|.$$

(b) Now let $\rho=|0\rangle\langle 0|$ and $\sigma=|+\rangle\langle +|$. Find the eigenvalues of $\rho-\sigma$, their trace distance, and the optimal guessing probability with equal priors.

(c) Give a two-outcome projective measurement and a guessing rule that attain this probability. Compare with the best guessing rule after a computational-basis measurement. *Hint: use the positive and negative eigenspaces of $\rho-\sigma$.*

6. Observables and correlations

Let $C=(X+Z)/\sqrt{2}$.

(a) Show that C is Hermitian and $C^2=I$. Find its eigenvalues and write its spectral projectors using I and C.

(b) Find the outcome probabilities and expected outcome when C is measured on $|0\rangle\langle 0|$ and on $I/2$. In each case, check that the expected outcome equals $\operatorname{Tr}(C\rho)$.

(c) Let $|\Phi^+\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$. Using $A_0=Z$, $A_1=X$, $B_0=(X+Z)/\sqrt{2}$, and $B_1=(Z-X)/\sqrt{2}$, compute

$$\langle\Phi^+|(A_0\otimes B_0+A_0\otimes B_1+A_1\otimes B_0-A_1\otimes B_1)|\Phi^+\rangle.$$

Hint: first simplify the operator using B_0+B_1 and B_0-B_1.

7. Quantum channels and dephasing

For $0\le p\le 1$, define $\mathcal{E}_p(\rho)=(1-p)\rho+pZ\rho Z$.

(a) Give Kraus operators K_0,K_1 for $\mathcal{E}_p$ and verify $\sum_k K_k^\dagger K_k=I$. Deduce trace preservation. To verify complete positivity, show that $(\mathrm{id}_R\otimes\mathcal{E}_p)(\omega_{RA})\ge 0$ for every auxiliary system R and every $\omega_{RA}\ge 0$, where id_R is the identity map on R.

(b) For a density matrix $\rho=\begin{pmatrix} a & c \\ \bar{c} & 1-a \end{pmatrix}$, compute $\mathcal{E}_p(\rho)$. Show that $\mathcal{E}_{1/2}$ is computational-basis measurement with its outcome discarded. What does it do to $|+\rangle\langle +|$?

(c) Append a qubit B in $|0\rangle$, apply CNOT with A as control, and discard B, as below. Show that the resulting channel on A is $\mathcal{E}_{1/2}$. You may use $\mathrm{CNOT}|x,y\rangle=|x,y\oplus x\rangle$ for $x,y\in\{0,1\}$.

