Quantum Complexity Theory

Problem set 1: A gentle introduction **Solutions**

Post-measurement kets are given up to global phase. Two-qubit vectors use the basis order $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, with the first qubit on the top wire.

1. States and bra-ket notation

(a) Since $\langle v \mid v\rangle = 1 + 4 = 5$, the normalisation constant is $c = 1/\sqrt{5}$, and

$$|\psi\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2i\end{pmatrix}, \qquad \langle\psi| = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2i\end{pmatrix}.$$

(b) $\langle w \mid \psi\rangle = (2 + i(2i))/5 = 0$ and $\langle \psi \mid w\rangle = \overline{\langle w \mid \psi\rangle} = 0$. Thus the states are orthogonal.

(c)

$$P = |\psi\rangle\langle\psi| = \frac{1}{5}\begin{pmatrix}1 & -2i\\2i & 4\end{pmatrix}, \qquad P|\psi\rangle = |\psi\rangle, \qquad P|w\rangle = \mathbf{0}.$$

The inner product $\langle\psi \mid \psi\rangle = 1$ is a scalar; the outer product is the rank-one projector $P = |\psi\rangle\langle\psi|$.

2. Unitary evolution

(a) Direct multiplication gives $H^\dagger H = H^2 = I$ and $S^\dagger S = I$. Hence $H^{-1} = H$ and $S^{-1} = S^\dagger = \mathrm{diag}(1, -i)$. The matrix P is not unitary, since $P^\dagger P = P^2 = P \neq I$.

(b) $\langle U\phi \mid U\psi\rangle = \langle\phi|U^\dagger U|\psi\rangle = \langle\phi \mid \psi\rangle$. Taking $\phi = \psi$ shows that normalisation is preserved.

(c) No: the inputs have inner product $\langle 0 \mid 1\rangle = 0$, whereas the proposed outputs have inner product $\langle + \mid +\rangle = 1$.

3. Composite systems and tensor products

(a)

$$|a\rangle \otimes |b\rangle = \frac{2|00\rangle - |01\rangle + 2i|10\rangle - i|11\rangle}{\sqrt{10}} = \frac{1}{\sqrt{10}}\begin{pmatrix}2\\-1\\2i\\-i\end{pmatrix}.$$

Its squared norm is $(4 + 1 + 4 + 1)/10 = 1$.

(b)

$$H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{pmatrix}, \qquad I \otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0 & 0\\1 & -1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & -1\end{pmatrix}.$$

Thus $(H \otimes I)|01\rangle = (|01\rangle + |11\rangle)/\sqrt{2}$ and $(I \otimes H)|01\rangle = (|00\rangle - |01\rangle)/\sqrt{2}$. They apply H to the first and second qubit, respectively.

(c) A product $(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$ equalling $|\mathrm{EPR}\rangle$ would require $\alpha\gamma = \beta\delta = 1/\sqrt{2}$, so all four coefficients would be nonzero. But it also requires $\alpha\delta = \beta\gamma = 0$, a contradiction.

4. Measurements in different bases

(a) Outcome 0 has probability $3/4$ and leaves $|0\rangle$; outcome 1 has probability $1/4$ and leaves $|1\rangle$.

(b) $\langle \pm \mid \chi\rangle = (\sqrt{3} \pm i)/(2\sqrt{2})$, so $p(+) = p(-) = 1/2$. The corresponding states are $|+\rangle$ and $|-\rangle$.

(c) Either intermediate state gives computational-basis outcomes uniformly. By total probability, $p(0) = \sum_{s\in\{+,-\}} p(s)p(0 \mid s) = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}$; likewise $p(1) = 1/2$, in contrast to $3/4, 1/4$ in (a).

5. Changing the measurement basis

(a) Orthonormality gives $U|j\rangle = |b_j\rangle$, and

$$U^\dagger U = \sum_{j,k} |j\rangle\langle b_j \mid b_k\rangle\langle k| = I, \qquad UU^\dagger = \sum_j |b_j\rangle\langle b_j| = I.$$

(b) Apply $U^\dagger$. The probability of outcome j is

$$\left|\langle j|U^\dagger|\psi\rangle\right|^2 = \left|\langle b_j \mid \psi\rangle\right|^2.$$

(c) Immediately after observing j, the state is $|j\rangle$. Applying U produces $|b_j\rangle$, the post-measurement state of a direct measurement in B.

(d) For $\{|+\rangle, |-\rangle\}$, apply H. For $\{|+i\rangle, |-i\rangle\}$, the basis-change unitary is $U = SH$, so preprocessing is $U^\dagger = HS^\dagger$: apply $S^\dagger$ **first, then** H.

6. Circuit I: relative phase and interference

(a) The successive states are

$$|\psi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},$$

$$|\psi_2\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix},$$

$$|\psi_3\rangle = \frac{(1+i)|0\rangle + (1-i)|1\rangle}{2} = \frac{1}{2}\begin{pmatrix}1+i\\1-i\end{pmatrix}.$$

(b) $p(0) = |1+i|^2/4 = 1/2$ and $p(1) = |1-i|^2/4 = 1/2$.

(c) Replacing S by Z gives

$$|0\rangle \xrightarrow{H} |+\rangle \xrightarrow{Z} |-\rangle \xrightarrow{H} |1\rangle.$$

The final H converts the relative sign into an observable outcome: $H|+\rangle = |0\rangle$, whereas $H|-\rangle = |1\rangle$.

7. Circuit II: an EPR pair and local gates

(a) The successive joint states are

$$|\phi_1\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}} = |+\rangle|0\rangle,$$

$$|\phi_2\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

$$|\phi_3\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$

$$|\phi_4\rangle = \frac{|00\rangle - |01\rangle + |10\rangle + |11\rangle}{2} = \frac{|0\rangle|-\rangle + |1\rangle|+\rangle}{\sqrt{2}}.$$

The initial state and $|\phi_1\rangle$ are product states; $|\phi_2\rangle$, $|\phi_3\rangle$ and $|\phi_4\rangle$ are entangled.

(b) Each of the four outcomes $00, 01, 10, 11$ has probability $1/4$.

(c) The first-qubit outcomes each have probability $1/2$. Conditional on 0, the second qubit is $|-\rangle$; conditional on 1, it is $|+\rangle$.