

Quantum Complexity Theory

Problem set 1: A gentle introduction

Show your working. Use the pure-state and projective-measurement postulates from the lecture. For two qubits, use the ordered basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, with the first qubit on the top wire. Circuits run from left to right. You may use

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}.$$

1. States and bra-ket notation

Let $|v\rangle = |0\rangle + 2i|1\rangle$ and $|w\rangle = (2|0\rangle - i|1\rangle)/\sqrt{5}$.

- (a) Find the positive real number c for which $|\psi\rangle = c|v\rangle$ is a valid quantum state. Write $|\psi\rangle$ as a column vector and $\langle\psi|$ as a row vector.
- (b) Compute $\langle w|\psi\rangle$ and $\langle\psi|w\rangle$. Are $|\psi\rangle$ and $|w\rangle$ orthogonal?
- (c) Write $P = |\psi\rangle\langle\psi|$ as a 2×2 matrix. Compute $P|v\rangle$ and $P|w\rangle$. Explain the difference between $\langle\psi||\psi\rangle$ and $|\psi\rangle\langle\psi|$.

2. Unitary evolution

- (a) Verify that H and S are unitary, and find their inverses. Is the matrix P from Problem 1 unitary? Justify your answer.
- (b) Show that a unitary U preserves inner products: $\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle$. Deduce that it preserves normalisation.
- (c) Can a single unitary map both $|0\rangle$ and $|1\rangle$ to $|+\rangle$? Explain.

3. Composite systems and tensor products

Let $|a\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$ and $|b\rangle = (2|0\rangle - |1\rangle)/\sqrt{5}$.

- (a) Expand $|a\rangle \otimes |b\rangle$ in the computational basis and write its four-component column vector. Check that it is normalised.
- (b) Write the matrices $H \otimes I$ and $I \otimes H$, where I is the 2×2 identity. Compute their actions on $|01\rangle$ and explain which qubit each operator acts on.
- (c) Show that $|\text{EPR}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ cannot be written as $|u\rangle \otimes |v\rangle$ for any single-qubit states $|u\rangle, |v\rangle$.

4. Measurements in different bases

A qubit is prepared in $|\chi\rangle = (\sqrt{3}|0\rangle + i|1\rangle)/2$.

- (a) If it is measured in the computational basis, give each outcome probability and the corresponding normalised post-measurement state.
- (b) Repeat for measurement in the basis $\{|+\rangle, |-\rangle\}$.
- (c) Suppose the measurement in (b) is performed first, its outcome is discarded, and the qubit is then measured in the computational basis. Find the final outcome probabilities and compare them with (a). Use the law of total probability.

5. Changing the measurement basis

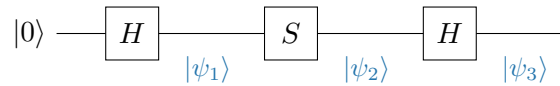
Let $B = \{|b_0\rangle, \dots, |b_{d-1}\rangle\}$ be an orthonormal basis of $\mathbb{C}^d$, and define

$$U = \sum_{j=0}^{d-1} |b_j\rangle\langle j|.$$

- Show that $U|j\rangle = |b_j\rangle$ and that U is unitary.
- To measure an arbitrary state $|\psi\rangle$ in the basis B , which should you apply before a computational-basis measurement: U or $U^\dagger$? Show that the probability of outcome j is $|\langle b_j|\psi\rangle|^2$.
- Immediately after that computational-basis measurement, what is the state? What further unitary produces the same post-measurement state as a direct measurement in B ?
- Give explicit gate sequences for the preprocessing in (b) for $\{|+\rangle, |-\rangle\}$ and for $\{|+i\rangle, |-i\rangle\}$, where $|\pm i\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$. Use only $H, S, S^\dagger$, and state the order of application.

6. Circuit I: relative phase and interference

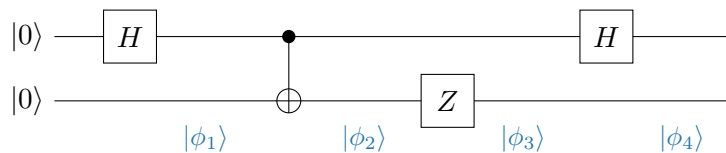
Consider the following circuit:



- Write the state after each gate, both in bra-ket notation and as a column vector.
- Find the final computational-basis measurement probabilities.
- Replace S by Z . What is the new final state? Explain how the last H makes the relative phase observable.

7. Circuit II: an EPR pair and local gates

The controlled-NOT gate acts as $\text{CNOT}|x, y\rangle = |x, y \oplus x\rangle$ for $x, y \in \{0, 1\}$, where $\oplus$ denotes addition modulo 2. The filled dot is the control.



- Write the joint two-qubit state after each gate. At which stages is it a product state, and at which stages is it entangled?
- If both qubits are measured in the computational basis at the end, find the probabilities of 00, 01, 10, 11.
- Instead, measure only the first qubit at the end. For each possible outcome, give its probability and the conditional state of the second qubit.