Quantum Cook-Levin Theorem

Yiyi Cai

Oct. 31st, 2025

Outline

- Classical Cook-Levin Theorem
- k-Local Hamiltonians & Examples
- Local Hamiltonians are in QMA
- Quantum Cook-Levin Theorem

Classical Cook-Levin Theorem

Theorem. SAT is NP-Complete.

For any $x \in L$, NP language L, we can encode it into a Boolean formula ϕ such that

$x \in L \leftrightarrow \phi$ is satisfiable

"Efficient verification" $\leftrightarrow$ "checking satisfying assignment"

Proof Sketch. Consider a deterministic TM M that runs in T(n). Encode computation as a tableau.

Check:

- Initialization
- Correct propagation (local)
- Correct output

Key ideas in Cook-Levin

- Each local clause checks a small neighborhood of the tableau
- Global correctness = all local checks are satisfied

Quantum Analog:

- Boolean variables → qubits
- Clauses → local Hamiltonian terms
- Satisfiability → ground-state energy = 0
- Nature is "local"

Local Hamiltonians

- Hamiltonians: energy operator that describes interactions of quantum systems
- Local Hamiltonians: each term only acts on constant number of qubits

Physics: "What is the ground state of a local Hamiltonian?"
Computer Science: "Can we efficiently verify such a state?"

- Embed k-SAT into local Hamiltonians

$$c = (x_1 \vee \neg x_2)$$

$|01\rangle$ is an unsatisfying assignment

$$H = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$$

Examples

- $\phi = c_1 \wedge c_2 \wedge c_3, \quad c_1 = (x_1 \vee x_2), c_2 = (\neg x_2 \vee x_3), c_3 = (x_3 \vee x_4)$

$$H = H_{c_1} \otimes I_{3,4} + I_1 \otimes H_{c_2} \otimes I_4 + I_{1,2} \otimes H_{c_3}$$

$$\langle x | H | x \rangle = 0 \iff \phi(x) = 1$$

$\langle x | H | x \rangle$ counts the number of unsatisfied clauses

- MAX CUT: partition edges of an undirected graph $G = (V, E)$ into two disjoint sets E_1, E_2 such that the maximum number of edges possible crosses between E_1 and E_2.

$$H_{ij} = I - Z_i \otimes Z_j \quad \forall (i,j) \in E$$

k-Local Hamiltonian Problem

$$H = \sum_{i=1}^{m} H_i, \quad supp(H_i) \leq k$$

Promise: efficiently computable $\alpha(n), \beta(n) \in \mathbb{R}$ satisfying $\alpha(n) - \beta(n) \geq 1/p(n)$

Output:

- If $\lambda_{\min}(H) \leq \alpha(n)$ or $\lambda_{\min}(H) \geq \beta(n)$, accept
- Accept/reject arbitrarily otherwise

Note:

- H does not need to be diagonal, no geometric restrictions
- inverse polynomial gap is important
- Number of samples needed to tell apart the states $\sim \dfrac{1}{\Delta}$

Quantum Cook-Levin Theorem

k-local Hamiltonian problems are QMA-complete.

Proof Sketch of k-LH is in QMA

$$Tr(H|\psi\rangle\langle\psi|) = \langle\psi|H|\psi\rangle = \sum_i \langle\psi|H_i|\psi\rangle$$

Verifier (given T copies of $|\psi\rangle$):

- Repeat T times:
  - Randomly pick a term H_i, measure it on $|\psi\rangle$, record its value
- Average the recorded scores

Main Obstacles

- Locally check quantum states
  - product states

$$|\phi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle,$$
$$|\phi'\rangle = |\phi_1'\rangle \otimes \cdots \otimes |\phi_n'\rangle$$

  - entangled states

$$|Cat_+\rangle = \frac{|0^n\rangle + |1^n\rangle}{\sqrt{2}}, |Cat_-\rangle = \frac{|0^n\rangle - |1^n\rangle}{\sqrt{2}}$$

$$Tr_{reg1}|Cat_+\rangle\langle Cat_+| = \frac{1}{2}|0^{n-1}\rangle\langle 0^{n-1}| + \frac{1}{2}|1^{n-1}\rangle\langle 1^{n-1}| = Tr_{reg1}|Cat_-\rangle\langle Cat_-|$$

Proof Sketch of Quantum Cook-Levin.

- $L \in QMA : \exists \{V_n\}$ such that if $x \in L_{yes}$, $\exists |\psi\rangle$ such that V_n accepts $|x\rangle \otimes |\psi\rangle$ w.p. at least 2/3..
- Circuit V_n can be broken down into unitaries $U_1, U_2, \cdots, U_T$

Natural attempt..

$$|\phi_0\rangle = |x\rangle \otimes |\psi\rangle$$

$$|\phi_1\rangle = U_1(|x\rangle \otimes |\psi\rangle)$$

$$\vdots$$

$$|\phi_T\rangle = U_T \cdots U_1(|x\rangle \otimes |\psi\rangle)$$

use local checks to verify $|\phi_{i+1}\rangle = U_{i+1}|\phi_i\rangle$.. hopeless even for $U = I$!

Culprit & Solution: Entanglement

$$|\eta\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle \otimes |0\rangle + |\psi'\rangle \otimes |1\rangle)$$

If the two states are equal, then $|\eta\rangle = |\psi\rangle \otimes |+\rangle$

$|\eta\rangle$ is the ground state of $H = I \otimes |-\rangle\langle -|$ iff $|\psi\rangle = |\psi'\rangle$

Check $|\psi'\rangle = U|\psi\rangle$?

Consider controlled unitary $W := I \otimes |0\rangle\langle 0| + U^{-1} \otimes |1\rangle\langle 1|$

$$W|\eta\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle \otimes |0\rangle + U^{-1}|\psi'\rangle \otimes |1\rangle)$$

$|\eta\rangle$ is the ground state of $H = W^\dagger(I \otimes |-\rangle\langle -|)W$ iff $|\psi'\rangle = U|\psi\rangle$

Proof Sketch Contd.

- Design Feymann-Kitaev Hamiltonians such that its ground state is

$$|\Omega\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle \otimes |\Omega_t\rangle$$

History state : $|\Omega_t\rangle = U_t U_{t-1} \cdots U_1 (|x\rangle \otimes |\psi\rangle \otimes |0\rangle)$

- $H = H_{start} + H_{prop} + H_{end}$
- Starts OK. $|\Omega_0\rangle = |x\rangle \otimes |\psi\rangle \otimes |0\rangle$
- Evolves OK based on unitaries.
- Ends OK. Measuring the output qubit of the final snapshot state $|\Omega_T\rangle$ yields $|1\rangle$ with high probability.

$x \in L_{yes}, \exists |\psi\rangle$ accepted by V_n, the history state $|\Omega\rangle$ satisfies all local terms

$x \in L_{no}$, all states violate at least one condition

Summary

- k-local Hamiltonian problems are QMA-complete with 1/poly gap promised
- Generic ground-state estimation problem is at least as hard as any QMA problem
- Find locality structure even when information is stored globally
- Quantum PCP Conjecture: constant gap instead of 1/poly is still QMA-hard!