# Quantum Complexity Theory

## Mixed states, observables, and channels

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# The full generality of QM

Generality

① Classical states $|i\rangle$

② Pure states $|\psi\rangle = \sum_i \alpha_i |i\rangle$ → Superposition

③ Mixed states $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ → distributions over pure states

## Why do we need this?

- Lack of knowledge / randomness
- $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$: what is the state of 1st qubit?
- Partial measurements / non-closed systems

__Spectral theorem__ Let $A \in \mathbb{C}^{d\times d}$ be Hermitian. $\exists$ ort. basis $\{|b_1\rangle, \ldots, |b_d\rangle\}$ of $\mathbb{C}^d$ and __real__ eigenvalue $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ s.t. $A = \sum_i \lambda_i |b_i\rangle\langle b_i|$.

__Positive-semidefinte (psd) matrices__ Hermitian $A$ is psd iff all its eigenvalues are __non-negative__.

Hermitians as high-dimensional numbers!

## Density matrices $\rho \in \mathbb{C}^{d\times d}$ psd s.t. $\mathrm{Tr}(\rho) = 1$

(Generalising probability distributions $\forall_i\ p_i \geq 0$, $\sum_i p_i = 1$)

Describing mixed states: pure $|\psi\rangle \in \mathbb{C}^d \longrightarrow |\psi\rangle\langle\psi|$

→ outer prod. as projection

mixture $\left\{\begin{array}{ll} |\psi_1\rangle & p_1 \\ \vdots & \vdots \\ |\psi_k\rangle & p_k \end{array}\right. \longrightarrow \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho$

- psd as convex comb. of psd.
- $\mathrm{Tr}(\rho) = \mathrm{Tr}\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right) = \sum_i p_i \mathrm{Tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$

Non-commutative probability theory

# Examples

$|0\rangle \rightarrow |0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ $\quad |1\rangle \rightarrow |1\rangle\langle 1| = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$

$\begin{cases}|0\rangle & 0.5\\|+\rangle & 0.5\end{cases} \longrightarrow \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +| = \begin{pmatrix}1/2 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}1/4 & 1/4\\1/4 & 1/4\end{pmatrix} = \begin{pmatrix}3/4 & 1/4\\1/4 & 1/4\end{pmatrix}$

$\begin{cases}|0\rangle & 0.5\\|1\rangle & 0.5\end{cases} \longrightarrow \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \frac{I}{2}$ → maximally mixed state

$\begin{cases}|+\rangle & 0.5\\|-\rangle & 0.5\end{cases} \longrightarrow \frac{1}{2}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\frac{1}{2} & -1/2\\-1/2 & \frac{1}{2}\end{pmatrix} = \frac{I}{2}$ → indistinguishable!

Note that $-|0\rangle \rightarrow \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \leftarrow |0\rangle$ → global phase is indistinguishable

## Projective measurements (PVM)

$M = \{M_1, \ldots, M_k\}$ is a $k$-outcome PVM if:

- $\forall_i$ $M_i$ is Hermitian projection $M_i^\dagger = M_i$, $M_i^2 = M_i$
- $\sum_i M_i = I$

pure

Measuring $|\psi\rangle$ using $M$ yields:

- Outcome $i$ w.p. $\| M_i |\psi\rangle \|^2 = \langle \psi | M_i | \psi \rangle$
- Post-measurement state $\dfrac{M_i|\psi\rangle}{\| M_i |\psi\rangle \|^2}$

# Density matrices encode all physical information

Unitary evolution: $\rho \mapsto U\rho U^\dagger$ $\qquad$ $\underset{\text{pure}}{|\psi\rangle \mapsto U|\psi\rangle}$ $\qquad$ $\underset{\text{mixed}}{|\psi\rangle\langle\psi| \mapsto U|\psi\rangle\langle\psi|U^\dagger}$

Measurement: Let $M = \{M_1, \ldots, M_k\}$ proj. Measuring $\rho$ with $M$

yields $i$ w.p. $\mathrm{Tr}(M_i \rho)$. $\qquad$ Ex $\rho = |\psi\rangle\langle\psi|$

$$\mathrm{Tr}(M_i |\psi\rangle\langle\psi|) = \mathrm{Tr}(\langle\psi| M_i |\psi\rangle)$$

$$= \langle\psi| M_i |\psi\rangle = \| M_i |\psi\rangle \|^2$$

Post-measurement state: $\rho \mapsto \dfrac{M_i \rho M_i}{\mathrm{Tr}(M_i \rho)}$

# Observables

An observable is a Hermitian that encodes a measurement.

$A \in \mathbb{C}^{d\times d}$, $A = \sum_j \lambda_j |b_j\rangle\langle b_j| \to$ basis $\{|b_j\rangle\}_j$

$\lambda_j$ are weights of outcomes

The expectation of $A$ w.r.t. $\rho$ is

$$\mathrm{Tr}(A\rho) = \mathrm{Tr}\left(\sum_j \lambda_j |b_j\rangle\langle b_j| \rho\right) = \sum_j \lambda_j \mathrm{Tr}(|b_j\rangle\langle b_j| \rho)$$

$$= \sum_j \lambda_j \Pr[\text{obtainig } |b_j\rangle \text{ measuring } \rho]$$

CHSH game: $A_0 = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad B_0 = \frac{1}{\sqrt{2}}(X+Z)$

$A_1 = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad B_1 = \frac{1}{\sqrt{2}}(Z-X)$

## Tracing out & purification

Reduced density matrix $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, where

$$\mathrm{Tr}_B(|a_1, b_1\rangle\langle a_2, b_2|) = |a_1\rangle\langle a_2| \cdot \langle b_2|b_1\rangle \quad \forall |a_1\rangle, |a_2\rangle, |b_1\rangle, |b_2\rangle$$

Analogous to a marginal distribution.

**Purification** $\forall \rho_A \quad \exists |\psi\rangle_{AB} \quad \rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|_{AB})$

mixed

pure (larger)

**Uhlmann's theorem** All purifications are related by a unitary.

# Trace distance

Quantum analogue of statistical distance

$$D(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1 = \frac{1}{2}\,\mathrm{Tr}(|\rho-\sigma|)$$

## Operational meaning

The maximum probability of distinguishing $\sigma$ and $\rho$ using any quantum operation (measurement, unitary)

# Quantum channels

General quantum dynamics: unitary, measurement, $\pm$ system

Captured by CPTP maps $\rho \mapsto S(\rho)$

(I) Positive: $\rho \geq 0 \quad \longmapsto \quad S(\rho) \geq 0$

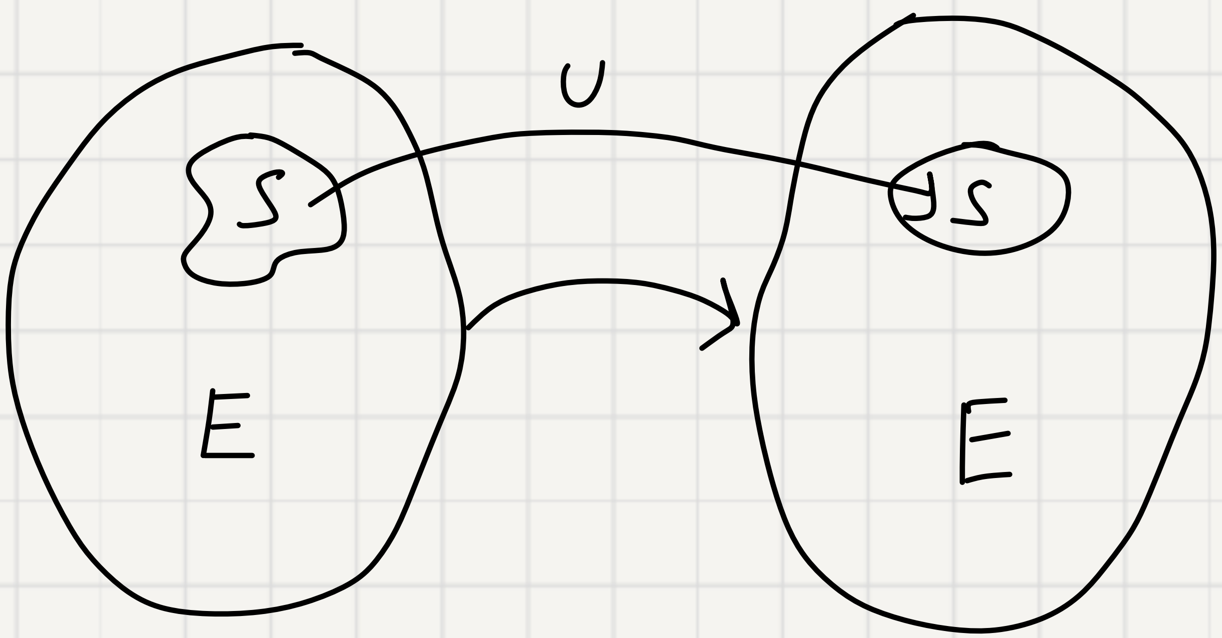
(II) Completely positive: $\sigma \geq 0$ s.t. $tr_R \sigma = \rho$

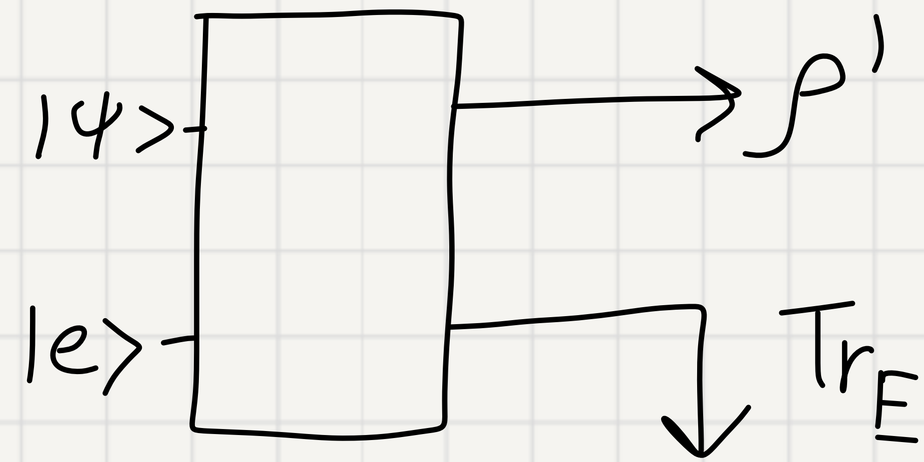
$$\Rightarrow \quad (I \otimes S)(\sigma) \geq 0$$

(III) Trace preserving: $Tr\, S(\rho) = Tr(\rho)$

Kraus representation $S(\rho) = \sum_k A_k \rho A_k^\dagger \; , \; \sum_k A_k A_k^\dagger = I$

# Open quantum systems




Channel / superoperator / CPTP map:

$|\psi\rangle\langle\psi|$

$$\rho \longrightarrow \rho' = \sum_k E_k \rho E_k^\dagger , \quad \sum_k E_k^\dagger E = I$$

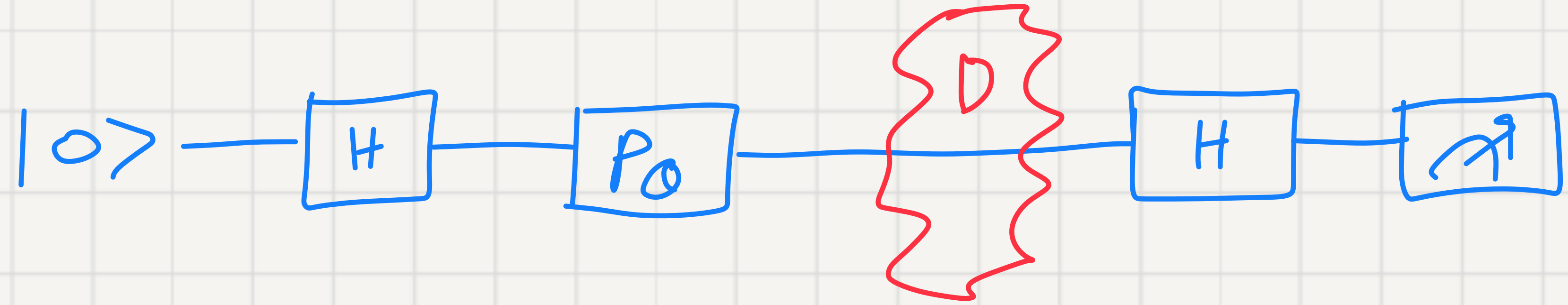
Kraus decomposition

Generalising unitary transformations:

$$\rho \longrightarrow \rho' = U \rho U^\dagger , \quad U^\dagger U = I$$

# Decoherence vs interference

Consider




Entanglement with env:

$$|0\rangle|e\rangle \to |0\rangle|e_0\rangle$$
$$|1\rangle|e\rangle \to |1\rangle|e_1\rangle$$
$$\langle e_0|e_1\rangle = V e^{i\alpha}$$

$$|0\rangle|e\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)|e\rangle \xrightarrow{P_\theta} \frac{1}{\sqrt{2}}\left(|0\rangle+e^{i\theta}|1\rangle\right)|e\rangle$$

$$\xrightarrow{D} \frac{1}{\sqrt{2}}\left[|0\rangle|e_0\rangle + e^{i\theta}|1\rangle|e_1\rangle\right] \xrightarrow{H} |0\rangle\frac{|e_0\rangle+e^{i\theta}|e_1\rangle}{2} + |1\rangle\frac{|e_0\rangle-e^{i\theta}|e_1\rangle}{2}$$

$$P_0 = \sqrt{\frac{1}{2}\left[1+V\cdot\cos(\theta+\alpha)\right]}$$