Proof Assistants

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Lent 2026

Part I

Isabelle

Material

- Isabelle part of this course based on book "Concrete Semantics with Isabelle/HOL" (2014) by Tobias Nipkow and Gerwin Klein

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- Slides shamelessly copied from Tobias Nipkow (errors are my own)

Chapter 1

Programming and Proving in Isabelle/HOL

1. Overview of Isabelle/HOL

2. Type and function definitions

3. Induction Heuristics

4. Simplification

HOL = Higher-Order Logic

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- Equalities ($term = term$)

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HOL Formulas:

- Equalities ($term = term$), e.g. $1 + 2 = 4$
- Later: $\wedge$, $\vee$, $\longrightarrow$, $\forall$, $\ldots$

1. Overview of Isabelle/HOL
   - Types and terms
   - By example: types *bool*, *nat* and *list*

Types

Basic syntax:

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Convention: $\tau_1 \Rightarrow \tau_2 \Rightarrow \tau_3 \equiv \tau_1 \Rightarrow (\tau_2 \Rightarrow \tau_3)$

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This language of terms is known as the *λ-calculus*.

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$$(\lambda x.\ t)\ u \ = \ t[u/x]$$

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- Isabelle performs β-reduction automatically.

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$t\ ::\ \tau$ means "t is a well-typed term of type τ".

$$\frac{t :: \tau_1 \Rightarrow \tau_2 \qquad u :: \tau_1}{t\ u :: \tau_2}$$

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User can help with *type annotations* inside the term.
Example: $f\ (x::nat)$

Overview_Demo.thy

(including an example of how to define a simple function and prove a lemma about it)

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   - Types and terms
   - By example: types $bool$, nat and $list$

Type *bool*

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if-and-only-if: $=$ or $\longleftrightarrow$

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You need type annotations: $1\ ::\ nat,\ x + (y{::}nat)$

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You need type annotations: $1 :: nat$, $x + (y{::}nat)$
unless the context is unambiguous: $Suc\ z$

Nat_Demo.thy

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Induction on natural numbers

To prove $P(n)$ for all natural numbers n, prove

- $P(0)$ and
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Type $'a\ list$

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- $x\ \#\ xs = Cons\ x\ xs$:
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- $[x_1, \ldots, x_n] = x_1\ \#\ \ldots\ x_n\ \#\ []$

Structural Induction for lists

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List_Demo.thy

An informal proof

Lemma $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$

Proof by induction on xs.

- Case Nil: $app\ (app\ Nil\ ys)\ zs = app\ ys\ zs = app\ Nil\ (app\ ys\ zs)$ holds by definition of app.
- Case $Cons\ x\ xs$: We assume $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$ (IH), and we need to show $app\ (app\ (Cons\ x\ xs)\ ys)\ zs = app\ (Cons\ x\ xs)\ (app\ ys\ zs)$.

  The proof is as follows:

  $app\ (app\ (Cons\ x\ xs)\ ys)\ zs$

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Predefined: $xs\ @\ ys$ (append), $length$,

Large library: `HOL/List.thy`

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Don't reinvent, reuse!

Predefined: $xs\ @\ ys$ (append), $length$, and map

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Introduces a *synonym* $name$ for type τ

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Type synonyms are expanded after parsing
and are not present in internal representation and output

datatype — the general case

$$\textbf{datatype}\ (\alpha_1, \ldots, \alpha_n)t \quad = \quad C_1\ \tau_{1,1} \ldots \tau_{1,n_1}$$
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- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)t$

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- *Injectivity:* $(C_i\ x_1\ldots x_{n_i} = C_i\ y_1\ldots y_{n_i}) = (x_1 = y_1 \wedge \cdots \wedge x_{n_i} = y_{n_i})$

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- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)t$
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- *Injectivity:* $(C_i\ x_1 \ldots x_{n_i} = C_i\ y_1 \ldots y_{n_i}) = (x_1 = y_1 \wedge \cdots \wedge x_{n_i} = y_{n_i})$

Distinctness and injectivity are applied automatically

Induction must be applied explicitly

Case expressions

Datatype values can be taken apart with *case*:

$$(\textit{case}\ xs\ \textit{of}\ \ [] \Rightarrow \ldots \ \ |\ \ y\#ys \Rightarrow \ldots\ y\ \ldots\ ys\ \ldots)$$

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Need ( ) in context

`Tree_Demo.thy`

The *option* type

datatype $'a\ option = None \mid Some\ 'a$

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2 Type and function definitions

Type definitions

Function definitions

Non-recursive definitions

Example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n\ =\ n{*}n$

Non-recursive definitions

Example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n\ =\ n{*}n$

No pattern matching, just $f\,x_1\ \dots\ x_n\ =\ \dots$

The danger of nontermination

How about $f\,x = f\,x + 1$?

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$\Longrightarrow 0 = 1$

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! All functions in HOL must be total **!**

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- Proves customized induction schema

Example: separation

fun $sep :: {'a} \Rightarrow {'a}\ list \Rightarrow {'a}\ list$ **where**
$sep\ \ a\ (x \# y \# zs) = x\ \#\ a\ \#\ sep\ \ a\ (y \# zs)\ \ |$
$sep\ \ a\ xs = xs$

Example: Ackermann

fun $ack :: nat \Rightarrow nat \Rightarrow nat$ **where**

$$
\begin{aligned}
&ack\ 0\ n = Suc\ n \mid \\
&ack\ (Suc\ m)\ 0 = ack\ m\ (Suc\ 0) \mid \\
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\end{aligned}$$

Terminates because the arguments decrease *lexicographically* with each recursive call:

- $(Suc\ m,\ 0) > (m,\ Suc\ 0)$
- $(Suc\ m,\ Suc\ n) > (Suc\ m,\ n)$
- $(Suc\ m,\ Suc\ n) > (m,\ \_)$

1. Overview of Isabelle/HOL

2. Type and function definitions

3. **Induction Heuristics**

4. Simplification

Basic induction heuristics

Theorems about recursive functions
are proved by induction

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Induction on argument number i of f
if f is defined by recursion on argument number i

A tail recursive reverse

Our initial reverse:

fun $rev :: {'a}\ list \Rightarrow {'a}\ list$ **where**

$rev\ [] \quad = [] \quad |$

$rev\ (x\#xs) \quad = rev\ xs\ @\ [x]$

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lemma $itrev\ xs\ [] = rev\ xs$

Induction_Demo.thy

Generalisation

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- Generalize free variables
  - by $arbitrary$ in induction proof
  - (or by universal quantifier in formula)

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In each induction step, 1 constructor is added.
In each recursive call, 1 constructor is removed.

Now: induction for complex recursion patterns.

Computation Induction

Example

fun $div2 :: nat \Rightarrow nat$ **where**
$div2\ 0 = 0\ \ |$
$div2\ (Suc\ 0) = 0\ \ |$
$div2\ (Suc(Suc\ n)) = Suc(div2\ n)$

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$\leadsto$ induction rule `div2.induct`:

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Induction follows course of (terminating!) computation
Motto: properties of f are best proved by rule $f.induct$

How to apply $f.induct$

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Heuristic:

- there should be a call $f\ a_1\ \ldots\ a_n$ in your goal

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- ideally the a_i should be variables.

Induction_Demo.thy

Computation Induction

1. Overview of Isabelle/HOL
2. Type and function definitions
3. Induction Heuristics
4. Simplification

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Using equations $l = r$ from left to right

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Simplification = (Term) Rewriting

An example

Equations:

$$
\begin{aligned}
0 + n &= n & (1)\\
(Suc\ m) + n &= Suc\ (m + n) & (2)\\
(Suc\ m \leq Suc\ n) &= (m \leq n) & (3)\\
(0 \leq m) &= True & (4)
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We can simplify $f(0)$ to $g(0)$ but
we cannot simplify $f(1)$ because $p(1)$ is not provable.

Termination

Simplification may not terminate.

Isabelle uses $simp$-rules (almost) blindly from left to right.

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$$n < m \Longrightarrow (n < Suc\ m) = True \quad \text{YES}$$
$$Suc\ n < m \Longrightarrow (n < m) = True \quad \text{NO}$$

Proof method *simp*

Goal: 1. $\llbracket\ P_1; \ldots; P_m\ \rrbracket \Longrightarrow C$

apply$(simp\ add{:}\ eq_1\ \ldots\ eq_n)$

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Variations:

- $(simp \ \ldots \ del\!: \ldots)$ removes $simp$-lemmas
- add and del are optional

$auto$ versus $simp$

- $auto$ acts on all subgoals
- $simp$ acts only on subgoal 1

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- *auto* can also be modified:
  $(auto\ simp\ add: \ldots\ simp\ del: \ldots)$

Rewriting with definitions

Definitions (**definition**) must be used explicitly:

$$(simp\ add{:}\ f\_def \ldots)$$

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f is the function whose definition is to be unfolded.

Case splitting with $simp/auto$

Automatic:

$$P\ (\textit{if}\ A\ \textit{then}\ s\ \textit{else}\ t)$$
$$=$$
$$(A \longrightarrow P(s)) \wedge (\neg A \longrightarrow P(t))$$

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By hand:

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Proof method: $(simp\ split\colon nat.split)$

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$$(e = 0 \longrightarrow P(a)) \wedge (\forall\, n.\ e = Suc\ n \longrightarrow P(b))$$

Proof method: $(simp\ split{:}\ nat.split)$

Or $auto$. Similar for any datatype t: $t.split$

Simp_Demo.thy

Chapter 2

Case Study: IMP Expressions

5 Case Study: IMP Expressions

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This section introduces

arithmetic and boolean expressions

of our imperative language IMP.

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IMP *commands* are introduced later.

5 Case Study: IMP Expressions

Arithmetic Expressions

Boolean Expressions

Concrete and abstract syntax

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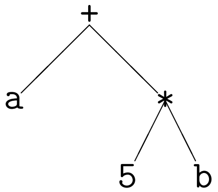
Abstract syntax: trees, eg



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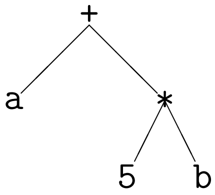

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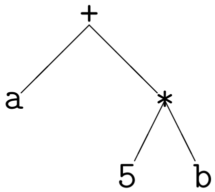

Parser: function from strings to trees

Linear view of trees: terms, eg $Plus\ a\ (Times\ 5\ b)$

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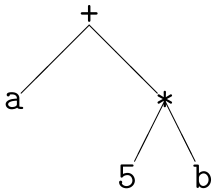

Parser: function from strings to trees

Linear view of trees: terms, eg *Plus a (Times 5 b)*

Abstract syntax trees/terms are datatype values!

Concrete syntax is defined by a context-free grammar, eg

$$a ::= n \mid x \mid (a) \mid a + a \mid a * a \mid \dots$$

where n can be any natural number and x any variable.

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We focus on *abstract* syntax
which we introduce via datatypes.

Datatype $aexp$

Variable names are strings, values are integers:

type_synonym $vname = string$
datatype $aexp = N\ int \mid V\ vname \mid Plus\ aexp\ aexp$

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Concrete	Abstract
5	$N\ 5$
`x`	$V\ ''x''$
`x+y`	$Plus\ (V\ ''x'')\ (V\ ''y'')$
`2+(z+3)`	$Plus\ (N\ 2)\ (Plus\ (V\ ''z'')\ (N\ 3))$

Warning

This is syntax, not (yet) semantics!

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$$N\,0 \;\neq\; Plus\;(N\,0)\;(N\,0)$$

The (program) state

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- The value of an expression
  depends on the value of its variables.
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- The state is a function from variable names to values:

  type_synonym $val = int$
  type_synonym $state = vname \Rightarrow val$

Function update notation

If $f :: \tau_1 \Rightarrow \tau_2$ and $a :: \tau_1$ and $b :: \tau_2$ then

$$f(a := b)$$

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is the function that behaves like f
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$$f(a := b) = (\lambda x.\ \mathit{if}\ x = a\ \mathit{then}\ b\ \mathit{else}\ f\ x)$$

How to write down a state

Some states:

- $\lambda x.\ 0$

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Nicer notation defined in `AExp.thy`:

$$<''a'' := 5,\ ''x'' := 3,\ ''y'' := 7>$$

How to write down a state

Some states:

- $\lambda x.\ 0$
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Nicer notation defined in `AExp.thy`:

$$<''a'' := 5,\ ''x'' := 3,\ ''y'' := 7>$$

Maps everything to 0, but $''a''$ to 5, $''x''$ to 3, etc.

AExp.thy

5 Case Study: IMP Expressions

Arithmetic Expressions

Boolean Expressions

BExp.thy

This was easy.

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Because evaluation of expressions always terminates.

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But execution of programs may *not* terminate.
Hence we cannot define it by a total recursive function.

We need more logical machinery
to define program execution and reason about it.

Chapter 3

Logic and Proof Beyond Equality

6 Logical Formulas

7 Proof Automation

8 Single Step Proofs

9 Inductive Definitions

Syntax (in decreasing precedence):

$$\begin{array}{lclclcl} form & ::= & (form) & \mid & term = term & \mid & \neg form \\ & \mid & form \wedge form & \mid & form \vee form & \mid & form \longrightarrow form \\ & \mid & \forall x.\ form & \mid & \exists x.\ form & & \end{array}$$

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Examples:

$$\neg\ A\ \wedge\ B\ \vee\ C\ \ \equiv\ \ ((\neg\ A)\ \wedge\ B)\ \vee\ C$$

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Examples:

$$\begin{array}{rcl} \neg\ A \wedge\ B \vee\ C & \equiv & ((\neg\ A) \wedge\ B) \vee\ C \\ s =\ t \wedge\ C & \equiv & (s =\ t) \wedge\ C \end{array}$$

Syntax (in decreasing precedence):

$$
\begin{array}{rclclcl}
\mathit{form} & ::= & (\mathit{form}) & | & \mathit{term} = \mathit{term} & | & \neg \mathit{form} \\
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\end{array}
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Examples:

$$
\begin{array}{rcl}
\neg\ A \wedge B \vee C & \equiv & ((\neg\ A) \wedge B) \vee C \\
s = t \wedge C & \equiv & (s = t) \wedge C \\
A \wedge B = B \wedge A & \equiv & {\color{red} A \wedge (B = B) \wedge A}
\end{array}
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Syntax (in decreasing precedence):

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Input syntax: $\longleftrightarrow$ (same precedence as $\longrightarrow$)

Variable binding convention:

$$\forall\, x\ y.\ P\ x\ y \ \equiv\ \forall\, x.\ \forall\, y.\ P\ x\ y$$

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Similarly for $\exists$ and λ.

Warning

Quantifiers have low precedence
and need to be parenthesized (if in some context)

$$! \quad P \wedge \forall x.\ Q\ x \ \rightsquigarrow\ P \wedge (\forall x.\ Q\ x) \quad !$$

Mathematical symbols

and their ascii representations

<table>
<tr><td>$\forall$</td><td>\<forall></td><td>ALL</td></tr>
<tr><td>$\exists$</td><td>\<exists></td><td>EX</td></tr>
<tr><td>λ</td><td>\<lambda></td><td>%</td></tr>
<tr><td>$\longrightarrow$</td><td>--></td><td></td></tr>
<tr><td>$\longleftrightarrow$</td><td><-></td><td></td></tr>
<tr><td>$\wedge$</td><td>/\</td><td>&</td></tr>
<tr><td>$\vee$</td><td>\/</td><td>|</td></tr>
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<tr><td>$\neq$</td><td>\<noteq></td><td>~=</td></tr>
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Sets over type $'a$

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$\in$	`\<in>`	`:`
$\subseteq$	`\<subseteq>`	`<=`
$\cup$	`\<union>`	`Un`
$\cap$	`\<inter>`	`Int`

Set comprehension

- $\{x.\ P\}$ where x is a variable

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- Instead: $\{t \mid x\ y\ z.\ P\}$
  is short for $\{v.\ \exists\, x\ y\ z.\ v = t \wedge P\}$
  where x, y, z are the free variables in t

simp and *auto*

simp: rewriting and a bit of arithmetic

auto: rewriting and a bit of arithmetic, logic and sets

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- Show you where they got stuck

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Exception: *auto* acts on all subgoals

fastforce

- rewriting, logic, sets, relations and a bit of arithmetic.

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blast

- A complete proof search procedure for FOL . . .

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Automating arithmetic

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- complete for first-order theory of nat and int (Presburger arithmetic)

Sledgehammer

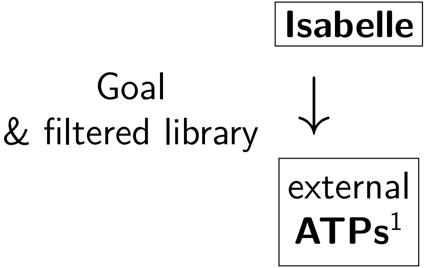
Architecture:

Isabelle

external
ATPs[1]

[1]Automatic Theorem Provers

Architecture:



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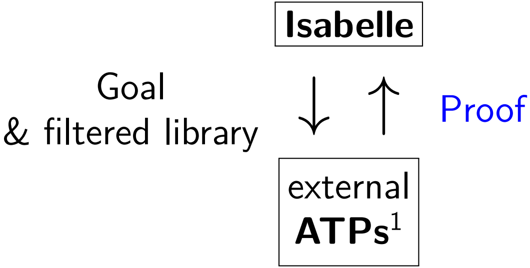

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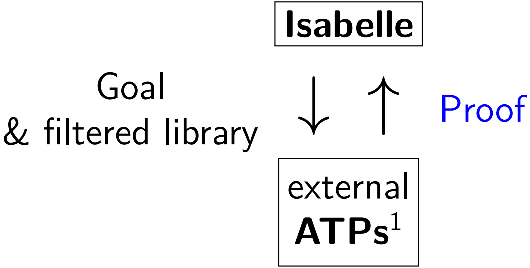

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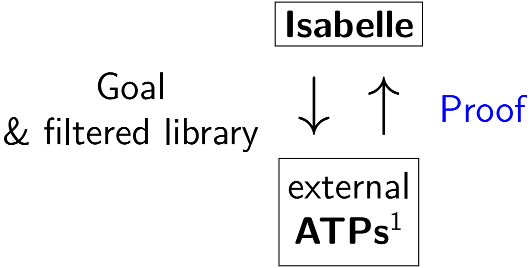

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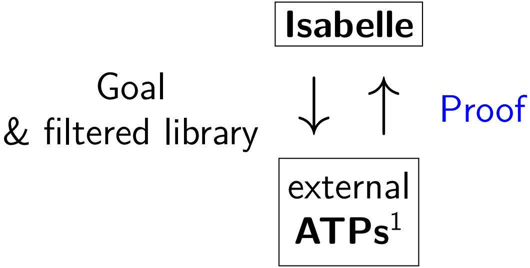

Characteristics:

- Sometimes it works,
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Do you feel lucky?

[1] Automatic Theorem Provers

by(*proof-method*)

$\approx$

apply(*proof-method*)
done

Auto_Proof_Demo.thy

Step-by-step proofs can be necessary if automation fails and you have to explore where and why it failed by taking the goal apart.

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  sets $?P$ to $a{=}b$ and $?Q$ to $False$.

Rule application

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Example: rule: $\llbracket ?P;\ ?Q \rrbracket \Longrightarrow ?P \wedge ?Q$

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The general case: applying rule $[\![\ A_1;\ \ldots\ ;\ A_n\ ]\!] \Longrightarrow A$ to subgoal $\ldots \Longrightarrow C$:

Rule application

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- Unify A and C

Rule application

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- Replace C with n new subgoals $A_1 \ldots A_n$

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apply($rule\ xyz$)

"Backchaining"

Typical backwards rules

$$\frac{?P \quad ?Q}{?P \land ?Q}\ \texttt{conjI}$$

Typical backwards rules

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$$\frac{?P \Longrightarrow ?Q}{?P \longrightarrow ?Q}\ \texttt{impI}$$

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Typical backwards rules

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They are known as introduction rules
because they *introduce* a particular connective.

Automating intro rules

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Can greatly increase the search space!

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The general case:

If r is a theorem $[\![\; A_1; \ldots; A_n \;]\!] \Longrightarrow A$
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$$a = a \wedge b = b$$

From now on: $?$ mostly suppressed on slides

Single_Step_Demo.thy

$\Longrightarrow$ versus $\longrightarrow$

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Phrase theorems like this $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$
not like this $A_1 \wedge \ldots \wedge A_n \longrightarrow A$

Example: even numbers

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inductive $ev :: nat \Rightarrow bool$
where
$ev\ 0 \quad |$
$ev\ n \Longrightarrow ev\ (n + 2)$

An easy proof: $ev\ 4$

$$ev\ 0 \Longrightarrow ev\ 2 \Longrightarrow ev\ 4$$

Consider

fun $evn :: nat \Rightarrow bool$ **where**
$evn\ 0 = True\ |$
$evn\ (Suc\ 0) = False\ |$
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To prove

$$ev\ n \Longrightarrow P\ n$$

by *rule induction* on $ev\ n$ we must prove

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- $P\ n \Longrightarrow P(n{+}2)$

Rule `ev.induct`:

$$\frac{ev\ n \quad P\ 0 \quad \bigwedge n.\ [\![\ ev\ n;\ P\ n\ ]\!] \Longrightarrow P(n{+}2)}{P\ n}$$

Format of inductive definitions

inductive $I :: \tau \Rightarrow bool$ **where**

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Note:

- I may have multiple arguments.

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Note:

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- Each rule may also contain *side conditions* not involving I.

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! Rule induction is absolutely central
to (operational) semantics
and the rest of this lecture course **!**

Inductive_Demo.thy

Inductively defined sets

inductive_set $I :: \tau\ set$ **where**

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Difference to **inductive**: I can later be used with set theoretic operators, eg $I \cup \ldots$

Chapter 4

Isar: A Language for Structured Proofs

Apply scripts

- unreadable

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No structure!

Apply scripts versus Isar proofs

Apply script $=$ assembly language program

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with assertions

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with assertions

But: **apply** still useful for proof exploration

A typical Isar proof

proof
  assume $formula_0$
  have $formula_1$ **by** $simp$
  $\vdots$
  have $formula_n$ **by** $blast$
  show $formula_{n+1}$ **by** $\ldots$
qed

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proves $formula_0 \Longrightarrow formula_{n+1}$

Isar core syntax

proof	=	**proof** [method] step* **qed**
	|	**by** method

Isar core syntax

$$\begin{array}{lcl} \text{proof} & = & \textbf{proof}\ [\text{method}]\ \ \text{step}^*\ \ \textbf{qed} \\ & | & \textbf{by}\ \text{method} \end{array}$$

$$\text{method} \ = \ (simp \ldots) \mid (blast \ldots) \mid (induction \ldots) \mid \ldots$$

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  | **assume** prop ($\Longrightarrow$)
  | [**from** fact$^+$] (**have** | **show**) prop proof

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prop = [name:] "formula"

Isar core syntax

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$$\begin{array}{lcll} \text{step} & = & \textbf{fix}\ \text{variables} & (\bigwedge) \\ & | & \textbf{assume}\ \text{prop} & (\Longrightarrow) \\ & | & [\textbf{from}\ \text{fact}^+]\ \ (\textbf{have} \mid \textbf{show})\ \text{prop}\ \ \text{proof} & \end{array}$$

$$\text{prop} \ = \ [\text{name:}]\ \ \text{"formula"}$$

$$\text{fact} \ = \ \text{name} \mid \ldots$$

Isar_Demo.thy

Isar by example

Further reading

- More detailed Isar introduction in Chapter 5 of "Concrete Semantics"
- Isabelle/Isar reference manual (`isar-ref.pdf`), in particular Chapter 6