Probabilistic Classification

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L101: Machine Learning for Language Processing

Andreas Vlachos

Previous lecture: the perceptron

Advantages:

- Intuitive
- Simple to implement

Disadvantages:

- No probabilities
- Can’t handle non-linear datasets

Why probabilities?

- Interpretability: scores don't tell us much about the confidence of the model
- Knowing what the model knows (and what it doesn't)
- Knowing when the input is ambiguous
- Ability to incorporate prior knowledge

Two approaches in today's lecture:

- Generative: Naive Bayes
- Discriminative: Logistic regression

Classification with Bayes

What we want to do:

$$\hat{y} = \arg\max_{y \in \mathcal{Y}} P(y|x)$$

Bayes Rule:

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)}$$

In plain English:

$$posterior = \frac{likelihood * prior}{evidence}$$

Should we care about the evidence?

- **No(?)** if we only want the class prediction
- **Yes** if we want to know what inputs our model knows about

Naive Bayes

With a feature function:

$$\hat{y} = \arg\max_{y \in \mathcal{Y}} P(\phi(x)|y)P(y)$$

Naive Bayes: assume each feature φ_i is **independent given the class:**

$$\hat{y} = \arg\max_{y \in \mathcal{Y}} P(y) \prod_i P(\phi_i(x)|y)$$

How do we train the model?

Supervised learning, in this case: **Count** and **divide!**

Maximum likelihood estimate for Naive Bayes

Given labeled training data of the form: $D = \{(x^1, y^1), \ldots (x^M, y^M)\}$

Find the parameters $\boldsymbol{w_1}, \boldsymbol{w_2}$ that maximize the likelihood L of D under the model:

Probability of one instance? $P(x, y) = P(x|y)P(y)$

$$w_1^\star, w_2^\star = \arg\max_{w_1, w_2} \prod_{(x,y)\in D} P(w_1; y) \prod_i P(w_2; \phi_i(x)|y)$$

- $\boldsymbol{w_1}$: **Count** the times each class appears, **divide** with the number of instances
- $\boldsymbol{w_2}$: **Count** the times each feature appears in instances of a class, **divide** with sum of feature occurrences for that class (use smoothing to avoid 0s)

What did we get by being naive?

Generative vs Discriminative

Generative models like Naive Bayes can generate text/instance given the class:

- Can ask the model what an instance of a certain class looks like
- Can be seen as a class conditional language model
- Help learning when we don’t have much training data

Discriminative models:

- Model the class prediction directly
- More flexibility in modelling features

From generative (back) to discriminative

All we want is to predict the class: $$\hat{y} = \arg\max_{y \in \mathcal{Y}} P(y|x)$$

And we are still happy to use a linear model $$w \cdot \phi(x)$$

Recall the binary linear classifier we learned with the perceptron: $$\hat{y} = sign(w \cdot \phi(x))$$

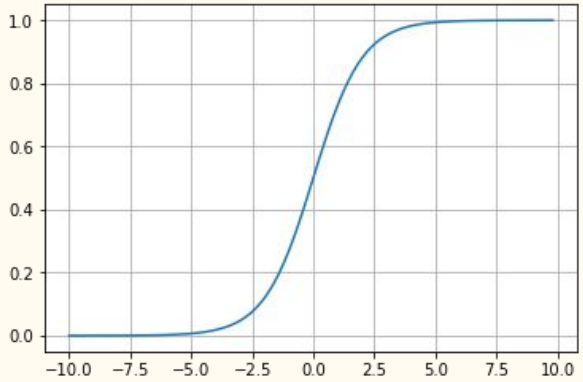
Logistic regression

Recall the binary linear classifier we learned with the perceptron:

$$\hat{y} = sign(w \cdot \phi(x))$$

Push the dot product through the sigmoid function:

$$\sigma(z) = \frac{1}{1+exp(-z)}$$



The binary logistic regression classifier (labels are 0, 1):

$$P(\hat{y} = 1) = \sigma(w \cdot \phi(x))$$

How to learn the parameters?

Supervised learning! Given labeled training data of the form:

$$D = \{(x^1, y^1), \ldots (x^M, y^M)\}$$

Learn weights w that **generalize** well to new instances

Naive Bayes has closed form solution for MLE (count and divide using the data)

In the case of the logistic regression this is not possible; we still define a learning **objective,** but then use a numerical optimization algorithm to find the weights.

Perceptron is a particular objective-algorithm combination

Objective

Maximize the likelihood of the data under the model:

$$L(\hat{y}, y) = P(\hat{y} = 1)^{y}(1 - P(\hat{y} = 1))^{1-y}$$

Recall that y has two discrete options, 1 and 0

Minimize the negative log likelihood (*NLL*):

$$NLL(\hat{y}, y) = -y \log P(\hat{y} = 1) - (1 - y) \log(1 - P(\hat{y} = 1))$$

Often referred to as the cross entropy loss

Objective

Plugging in the logistic regression function: $P(\hat{y} = 1) = \sigma(w \cdot \phi(x))$

And all the training data:

$$D = \{(x^1, y^1), \ldots (x^M, y^M)\}$$

$$w^\star = \arg\min_w \sum_{(x,y) \in D} -y \log \sigma(w \cdot \phi(x)) - (1 - y) \log(1 - \sigma(w \cdot \phi(x)))$$

Unlike the perceptron, it is not enough for the correct label to be the highest scoring; the incorrect one must score as low as possible

Optimizing the objective

We have a function that we want to minimize wrt some parameters.

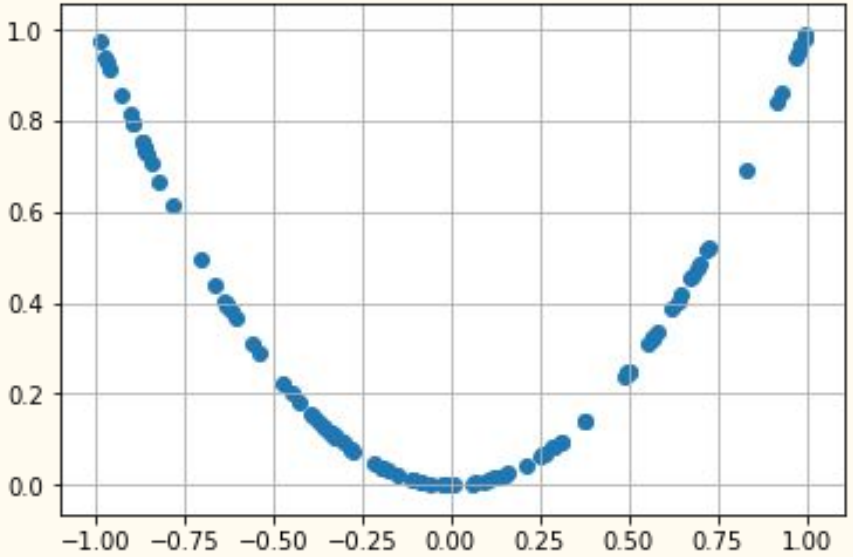
Sometimes there are closed form solutions (naive Bayes), otherwise?

- Random guesses at parameters and objective evaluations

- **Take into account the shape of the function**

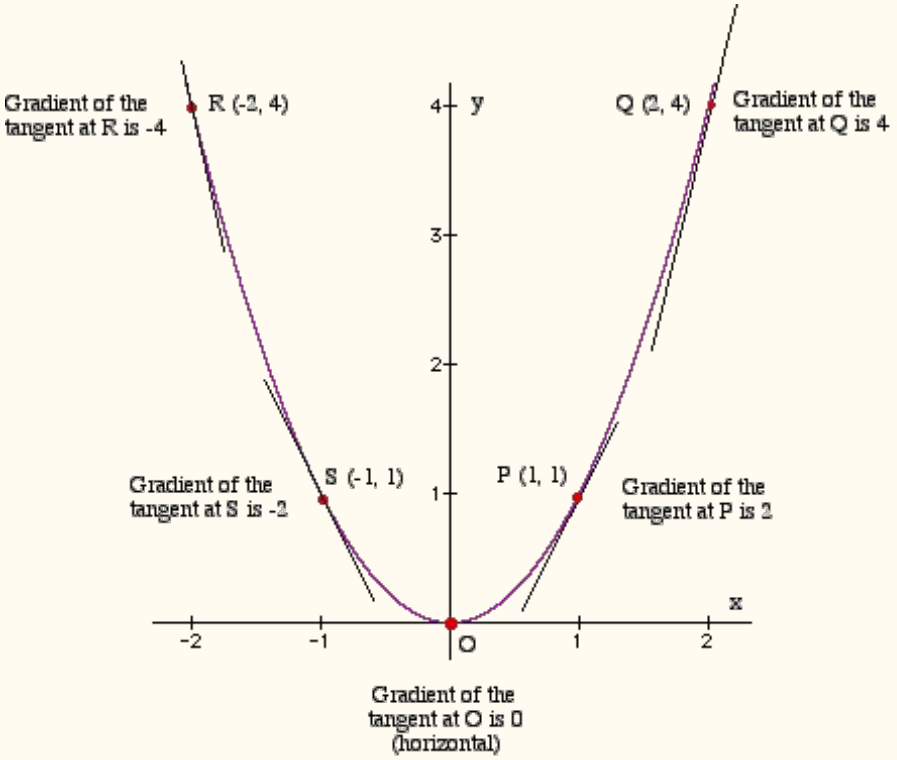
Optimizing a simple function

$$f(x) = x^2$$



With a number of random guesses at x we can get close to the minimum

Gradients



$$f(x) = x^2$$

$$\nabla_x f(x) = 2x$$

Gradients guide us to the minimum, where the gradient in this case is 0

http://bestmaths.net/online/index.php/year-levels/year-12/year-12-topic-list/first-principles/

Gradients for logistic regression

Objective (reminder):

$$NLL(w;\mathcal{D}) = \sum_{(x,y)\in\mathcal{D}} -y \log \sigma(w \cdot \phi(x)) - (1-y)\log(1-\sigma(w \cdot \phi(x)))$$

Gradient with respect to weight w_j for feature φ_j:

$$\frac{\partial NLL(w;\mathcal{D})}{\partial w_j} = \sum_{(x,y)\in D} (\sigma(w \cdot \phi(x)) - y)\phi_j(x)$$

Interpretation: the weight should be updated proportionally to the loss of the model multiplied by the value of the feature for each instance

Binary to Multiclass

The sigmoid “squishes” a real number z to the 0..1 range

$$\sigma(z) = \frac{1}{1+exp(-z)}, z \in \Re$$

The softmax “squishes” a vector $\boldsymbol{z}$ of k real numbers to the probability simplex

$$softmax(z_i) = \frac{\exp(z_i)}{\sum_{j=1}^{k} \exp(z_j)}, z \in \Re^k$$

Multinomial logistic regression:

$$P(\hat{y} = y) = \frac{\exp(w_y \cdot \phi(x))}{\sum_{y' \in \mathcal{Y}} \exp(w_{y'} \cdot \phi(x))}$$

Still a linear classifier:

$$\arg\max_{y \in \mathcal{Y}} P(\hat{y} = y) = \arg\max_{y \in \mathcal{Y}} w_y \cdot \phi(x)$$

Generative vs Discriminative

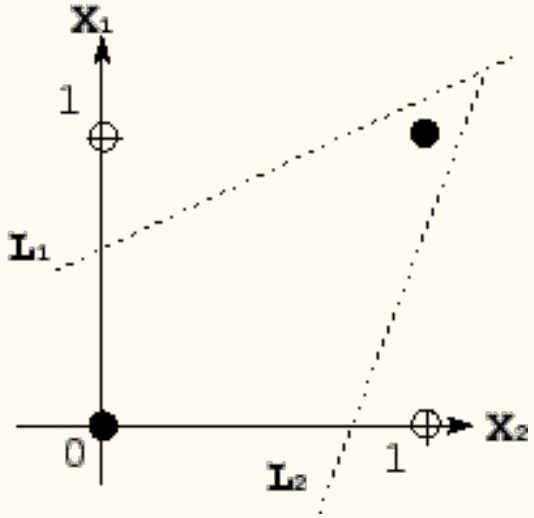
Which one would you choose?

- If we do not have a lot of training data, generative can avoid overfitting (must learn to generate the data too)
- If a lot of features are likely to matter and not sure about their correlations, discriminative is more intuitive (no need to generate the data)
- Naive Bayes is trivial to train
- Logistic regression is the standard at this point for linear classifiers
    - Linear support vector machines are good too (but not probabilistic)

Limitations of linear classifiers

X_1	X_2	Y
0	0	0
0	1	1
1	0	1
1	1	0

$Y = X_1 \oplus X_2$



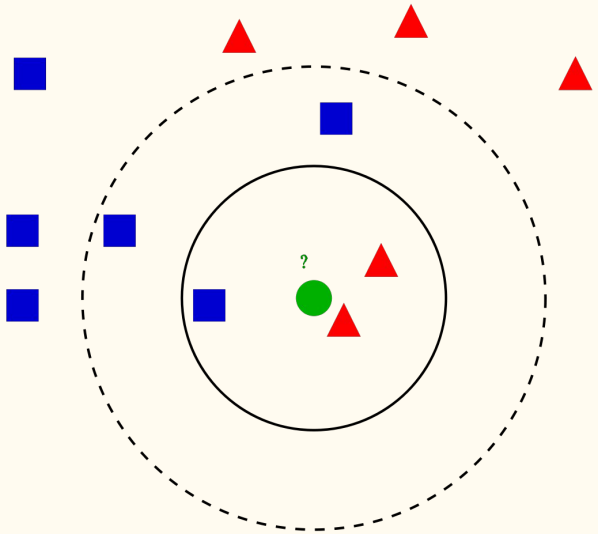
They assume that our data as represented by $\varphi(x)$ is linearly separable. It is easy construct datasets that are not.

http://www.ece.utep.edu/research/webfuzzy/docs/kk-thesis/kk-thesis-html/node19.html

Alternatives?

- Engineer better features
- K-nearest neighbors
- Kernel methods
- Neural networks

K-nearest neighbors



- Assumes a distance metric (can be learned)
- No training, just memorize the training data
- Classify according to the nearest neighbor(s)

https://en.wikipedia.org/wiki/K-nearest_neighbors_algorithm

Advantages:

- Non-linear
- Non-parametric

Disadvantages:

- With large datasets can be computation/memory heavy
- No feature learning

Kernel methods

We can replace the weights with calculations involving the training instances:

Binary perceptron

$$\hat{y} = sign(w \cdot \phi(x))$$

α^i: the times (x^i, y^i) was misclassified in training

$$\hat{y} = sign \sum_{(x,y)^i \in \mathcal{D}} \alpha^i y^i (\phi(x^i) \cdot \phi(x))$$

Perceptron with (non-linear) kernels

$$\hat{y} = sign \sum_{(x,y)^i \in \mathcal{D}} \alpha^i y^i K(x^i, x)$$

Support vector machines also use kernels, but in addition they find the max margin separating hyperplane

Bibliography

- Jurafsky and Martin chapter 4 on logistic regression
- André Martins gave a 3hr lecture covering a lot of what we discussed at LxMLS
- Work on modelling the evidence ***P(x)*** by Nalisnick et al. (2019) with references on learning the evidence; TLDR: doesn't behave as needed, WIP
- Ng and Jordan (2002) on generative vs discriminative
- K-nearest neighbours are coming back? Generalization through memorization: Nearest neighbor language models applied to machine translation
- On interpreting probabilities (Baan et al. 2024)
- Historical note: until 2010 or so, logistic regression was referred to as maximum entropy or maxent