Introduction to Probability

Lecture 1: Conditional probabilities and Bayes' theorem

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

UNIVERSITY OF
CAMBRIDGE

Outline

Lecturers

Mateja Jamnik

Thomas Sauerwald

Course logistics

Rough syllabus:

- Introduction to probability: 1 lecture
- Discrete and continuous random variables: 6 lectures
- Moments and limit theorems: 3 lectures
- Applications/statistics: 2 lectures

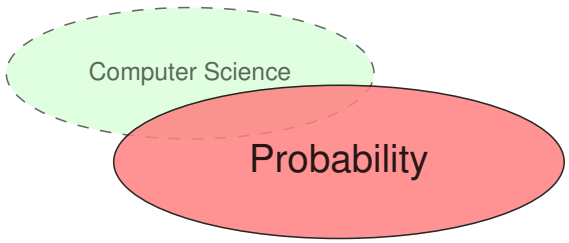
Recommended reading:

- **Ross, S.M. (2014). A First course in probability. Pearson (9th ed.).**
- **Dekking, F.M., et. al. (2005) A modern introduction to probability and statistics. Springer.**
- Bertsekas, D.P. & Tsitsiklis, J.N. (2008). Introduction to probability. Athena Scientific.
- Grimmett, G. & Welsh, D. (2014). Probability: an Introduction. Oxford University Press (2nd ed.).

Why probability?

- Gives us mathematical tools to deal with uncertain events.
- It is used everywhere, especially in applications of machine learning.
- Machine learning: use probability to compute predictions about and from data.
- Probability is not statistics:
  - Both about random processes.
  - Probability: logically self-contained, few rules for computing, one correct answer.
  - Statistics: messier, more art, get experimental data and try to draw probabilistic conclusions, no single correct answer.

Applications of probability



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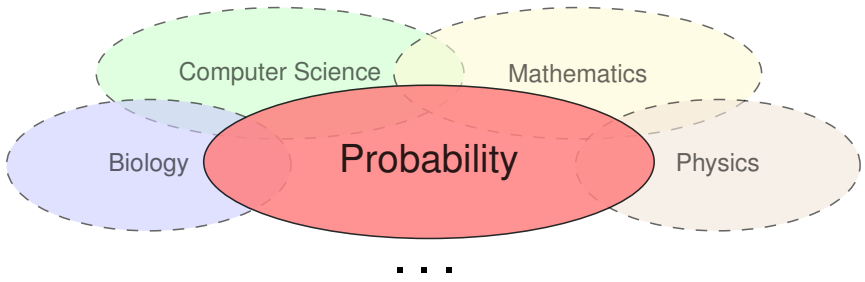
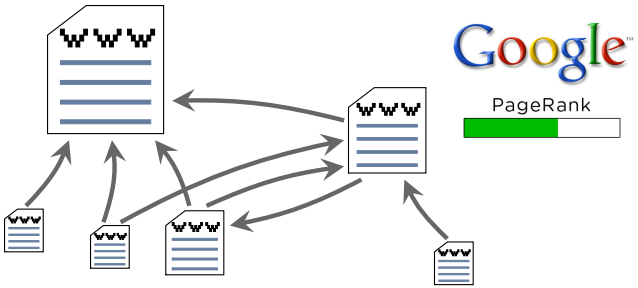
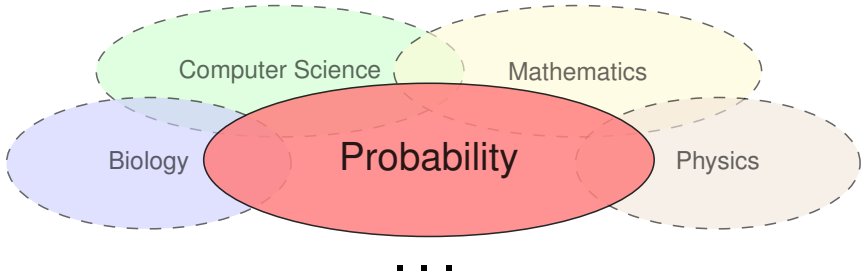


Applications of probability

Ranking Websites





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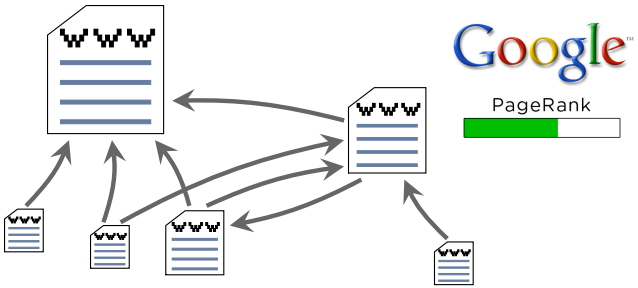

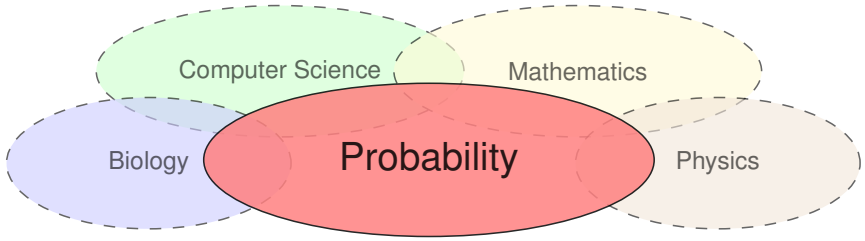

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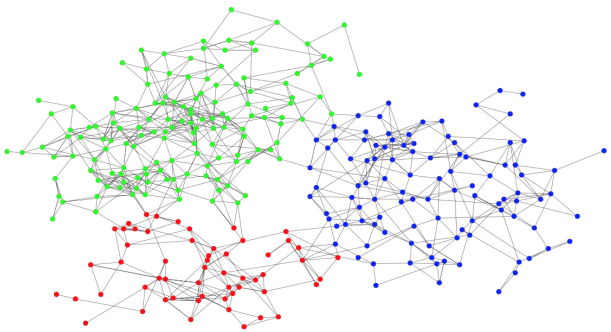
Data Mining



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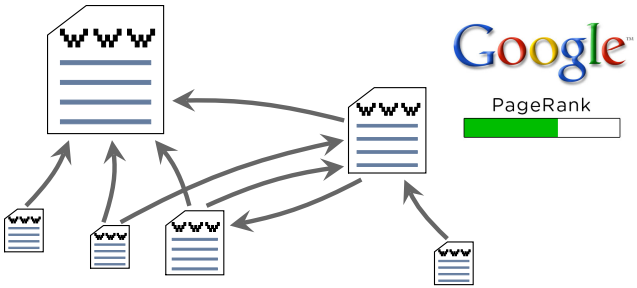

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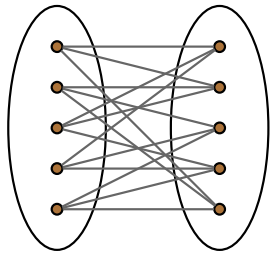

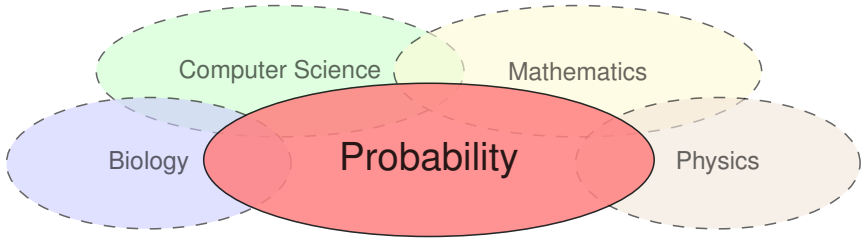

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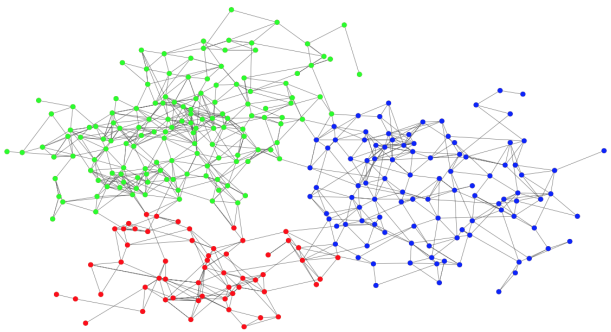

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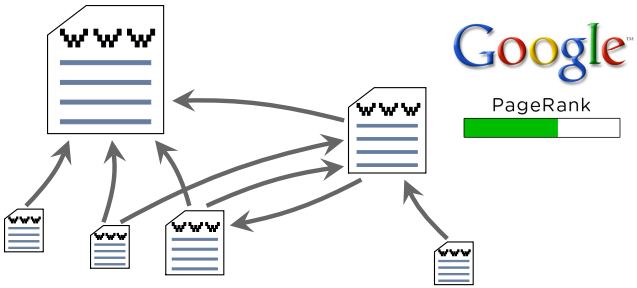

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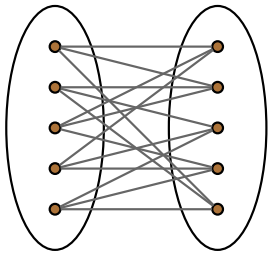

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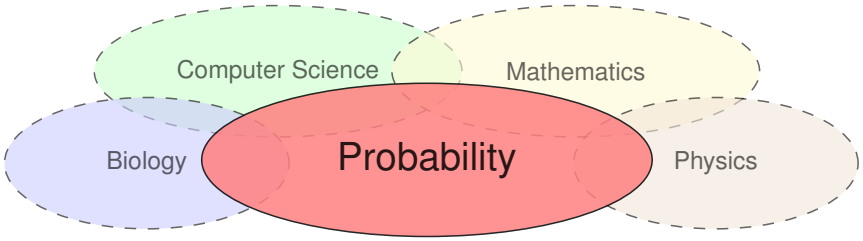

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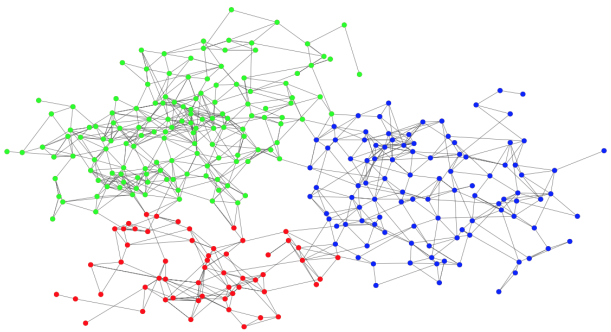

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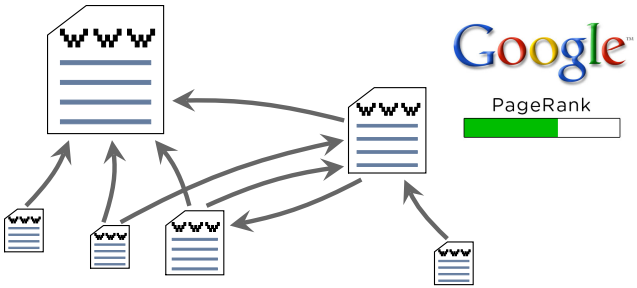

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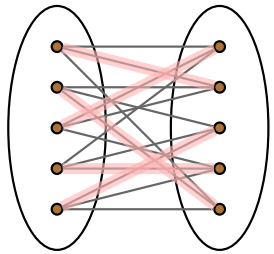

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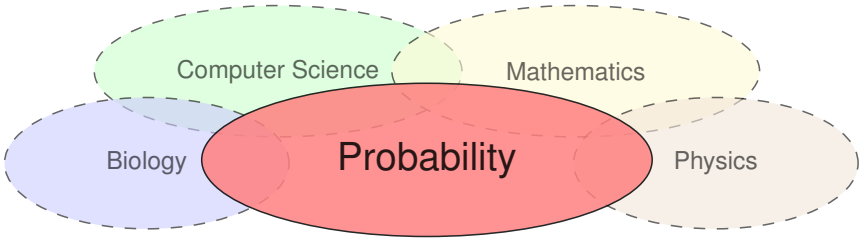

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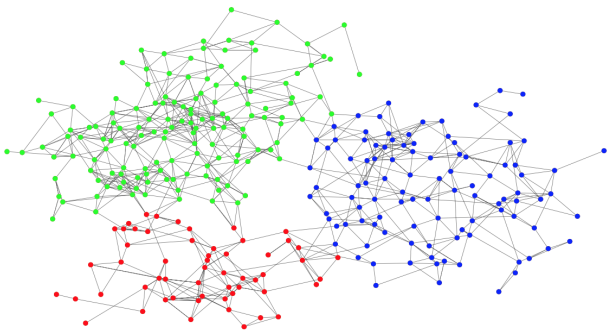

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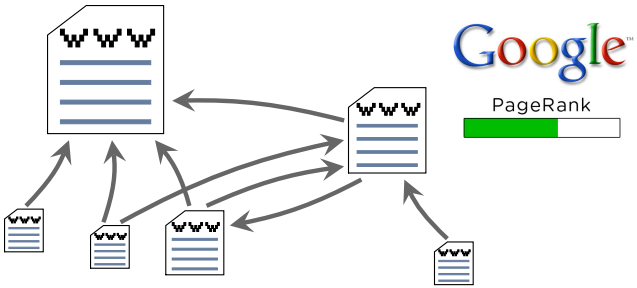

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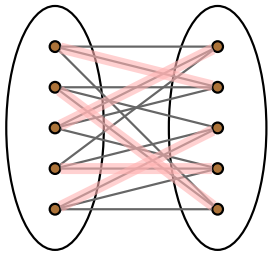

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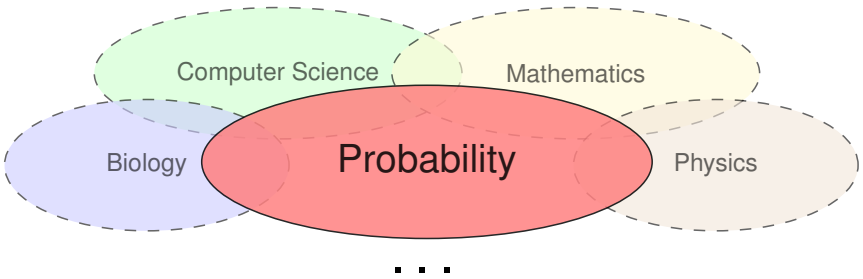

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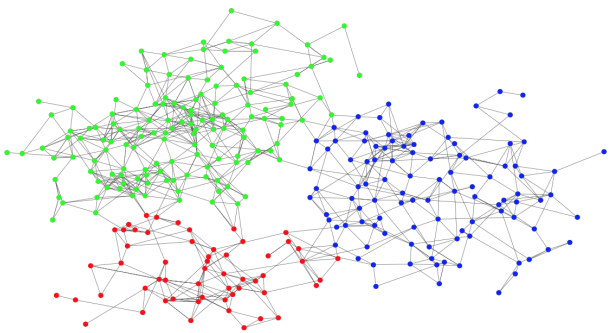
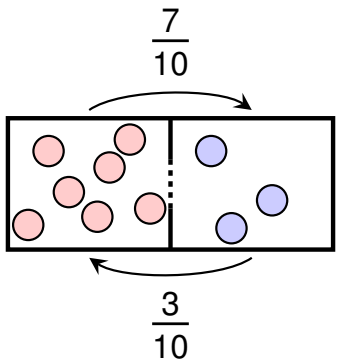


Particle Processes



Applications of probability

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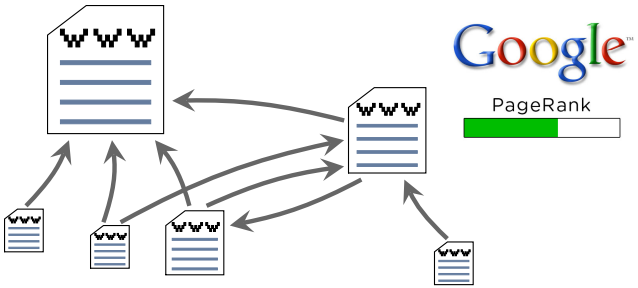

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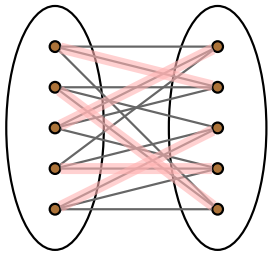

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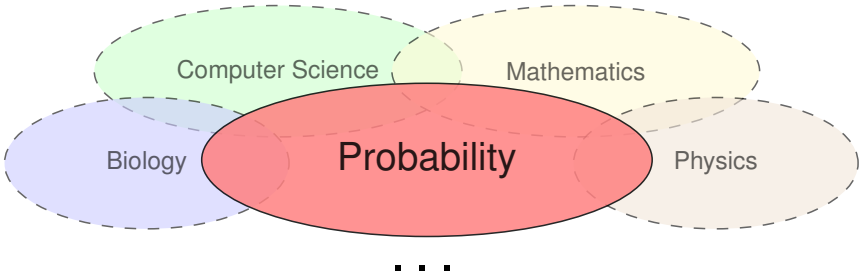

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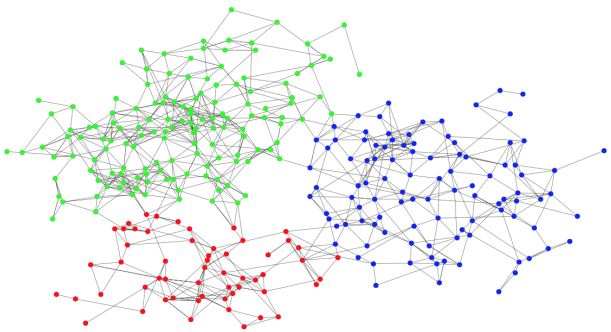
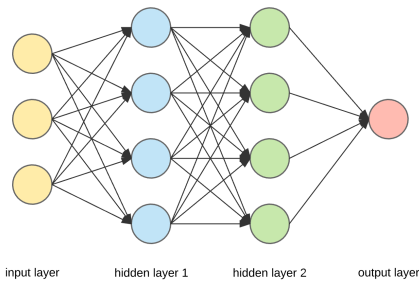
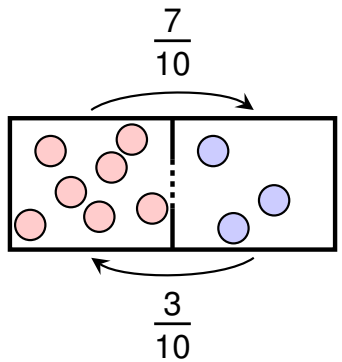
Data Mining



Deep Learning



Particle Processes



Applications of probability

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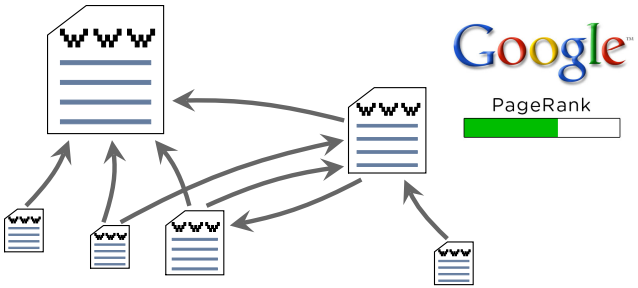

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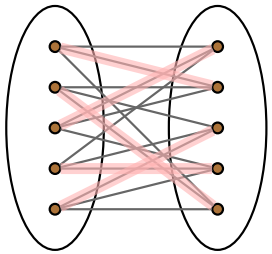

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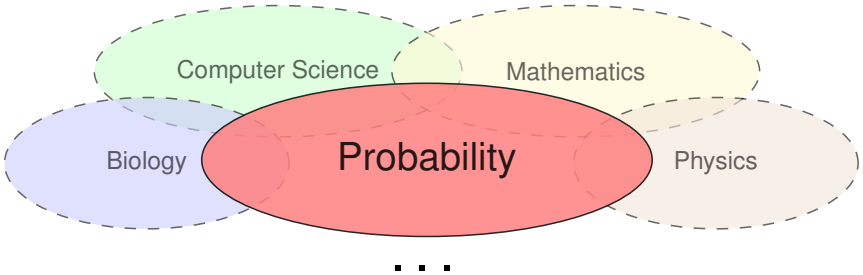
Finance



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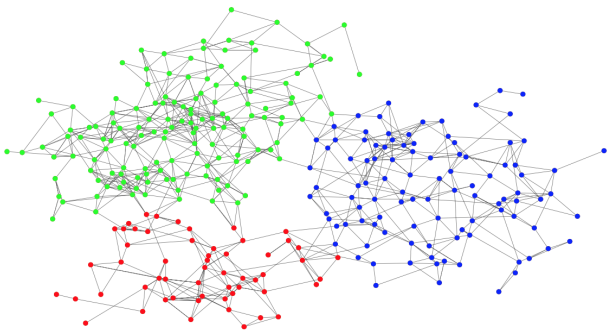
Data Mining



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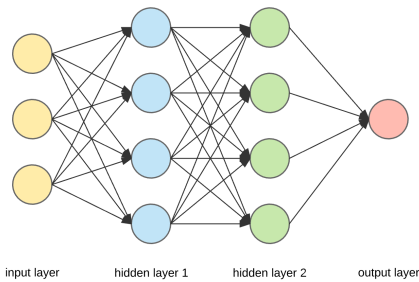

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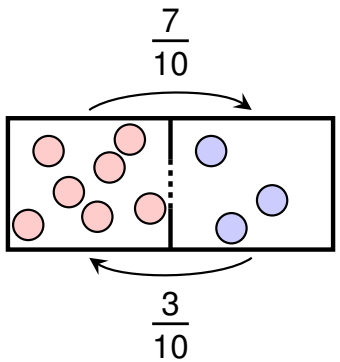

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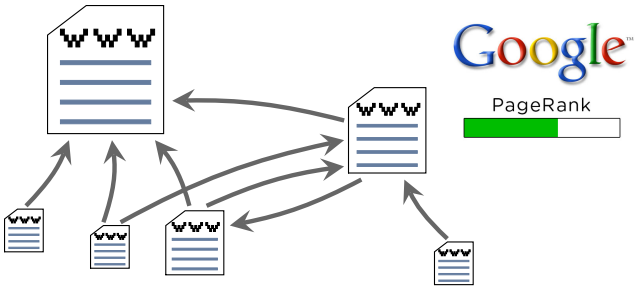

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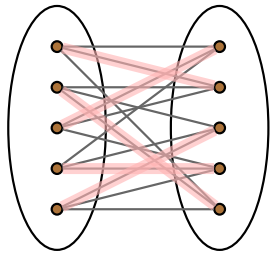

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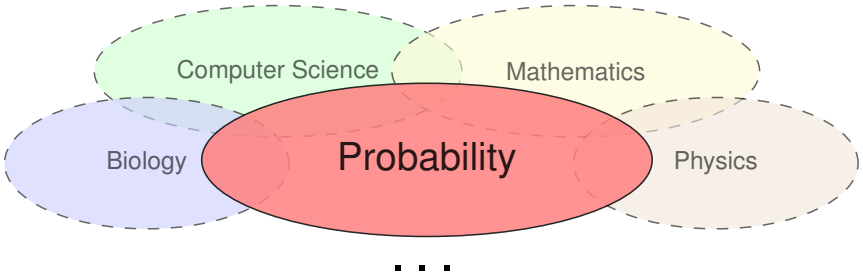
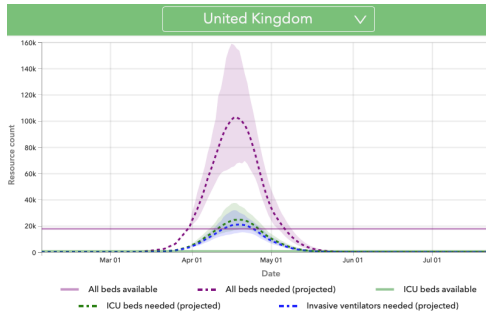
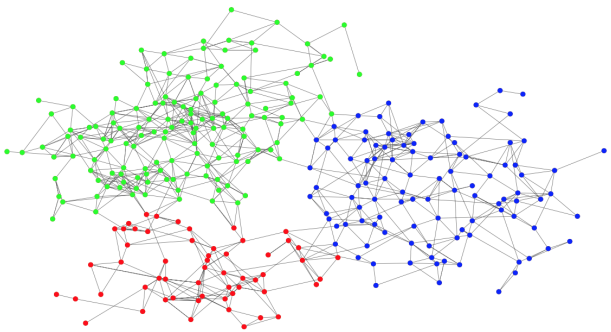


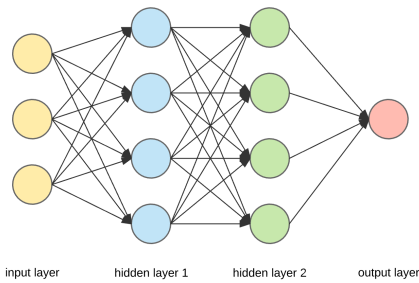
Medicine



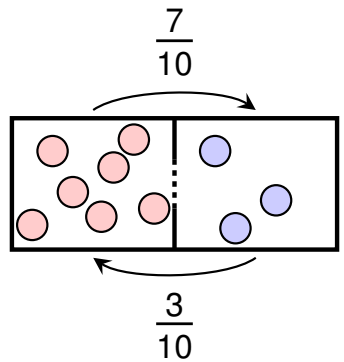
Data Mining



Deep Learning



Particle Processes



Prerequisite background

- Set theory
- Counting: product rule, sum rule, inclusion-exclusion
- Combinatorics: permutations
- Probability space: sample space, event space
- Axioms
- Union bound

- Look for revision material of above on the course website:

  `https://www.cl.cam.ac.uk/teaching/2526/IntroProb/`

Outline

Definition

Conditional probability

Consider an experiment with sample space S, and two events E and F. Then, the (conditional) probability of event E given F has occurred (denoted $\mathbf{P}[\, E|F \,]$) with $\mathbf{P}[\, F \,] > 0$ is defined by

$$\mathbf{P}[\, E|F \,] = \frac{\mathbf{P}[\, E \cap F \,]}{\mathbf{P}[\, F \,]} = \frac{\mathbf{P}[\, EF \,]}{\mathbf{P}[\, F \,]}$$

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Example

Example

Two dice are rolled yielding value D_1 and D_2. Let E be event that $D_1 + D_2 = 4$.

1. What is $\mathbf{P}[\, E \,]$?
2. Let event F be $D_1 = 2$. What is $\mathbf{P}[\, E | F \,]$?

Answer

Rules revisited

Chain rule

Rearranging the definition of conditional probability gives us:

$$\mathbf{P}[\, EF \,] = \mathbf{P}[\, E|F \,]\,\mathbf{P}[\, F \,]$$

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Generalisation of the Chain rule:

Multiplication rule

$$\mathbf{P}[\, E_1 E_2 \cdots E_n \,] = \mathbf{P}[\, E_1 \,]\,\mathbf{P}[\, E_2|E_1 \,]\,\mathbf{P}[\, E_3|E_2 E_1 \,] \cdots \mathbf{P}[\, E_n|E_1 \cdots E_{n-1} \,]$$

Example

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An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. What is the probability that each pile has exactly 1 ace?

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E_1 = *ace*♥ is in any one pile

E_2 = *ace*♥ and *ace*♠ are in different piles

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$$\mathbf{P}[\, E_1 E_2 E_3 E_4 \,] = \mathbf{P}[\, E_1 \,]\mathbf{P}[\, E_2 | E_1 \,]\mathbf{P}[\, E_3 | E_1 E_2 \,]\mathbf{P}[\, E_4 | E_1 E_2 E_3 \,]$$

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We have $\mathbf{P}[\, E_1 \,] = 1$. For rest we consider complement of next ace being in the same pile and thus have:

Outline

Logistics, motivation, background

Conditional probability

Bayes' Theorem

Independence

Law of total probability

The law of total probability (a.k.a. Partition theorem)

For events E and F where $\mathbf{P}[F] > 0$, then for any event E

$$\mathbf{P}[E] = \mathbf{P}[EF] + \mathbf{P}[EF^c] = \mathbf{P}[E|F]\mathbf{P}[F] + \mathbf{P}[E|F^c]\mathbf{P}[F^c]$$

In general, for disjoint events $F_1, F_2, \ldots F_n$ s.t. $F_1 \cup F_2 \cup \cdots \cup F_n = S$,

$$\mathbf{P}[E] = \sum_{i=1}^{n} \mathbf{P}[E|F_i]\mathbf{P}[F_i]$$

Intuition:

Want to know probability of E. There are two scenarios, F and F^c. If we know these and the probability of E conditioned on each scenario, we can compute the probability of E.

Lightbulb example

Example

There are 3 boxes each containing a different number of light bulbs. The first box has 10 bulbs of which 4 are dead, the second has 6 bulbs of which 1 is dead, and the third box has 8 bulbs of which 3 are dead. What is the probability of a dead bulb being selected when a bulb is chosen at random from one of the 3 boxes (each box has equal chance of being picked)?

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$$\mathbf{P}[\, E|F_1 \,] = \frac{4}{10}, \mathbf{P}[\, E|F_2 \,] = \frac{1}{6}, \mathbf{P}[\, E|F_3 \,] = \frac{3}{8}$$

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$$\mathbf{P}[\,E\,] = \sum_{i=1}^{n} \mathbf{P}[\,E|F_i\,]\mathbf{P}[\,F_i\,] = \frac{4}{10}\frac{1}{3} + \frac{1}{6}\frac{1}{3} + \frac{3}{8}\frac{1}{3} = \frac{113}{360} \approx 0.31$$

Bayes' theorem

How many spam emails contain the word "Dear"?

$$\mathbf{P}[\, E|F \,] = \mathbf{P}[\, \text{"Dear"}|\text{spam} \,]$$

But how about what is the probability that an email containing "Dear" is spam?

$$\mathbf{P}[\, F|E \,] = \mathbf{P}[\, \text{spam}|\text{"Dear"} \,]$$

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Bayes' theorem

For any events E and F where $\mathbf{P}[\, E \,] > 0$ and $\mathbf{P}[\, F \,] > 0$,

$$\mathbf{P}[\, F|E \,] = \frac{\mathbf{P}[\, E|F \,]\mathbf{P}[\, F \,]}{\mathbf{P}[\, E \,]}$$

and in expanded form,

$$\mathbf{P}[\, F|E \,] = \frac{\mathbf{P}[\, E|F \,]\mathbf{P}[\, F \,]}{\mathbf{P}[\, E|F \,]\mathbf{P}[\, F \,] + \mathbf{P}[\, E|F^c \,]\mathbf{P}[\, F^c \,]} = \frac{\mathbf{P}[\, E|F \,]\mathbf{P}[\, F \,]}{\sum_{i=1}^{n} \mathbf{P}[\, E|F_i \,]\mathbf{P}[\, F_i \,]}$$

using the Law of Total Probability. Note that all events F_i must be mutually exclusive (non-overlapping) and exhaustive (their union is the complete sample space) .

Example – Do it at home

Example

60% of all email in 2022 is spam. 20% of spam contains the word "Dear". 1% of non-spam contains the word "Dear". What is the probability that an email is spam given it contains the word "Dear"?

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Answer

- Let event E ="Dear", event F = spam.

Bayes' terminology

$$\mathbf{P}[\,F|E\,] \quad = \quad \frac{\mathbf{P}[\,E|F\,] \quad \cdot \quad \mathbf{P}[\,F\,]}{\mathbf{P}[\,E\,]}$$

F: hypothesis, E: evidence

Bayes' terminology

posterior

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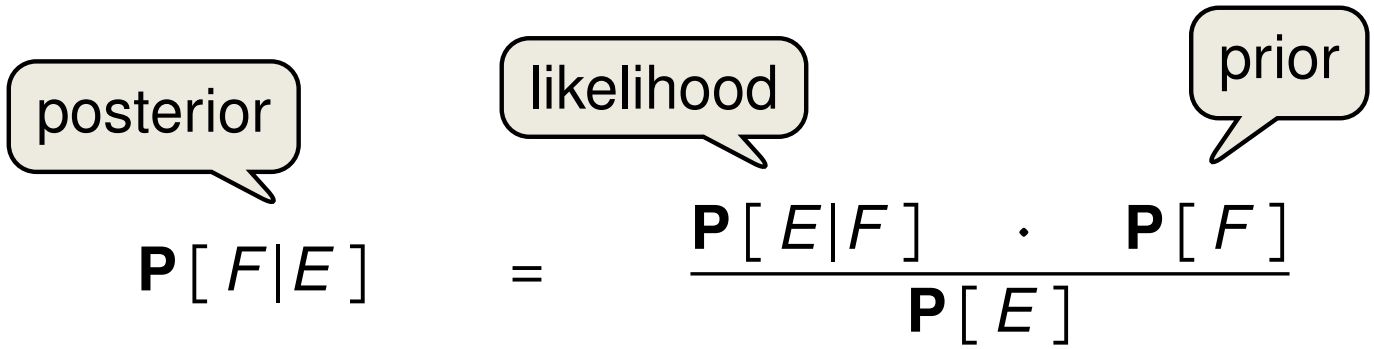

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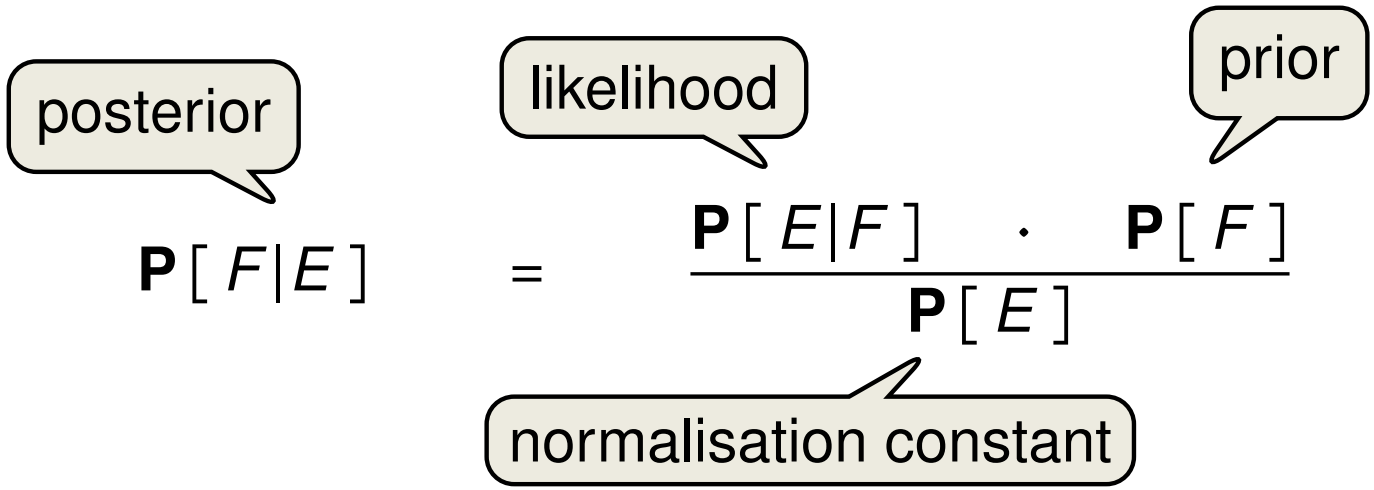

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$\mathbf{P}[\,E\,]$: calculated by making sure that probabilities of all outcomes sum to 1 (they are "normalised")

Confusion matrix (error matrix)

Used in classification tasks for predicting output error.

		True condition	
	Total population	Condition positive F	Condition negative F^c
Predicted condition	Predicted condition positive E	True positive $\mathbf{P}\left[E \mid F \right]$	False positive $\mathbf{P}\left[E \mid F^c \right]$
	Predicted condition negative E^c	False negative $\mathbf{P}\left[E^c \mid F \right]$	True negative $\mathbf{P}\left[E^c \mid F^c \right]$

Medical testing example

Example

- A test is 98% effective at detecting the disease COVID-19 ("true positive").
- The test has a "false positive" rate of 1%.
- 0.5% of the population has COVID-19.
- What is the likelihood you have COVID-19 if you test positive?

Answer

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Answer

- Let E: test positive, F: actually have COVID-19.
- Need to find $\mathbf{P}[\, F \,|\, E \,]$.

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	F yes disease	F^c no disease
E test+	True positive $\mathbf{P}[E\|F] = 0.98$	False positive $\mathbf{P}[E\|F^c] = 0.01$
E^c test-	False negative $\mathbf{P}[E^c\|F] = 0.02$	True negative $\mathbf{P}[E^c\|F^c] = 0.99$

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$$\mathbf{P}[F|E^c] = \frac{\mathbf{P}[E^c|F]\mathbf{P}[F]}{\mathbf{P}[E^c|F]\mathbf{P}[F] + \mathbf{P}[E^c|F^c]\mathbf{P}[F^c]} \approx 0.0001$$

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- We update our beliefs with Bayes' theorem:
  I have 0.5% chance of having COVID-19. I take the test:
  - Test is positive: I now have 33% chance of having COVID-19.
  - Test is negative: I now have 0.01% chance of having COVID-19.
- So it makes sense to take the test.

Outline

Independent events

Independence

Two events E and F are independent if and only if

$$\mathbf{P}[\, EF \,] = \mathbf{P}[\, E \,]\,\mathbf{P}[\, F \,]$$

Otherwise, they are called dependent events.

In general, n events $E_1, E_2, \ldots, E_n$ are mutually independent if for every subset of these events with r elements (where $r \leq n$) it holds that

$$\mathbf{P}[\, E_a E_b \cdots E_r \,] = \mathbf{P}[\, E_a \,]\,\mathbf{P}[\, E_b \,] \cdots \mathbf{P}[\, E_r \,]$$

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$$\mathbf{P}[\,E_a E_b \cdots E_r\,] = \mathbf{P}[\,E_a\,]\mathbf{P}[\,E_b\,]\cdots\mathbf{P}[\,E_r\,]$$

Therefore for 3 events E, F, G to be independent, we must have

$$\begin{aligned}
\mathbf{P}[\,EFG\,] &= \mathbf{P}[\,E\,]\mathbf{P}[\,F\,]\mathbf{P}[\,G\,] \\
\mathbf{P}[\,EF\,] &= \mathbf{P}[\,E\,]\mathbf{P}[\,F\,] \\
\mathbf{P}[\,EG\,] &= \mathbf{P}[\,E\,]\mathbf{P}[\,G\,] \\
\mathbf{P}[\,FG\,] &= \mathbf{P}[\,F\,]\mathbf{P}[\,G\,]
\end{aligned}$$

Independence of complement

Notice an equivalent definition for independent events E and F ($\mathbf{P}[\,F\,] > 0$)

$$\mathbf{P}[\,E|F\,] = \mathbf{P}[\,E\,]$$

Proof:

Independence of complement

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If events E and F are independent, then E and F^c are independent:

$$\mathbf{P}[\,EF^c\,] = \mathbf{P}[\,E\,]\mathbf{P}[\,F^c\,]$$

Proof:

Example

Example

Each roll of a die is an independent trial. We have two rolls of D_1 and D_2. Let event $E : D_1 = 1$, $F : D_2 = 6$ and event $G : D_1 + D_2 = 7$ (thus $G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$).

1. Are E and F independent?
2. Are E and G independent?
3. Are E, F, G independent?

Answer

Conditional independence – Convince yourself at home

Conditional independence

Two events E and F are called conditionally independent given a third event G if

$$\mathbf{P}[\,EF|G\,] = \mathbf{P}[\,E|G\,]\,\mathbf{P}[\,F|G\,]$$

Or equivalently,

$$\mathbf{P}[\,E|FG\,] = \mathbf{P}[\,E|G\,]$$

Notice that:

- Dependent events can become conditionally independent.
- Independent events can become conditionally dependent.
- Knowing when conditioning breaks or creates independence is a big part of building complex probabilistic models.

Example revisited

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1. Are E and F independent?
2. Are E and F independent given G?

Answer

Summary of conditional probability

Conditioning on event G:

Name of rule	Original rule	Conditional rule
1st axiom of probability	$0 \leq \mathbf{P}[\,E\,] \leq 1$	$0 \leq \mathbf{P}[\,E\,|\,G\,] \leq 1$
Complement	$\mathbf{P}[\,E\,] = 1 - \mathbf{P}[\,E^c\,]$	$\mathbf{P}[\,E\,|\,G\,] = 1 - \mathbf{P}[\,E^c\,|\,G\,]$
Chain rule	$\mathbf{P}[\,EF\,] = \mathbf{P}[\,E\,|\,F\,]\mathbf{P}[\,F\,]$	$\mathbf{P}[\,EF\,|\,G\,] = \mathbf{P}[\,E\,|\,FG\,]\mathbf{P}[\,F\,|\,G\,]$
Bayes' theorem	$\mathbf{P}[\,F\,|\,E\,] = \dfrac{\mathbf{P}[\,E\,|\,F\,]\mathbf{P}[\,F\,]}{\mathbf{P}[\,E\,]}$	$\mathbf{P}[\,F\,|\,EG\,] = \dfrac{\mathbf{P}[\,E\,|\,FG\,]\mathbf{P}[\,F\,|\,G\,]}{\mathbf{P}[\,E\,|\,G\,]}$

Introduction to Probability

Lecture 2: Random variables, probability mass function, expectation

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

Outline

What is a random variable?

Random variable

A random variable X is a function from the sample space to the real numbers.

- We can interpret X as a quantity whose value depends on the outcome of an experiment (some probabilistic process).
  - Roll two dice, X: sum of dice
  - Toss 3 coins, X: number of heads
  - Give a student a test, X: score
  - Stock market index

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  - Roll two dice, X: sum of dice
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  - Stock market index
- Or can think of X as a variable in a programming language that takes on values, has a type, and has a domain over which it is applicable.

What is a random variable?

Random variable

A random variable X is a function from the sample space to the real numbers.

- We can interpret X as a quantity whose value depends on the outcome of an experiment (some probabilistic process).
  - Roll two dice, X: sum of dice
  - Toss 3 coins, X: number of heads
  - Give a student a test, X: score
  - Stock market index
- Or can think of X as a variable in a programming language that takes on values, has a type, and has a domain over which it is applicable.
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- Many different types of RV: indicator, binary, choice, Bernoulli, etc.
- Random variable can be discrete or continuous:
  - X has finitely many possible values: discrete.
  - X has every integer as a possible value: discrete.
  - X amount of time it takes to finish a race: continuous (possible value: $\{t : 0 \leq t < \infty\} = [0, \infty)$).

Examples of random variables

Example

We toss 3 fair coins. Let a random variable X be the total number of heads on the 3 coins. What are the probabilities of X taking on the following values: $X = 0, X = 1, X = 2, X = 3, X \geq 4$?

Answer

Random variables are NOT events

random variables ≠ events

Tossing 3 fair coins example

$X = x$	$\mathbf{P}[X = x]$	Set of outcomes	Possible event E
$X = 0$	$\frac{1}{8}$	$\{(T, T, T)\}$	Toss 0 heads
$X = 1$	$\frac{3}{8}$	$\{(H, T, T), (T, H, T), (T, T, H)\}$	Toss exactly 1 head
$X = 2$	$\frac{3}{8}$	$\{(H, H, T), (T, H, H), (H, T, H)\}$	Event where $X = 2$
			Toss exactly 2 heads
$X = 3$	$\frac{1}{8}$	$\{(H, H, H)\}$	Toss 0 tails
$X \geq 4$	0	$\{\}$	Toss 4 or more heads

We can define events by condition of the value of a random variable (RV takes on values that satisfy a numerical test).

Another example

Example

Tossing a coin has the probability p that it comes up heads. Toss a coin 5 times. Let X: the number of heads in 5 tosses. What is the range of X (i.e., what are the values that X can take on with non-zero probability)? What is $\mathbf{P}[\, X = k \,]$ where k is in the range of X?

Answer

- Notice that each coin toss is an independent trial.

Outline

Probability mass function definition (PMF)

Discrete random variable

A random variable X is discrete if its range has countably many values

$$X = x \text{ where } x \in \{x_1, x_2, x_3, \ldots\}$$

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Recall that probabilities must sum to 1: $\sum_{i=1}^{\infty} p(a_i) = 1$.

Example for a single die

- Let X be a RV representing a single die roll.
- Range of X : $\{1, 2, 3, 4, 5, 6\}$, thus X is a discrete RV.

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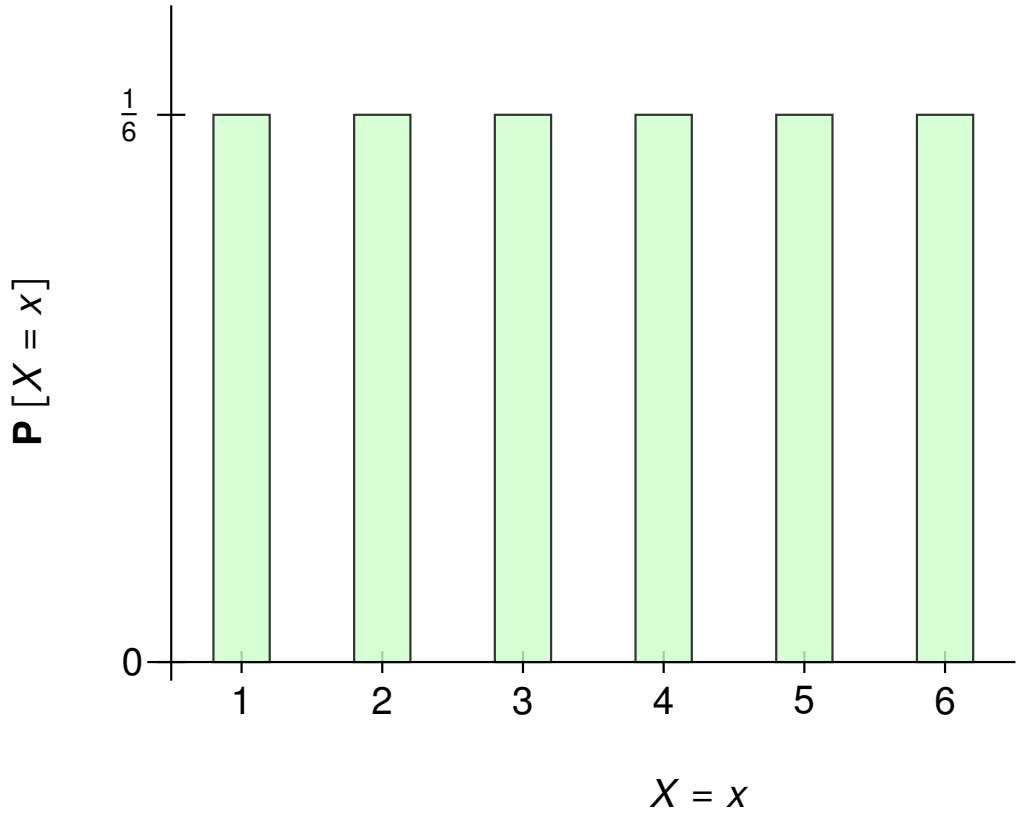

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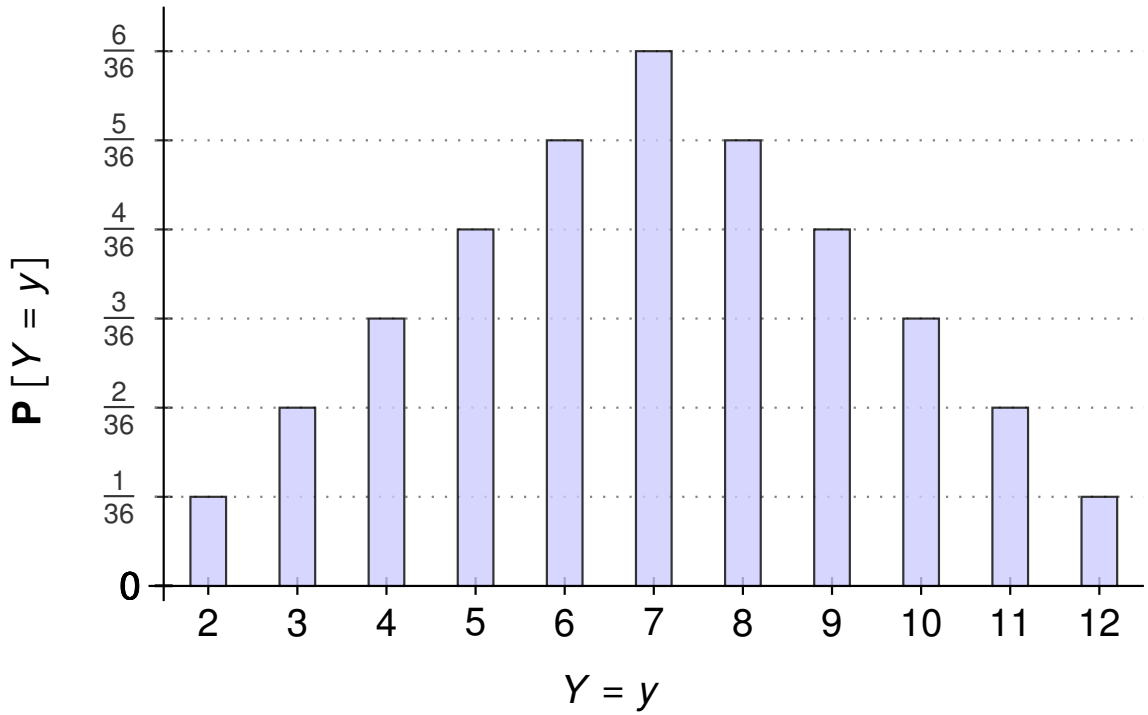

Properties of PMF

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5. For continuous RVs, these sums are replaced by integrals.

Outline

Cumulative distribution function definition (CDF)

Another useful way to analyse probabilities.

Cumulative distribution function

The cumulative distribution function (CDF) of a random variable X is defined as

$$F(a) = F_X(a) = \mathbf{P}[\, X \leq a \,] \text{ where } -\infty < a < \infty$$

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For a **discrete** random variable X, the CDF is

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Note that for a discrete RV the CDF is a step function, i.e., the value of F is constant in the intervals (x_{i-1}, x_i) and then takes a step of size $p(x_i)$ at x_i.

Example

- Let the PMF for X be given by $p(1) = \frac{1}{4}, p(2) = \frac{1}{2}, p(3) = \frac{1}{8}, p(4) = \frac{1}{8}$.

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- Then CDF is:

$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \leq a < 2 \\ \frac{3}{4} & 2 \leq a < 3 \\ \frac{7}{8} & 3 \leq a < 4 \\ 1 & 4 \leq a \end{cases}$$

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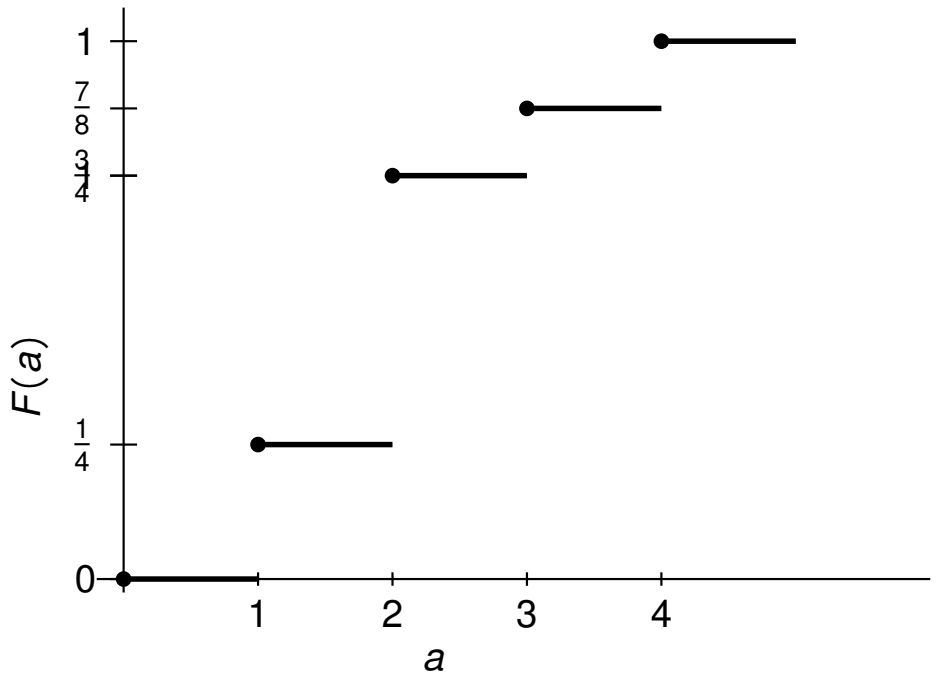
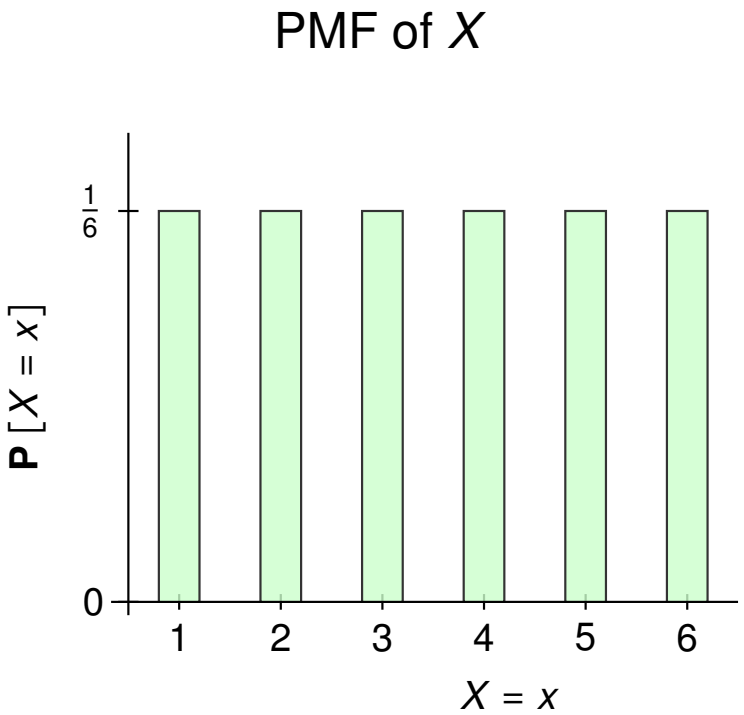
- Graphical depiction of function:

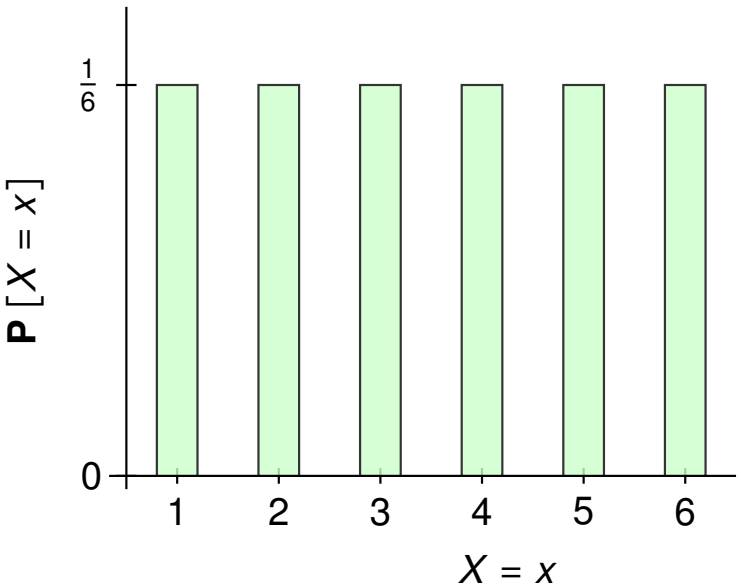


Example for a single die



Example for a single die

PMF of X



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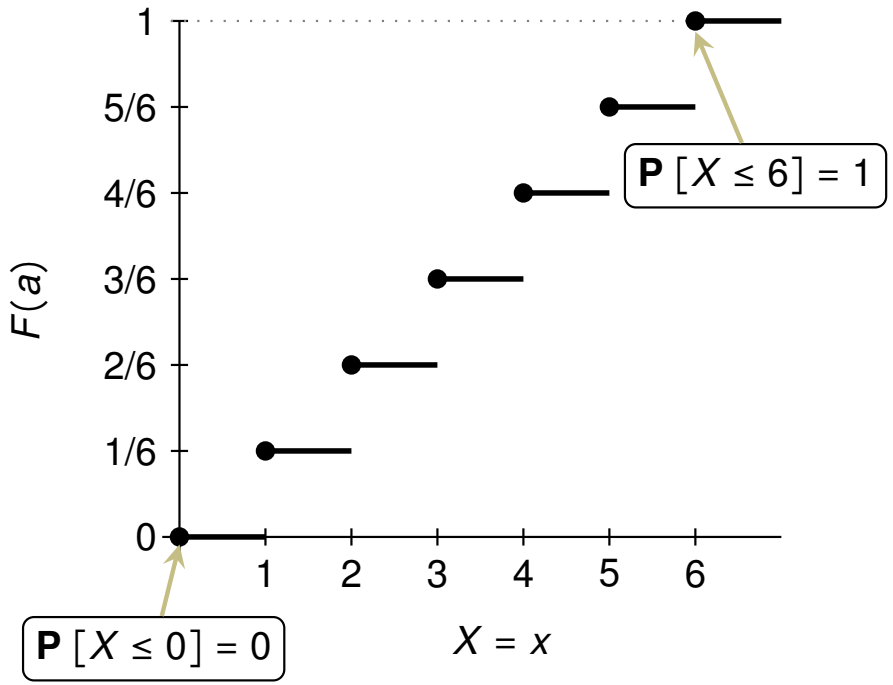

Properties of CDF

1. $0 \leq F(x) \leq 1$ for all x
2. $\lim_{x \to -\infty} F(x) = 0$
3. $\lim_{x \to \infty} F(x) = 1$
4. $F(x)$ is a non-decreasing function of x (if $x_1 < x_2$ then $F(x_1) \leq F(x_2)$)

Outline

Expectation - expected value

Expectation

The expectation of a discrete random variable X is defined as

$$\mathbf{E}[X] = \sum_{x:\mathbf{P}[x]>0} x\mathbf{P}[x]$$

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The expectation of a discrete random variable X is defined as

$$\mathbf{E}[X] = \sum_{x:\mathbf{P}[x]>0} x\mathbf{P}[x]$$

- Expectation is the average value of the random variable over many repetitions of the experiment it represents.
- It is the sum over all values of $X = x$ that have non-zero probability.
- AKA: mean, expected value, weighted average, centre of mass, first moment.

Example of a die roll

What is the expected value of a 6-sided die roll (i.e., what is the average value of a die roll)?

1. Define random variables:

$$X = \text{RV for value of roll}$$

$$\mathbf{P}[\, X = x \,] = \begin{cases} \frac{1}{6} & x \in \{1, \ldots, 6\} \\ 0 & \text{otherwise} \end{cases}$$

2. Solve:

Example of school classes

Example

A school has 3 classes with 5, 10 and 150 students. What is the average class size?

Answer

Interpretation 1: Randomly choose a class with equal probability. Thus, X = size of chosen class

Interpretation 2: Randomly choose a student with equal probability. Thus, Y = size of chosen class

This is a general phenomenon: it occurs because the more students are in a class, the more likely it is that a randomly chosen student would be in that class. As a result, bigger classes are given more weight than smaller classes.

Example of Roulette Version 1

Example

A roulette wheel has 36 places numbered from 1 to 36. In addition, 18 of them are coloured red and the other 18 are coloured black. A ball is thrown to take one of 36 places. A gambler can bet:

- on the colour of the place that the ball takes. A correct, either red or black, place wins them a 1 to 1 ratio payout;
- on the number of the place that the ball takes. A correct number wins them a 35 to 1 ratio payout.

What is the expected value if a gambler bets on

1. the colour of the place in the roulette;
2. the number of the place in the roulette that the ball will fall.

Are the two different betting games fair?

Answer

Example of Roulette Version 1 Cont.

Example

What is the expected value if a gambler bets on

1. the colour of the place in the roulette;
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Are the two different betting games fair?

Answer

1. Let E_X : bet on colour.

2. Let E_Y : bet on number.

Example of Roulette Version 2

Example

Change the game to add two green places, 0 and 00. Now there are a total of 38 places. Payouts are the same as before. What are the expected values now?

Answer

1. Let E_X : bet on red colour.

2. Let E_Y : bet on number 10.

Introduction to Probability

Lecture 3: Expectation properties, variance, discrete distributions

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

Outline

Properties of expectation: linearity

Linearity of expectation

Expectations preserve linearity: if a and b are constants, then

$$\mathbf{E}[\, aX + b \,] = a\,\mathbf{E}[\, X \,] + b$$

Proof:

Properties of expectation: linearity

Linearity of expectation

Expectations preserve linearity: if a and b are constants, then

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Proof:

Example

Let the event be a roll of a 6-sided die, X its random variable, and Y another random variable where $Y = 3X + 1$. What are the expected values $\mathbf{E}[\, X \,]$ and $\mathbf{E}[\, Y \,]$?

Answer

Properties of expectation: additivity

Additivity of expectation

Expectation of a sum is equal to the sum of expectations: if X and Y are any random variables on the same sample space then

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

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Example

Let the events be rolls of 2 dice, and X the random variable for the roll of die 1, and Y for the roll of die 2. What is the expected value of the sum of the rolls of the two dice?

Answer

Properties of expectation: LOTUS

Law of the unconscious statistician (LOTUS)

Let X be a random variable, and Y another random variable that is a function of X, so $Y = g(X)$. Let $p(x)$ be a PMF of X. Then

$$\mathbf{E}[\,Y\,] = \mathbf{E}[\,g(X)\,] = \sum_{x:p(x)>0} g(x)p(x)$$

Note how now we no longer need to know PMF of Y.

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- LOTUS is also known as **expected value of a function of a random variable**.
- Note that the properties of expectation let you avoid defining difficult PMFs.
- Let X be a discrete RV, then:
  - $\mathbf{E}\left[\,X^2\,\right]$ is know as the second moment of X.
  - $\mathbf{E}\left[\,X^n\,\right]$ is know as the n^{th} moment of X.

Second moment example

Example

Let X be a discrete random variable that ranges over the values $\{-1, 0, 1\}$, and respective probabilities $\mathbf{P}[\, X = -1 \,] = 0.2$, $\mathbf{P}[\, X = 0 \,] = 0.5$ and $\mathbf{P}[\, X = 1 \,] = 0.3$. Let another random variable $Y = X^2$ (second moment). What is $\mathbf{E}[\, Y \,]$?

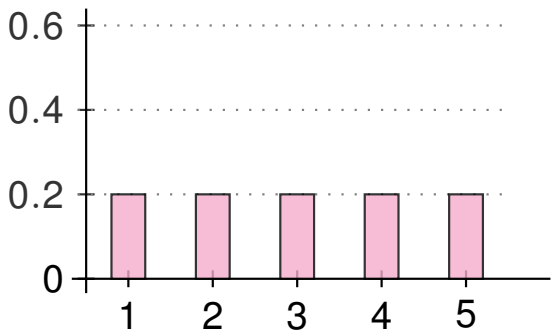
Answer

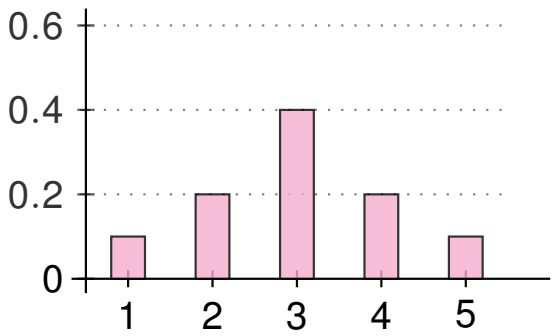
Note that $Y = g(X) = X^2$ and $\mathbf{E}[\, Y \,] = \mathbf{E}[\, g(X) \,] = \sum_{x:p(x)>0} g(x)p(x)$, thus

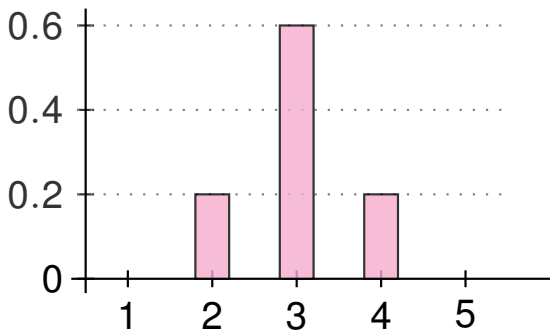
Outline

Spread in the distribution

Expectation is a useful statistic, but it does not give a detailed view of the PMF. Consider these 3 distributions (PMFs).







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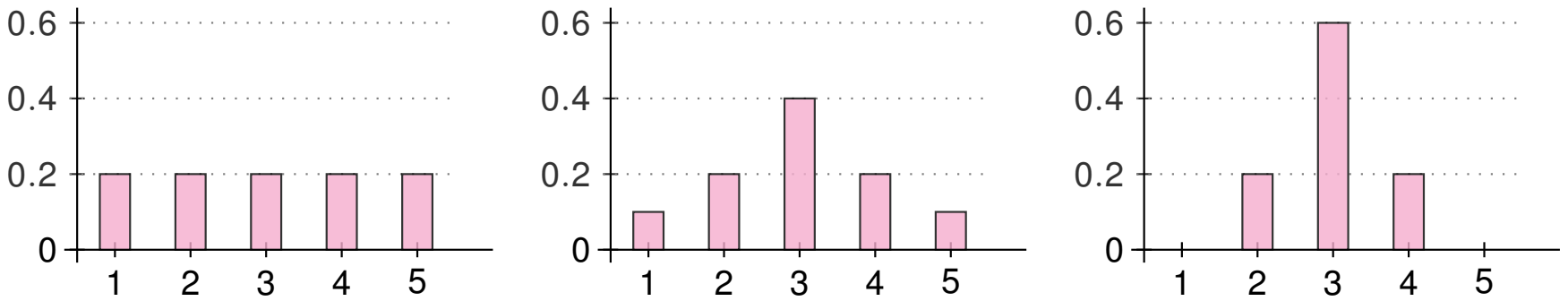

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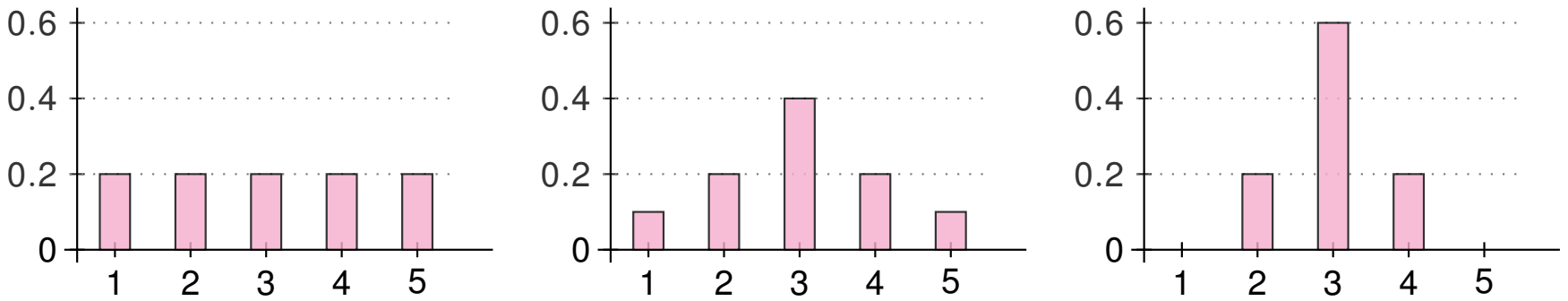

- Expectation is the same for all distributions: $\mathbf{E}[X] = 3$.
- First has the greatest spread, the third has the least spread.
- But the "spread" or "dispersion" of X in the distribution is very different!
- Variance, $\mathbf{V}[X]$ defines a formal quantification of "spread".
- Several ways to quantify: it uses average square distance from the mean.

Definition of variance

Variance

The variance of a discrete random variable X with expected value (mean) μ is:

$$\mathbf{V}[X] = \mathbf{E}\left[(X - \mu)^2\right]$$

When computing the variance, we often use a different form of the same equation:

$$\mathbf{V}[X] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2$$

Proof:

Note:

- $\mathbf{V}[X] \geq 0$
- AKA: Second central moment, or square of the standard deviation

Example with a die roll

Example

Let X be the value on one roll of a 6-sided fair die. Recall that $\mathbf{E}[X] = \frac{7}{2} = 3.5$. What is $\mathbf{V}[X]$?

Answer

Using $\mathbf{V}[X] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2$:

Using $\mathbf{V}[X] = \mathbf{E}\left[(X-\mu)^2\right] = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right]$:

Example of spread

Example

Let X, Y and Z be discrete random variables with the range $X : \{10\}$ and probability 1, and $Y : \{11, 9\}$ and $Z : \{110, -90\}$ with equal probabilities $\frac{1}{2}$. Compute expectation and variance for X, Y and Z.

Answer

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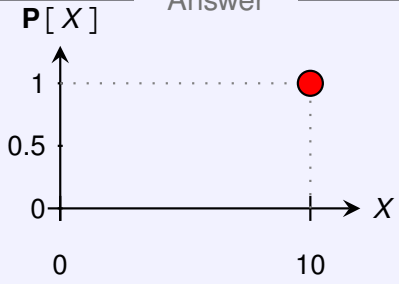
Answer

a) $\mathbf{E}[X] = \sum_x xp(x) = 10 \cdot 1 = 10$

$$\mathbf{V}[X] = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = \mathbf{E}\left[(X - 10)^2\right]$$
$$= (X - 10)^2 p(x) = 0^2 \cdot 1 = 0$$



Example of spread

Example

Let X, Y and Z be discrete random variables with the range $X : \{10\}$ and probability 1, and $Y : \{11, 9\}$ and $Z : \{110, -90\}$ with equal probabilities $\frac{1}{2}$. Compute expectation and variance for X, Y and Z.

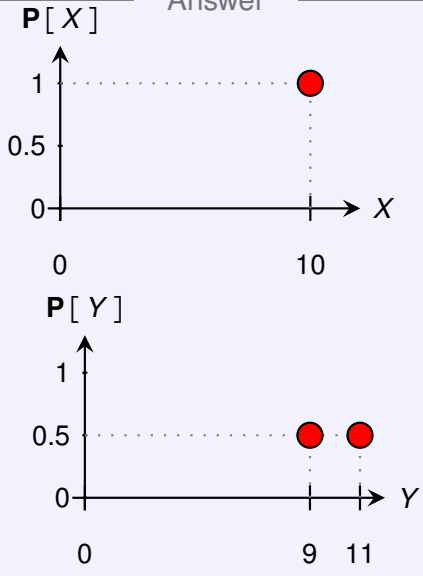
Answer

a) $\mathbf{E}[X] = \sum_x xp(x) = 10 \cdot 1 = 10$

$$\mathbf{V}[X] = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = \mathbf{E}\left[(X - 10)^2\right]$$
$$= (X - 10)^2 p(x) = 0^2 \cdot 1 = 0$$

b) $\mathbf{E}[Y] = (11)(0.5) + (9)(0.5) = 10$

$$\mathbf{V}[Y] = \mathbf{E}\left[(Y - \mathbf{E}[Y])^2\right] = \mathbf{E}\left[(Y - 10)^2\right]$$
$$= (11 - 10)^2(0.5) + (9 - 10)^2(0.5) = 1$$



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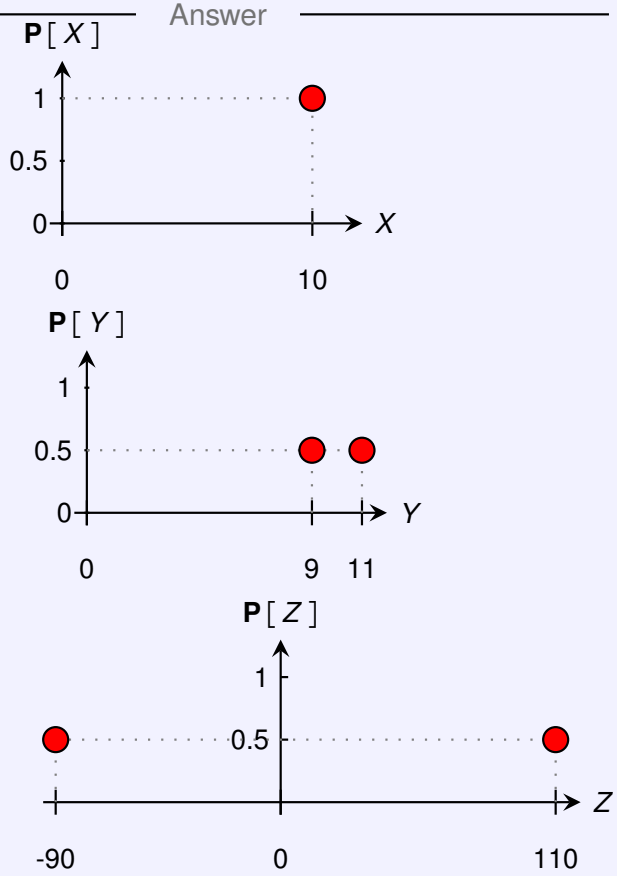
$= (110-10)^2(0.5) + (-90-10)^2(0.5)$

$= 100^2 = 10000$



Standard deviation

- Standard deviation is a kind of average distance of a sample of the mean, i.e., a root mean square (RMS) average.
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Standard deviation

Standard deviation is defined as a square root of variance:

$$\mathbf{SD}[X] = \sqrt{\mathbf{V}[X]}$$

Note:

- $\mathbf{E}[X]$ and $\mathbf{V}[X]$ are real numbers, not RVs.
- $\mathbf{V}[X]$ is expressed in units of the values in the range of X^2.
- $\mathbf{SD}[X]$ is expressed in units of the values in the range of X.
- For the spread example above: $\mathbf{SD}[X] = 0, \mathbf{SD}[Y] = 1, \mathbf{SD}[Z] = 100$.

Properties of variance

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- **Property 2:** variance is not linear: $\mathbf{V}[aX + b] = a^2\mathbf{V}[X]$

  Proof:

$$
\begin{aligned}
\mathbf{V}[(aX + b] &= \mathbf{E}\left[(aX + b)^2\right] - (\mathbf{E}[aX + b])^2 \\
&= \mathbf{E}\left[a^2X^2 + 2abX + b^2\right] - (a\mathbf{E}[X] + b)^2 \\
&= a^2\mathbf{E}\left[X^2\right] + 2ab\mathbf{E}[X] + b^2 - (a^2(\mathbf{E}[X])^2 + 2ab\mathbf{E}[X] + b^2) \\
&= a^2\mathbf{E}\left[X^2\right] - (a^2(\mathbf{E}[X])^2) = a^2(\mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2) \\
&= a^2\mathbf{V}[X]
\end{aligned}
$$

Summary of expectation and variance for discrete RV

$$\mathbf{E}[X] = \sum_{x:\mathbf{P}[x]>0} x\mathbf{P}[x] = \sum_{x} xp(x)$$

Properties of Expectation

$$\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

$$\mathbf{E}[aX+b] = a\mathbf{E}[X] + b$$

$$\mathbf{E}[g(X)] = \sum_{x} g(x)p_X(x)$$

Properties of Variance

$$\mathbf{V}[X] = \mathbf{E}\left[(X-\mu)^2\right]$$

$$\mathbf{V}[X] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2$$

$$\mathbf{V}[aX+b] = a^2\mathbf{V}[X]$$

Parametric/standard discrete random variables

- There is deluge of classic RV abstractions that show up in problems.
- They give rise to significant discrete distributions.
- If problem fits, use precalculated (parametric) PMF, expectation, variance and other properties by providing parameters of the problem.
- We will cover the following RVs:
  1. Bernoulli
  2. Binomial
  3. Poisson
  4. Geometric
  5. Negative Binomial
  6. Hypergeometric

Outline

Bernoulli

Bernoulli discrete random variable

A Bernoulli RV X maps "success" of an experiment to 1 and "failure" to 0. It is AKA indicator RV, boolean RV. X is "Bernoulli RV with parameter p", where $\mathbf{P}[\text{"}sucess\text{"}] = p$ and so PMF $p(1) = p$.

$$\mathbf{X} \sim \mathbf{Ber}(\mathbf{p})$$

$$\begin{aligned}
\text{Range:} \quad & \{0, 1\} \\
\text{PMF:} \quad & \mathbf{P}[X = 1] = p(1) = p \\
& \mathbf{P}[X = 0] = p(0) = 1 - p \\
\text{Expectation:} \quad & \mathbf{E}[X] = p \\
\text{Variance:} \quad & \mathbf{V}[X] = p(1 - p)
\end{aligned}$$

Examples: coin toss, random binary digit, if someone likes a film, the gender of a newborn baby, pass/fail of you taking an exam.

Bernoulli examples

Example

You watch a film on Netflix. At the end you click "like" with probability p. Define a RV representing this event.

Answer

Bernoulli examples

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Example

Two fair 6-sided dice are rolled. Define a random variable X for a successful roll of two 6's, and failure for anything else.

Answer

Outline

Binomial

Binomial discrete random variable

A Binomial RV X represents the number of successes in n successive independent trials of a Bernoulli experiment. $X \sim Bin(n, p)$ is a Binomial RV, where p is the probability of success in a given trial:

$$\mathbf{X} \sim \mathbf{Bin}(\mathbf{n}, \mathbf{p})$$

Range: $\{0, 1, \ldots, n\}$

PMF: $k \in \{0, 1, \ldots, n\}$

$$\mathbf{P}[\, X = k \,] = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Expectation: $\mathbf{E}[\, X \,] = np$

Variance: $\mathbf{V}[\, X \,] = np(1-p)$

Examples: # heads in n coin tosses, # of 1's in randomly generated length n bit string

Note: by Binomial theorem (revision), we can prove $\sum_{k=0}^{n} \mathbf{P}[\, X = k \,] = 1$.

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Probability Mass Function for a Binomial

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Binomial example

Example

Let X be the number of heads after a coin is tossed three times: $X \sim Bin(3, 0.5)$. What is the probability of each of the different values of X?

Answer

Binomial RV is sum of Bernoulli RVs

Let X be a Bernoulli RV: $X \sim Ber(p)$. Let Y be a Binomial RV: $Y \sim Bin(n, p)$. Binomial RV = sum of n independent Bernoulli RVs:

$$Y = \sum_{i=1}^{n} X_i, \quad X_i \sim Ber(p)$$

$$\mathbf{E}[\, Y \,] = \mathbf{E}\left[\sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} \mathbf{E}[\, X_i \,] = np$$

Note: $Ber(p) = Bin(1, p)$

Another example

Example

An off-licence sells cases of wine, each containing 20 bottles. The probability that a bottle is bad is 0.05. The off-licence gives a money-back guarantee that the case will contain no more than one bad bottle. What is the probability that the off-licence will have to give money back?

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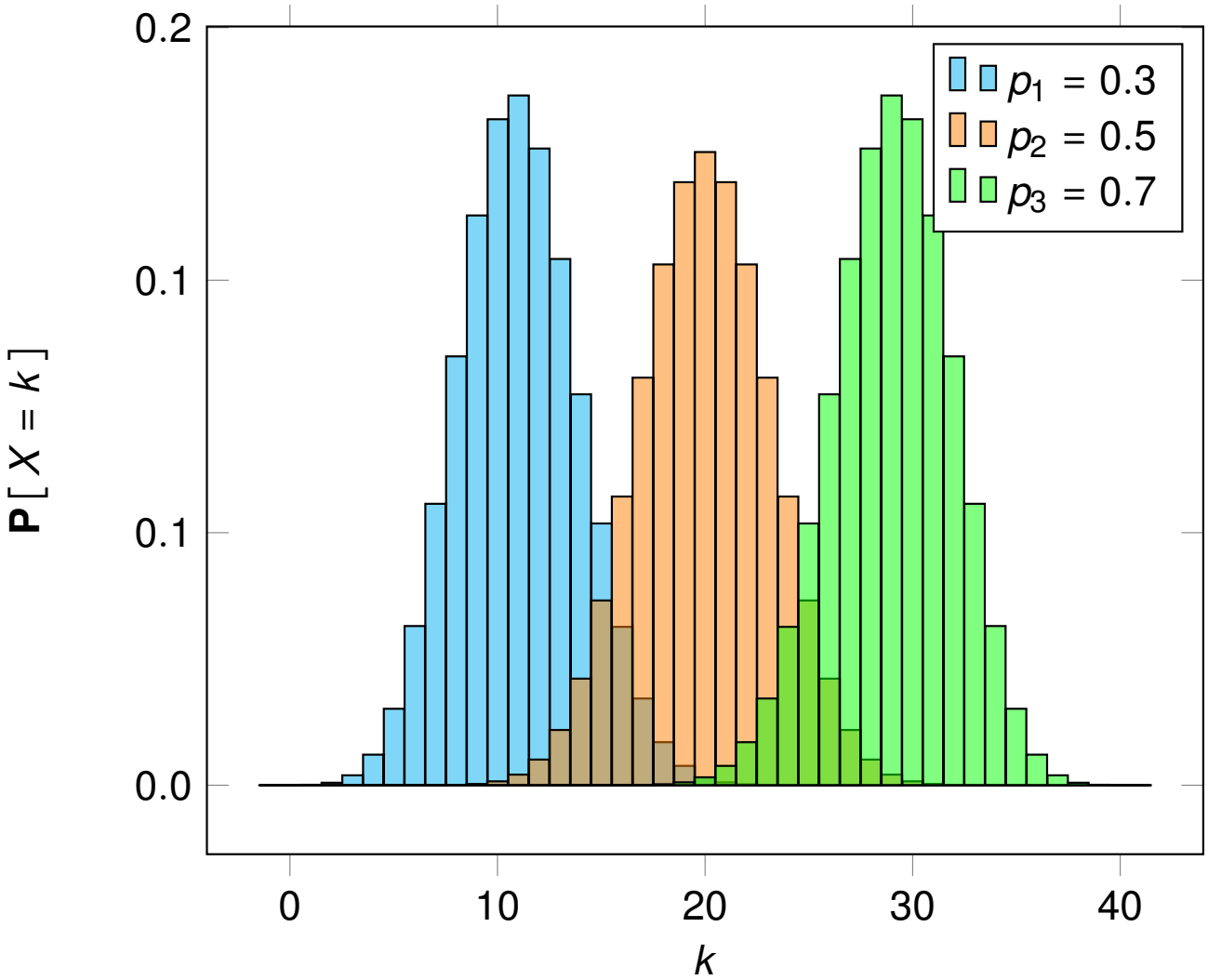
- Bernoulli trial: check if a bottle is bad
- $\mathbf{P}[\text{ success }] = \mathbf{P}[\text{ bottle is bad }] = 0.05$
  $\mathbf{P}[\text{ failure }] = \mathbf{P}[\text{ bottle is good }] = 0.95$
- Recall, when $X \sim Bin(n, p)$ then $\mathbf{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$ thus

$$\mathbf{P}[X \geq 2] = 1 - \mathbf{P}[X = 0] - \mathbf{P}[X = 1]$$

Visualising Binomial PMFs

$X \sim Bin(40, 0.3); \quad X \sim Bin(40, 0.5); \quad X \sim Bin(40, 0.7)$



Introduction to Probability

Lecture 4: More discrete distributions – Poisson, Geometric, Negative Binomial, Hypergeometric

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

UNIVERSITY OF CAMBRIDGE

Outline

Preliminaries:

The natural exponent e

e is a mathematical constant AKA the Euler number. e is very important for exponential functions. Here are some important identities:

$$e \approx 2.71828$$

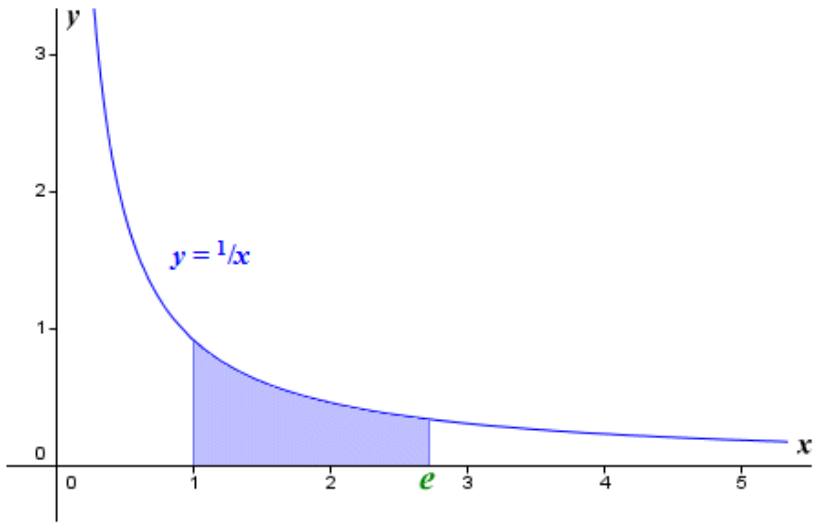
$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$e^{-\lambda} = \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n$$

$$e^r = \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n$$

$$e^r = \sum_{n=0}^{\infty} \frac{r^n}{n!}$$



Binomial RV example: large n, small p

We are trying to predict footfall in a store. We know, based on previous data, that on average 8 people enter the store per hour. What is the probability of k people entering the store in the next 1 hour?

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1. Break an hour into **minutes**.
   - At each **minute**, independent Bernoulli trial with 1 for a person entering the store and 0 for nobody entering the store.
   - X is a Binomial RV: # people entering in an hour, so $\mathbf{E}[X] = np = \lambda = 8$.
   - $X \sim Bin(n = 60, p = \frac{\lambda}{n})$, so $\mathbf{P}[X{=}k] = \binom{n}{k} p^k (1-p)^{n-k} = \binom{60}{k} \left(\frac{8}{60}\right)^k \left(1 - \frac{8}{60}\right)^{n-k}$

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What if 2 people enter in the same minute?

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2. Break an hour into **milliseconds**.
   - At each **millisecond**, independent Bernoulli trial: 1 for enter, 0 for not enter.
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   - $X \sim Bin(n = 3600000, p = \frac{\lambda}{n})$, so $\mathbf{P}[X{=}k] = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$

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Binomial RV example: large n, small p

We are trying to predict footfall in a store. We know, based on previous data, that on average 8 people enter the store per hour. What is the probability of k people entering the store in the next 1 hour?

1. Break an hour into **minutes**.
   - At each **minute**, independent Bernoulli trial with 1 for a person entering the store and 0 for nobody entering the store.
   - X is a Binomial RV: # people entering in an hour, so $\mathbf{E}[X] = np = \lambda = 8$.
   - $X \sim Bin(n = 60, p = \frac{\lambda}{n})$, so $\mathbf{P}[X{=}k] = \binom{n}{k} p^k (1-p)^{n-k} = \binom{60}{k} \left(\frac{8}{60}\right)^k \left(1 - \frac{8}{60}\right)^{n-k}$
2. Break an hour into **milliseconds**.
   - At each **millisecond**, independent Bernoulli trial: 1 for enter, 0 for not enter.
   - X is a Binomial RV: # people entering in an hour, so $\mathbf{E}[X] = np = \lambda = 8$.
   - $X \sim Bin(n = 3600000, p = \frac{\lambda}{n})$, so $\mathbf{P}[X{=}k] = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$
3. Break an hour into **infinitely small units**.
   - At each **unit**, independent Bernoulli trial: 1 for enter, 0 for not enter.
   - X is a Binomial RV: # people entering in an hour, so $\mathbf{E}[X] = np = \lambda = 8$.
   - $X \sim Bin(n, p = \frac{\lambda}{n})$, thus $\mathbf{P}[\mathbf{X{=}k}] = \lim_{\mathbf{n}\to\infty} \binom{\mathbf{n}}{\mathbf{k}} \left(\frac{\lambda}{\mathbf{n}}\right)^{\mathbf{k}} \left(\mathbf{1} - \frac{\lambda}{\mathbf{n}}\right)^{\mathbf{n-k}}$

Computing Binomial in the limit

$$\mathbf{P}[\,X = k\,] = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} \underset{\textit{expand}}{=}$$

Computing Binomial in the limit

$$\mathbf{P}[\,X = k\,] = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} \underset{\textit{expand}}{=} \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

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$$\underset{\textit{rearrange}}{=} \lim_{n\to\infty} \frac{n!}{n^k(n-k)!} \frac{\lambda^k}{k!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

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$$\text{as } n \to \infty \qquad \frac{n(n-1)\cdots(n-k+1)}{n^k} \approx \frac{n^k}{n^k} = 1$$

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$$\mathbf{P}[X=k] = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} \underset{expand}{=} \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

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Therefore, in our store footfall example: the probability of k people entering the store in the next 1 hour is:

$$\mathbf{P}[X=k] = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!}e^{-\lambda}$$

Poisson

Poisson discrete random variable

A Poisson RV X approximates Binomial where n is large, p is small, and $\lambda = np$ is "moderate". Thus we no longer need to know n and p, we only need to provide **rate** λ. X is the number of successes over the duration of the experiment.

$$\mathbf{X} \sim \mathbf{Pois}(\lambda)$$

$$\begin{aligned}
\text{Range:} \quad & \{0, 1, 2, \ldots\} \\
\text{PMF:} \quad & \mathbf{P}[\, X = k \,] = \frac{\lambda^k}{k!} e^{-\lambda} \\
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Examples: # earthquakes in a given year, # goals scored during a 90 minute football game, # misprints per page in a book, # emails per day.

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Examples: # earthquakes in a given year, # goals scored during a 90 minute football game, # misprints per page in a book, # emails per day.

Key idea: Divide time into a **large number** of small increments. Assume that during each increment, there is some **small probability** of the event happening (independent of other increments).

Earthquake example

Example

Suppose there are an average of 2.79 major earthquakes in the world each year. What is the probability of getting 3 major earthquakes next year?

Answer

Earthquake example

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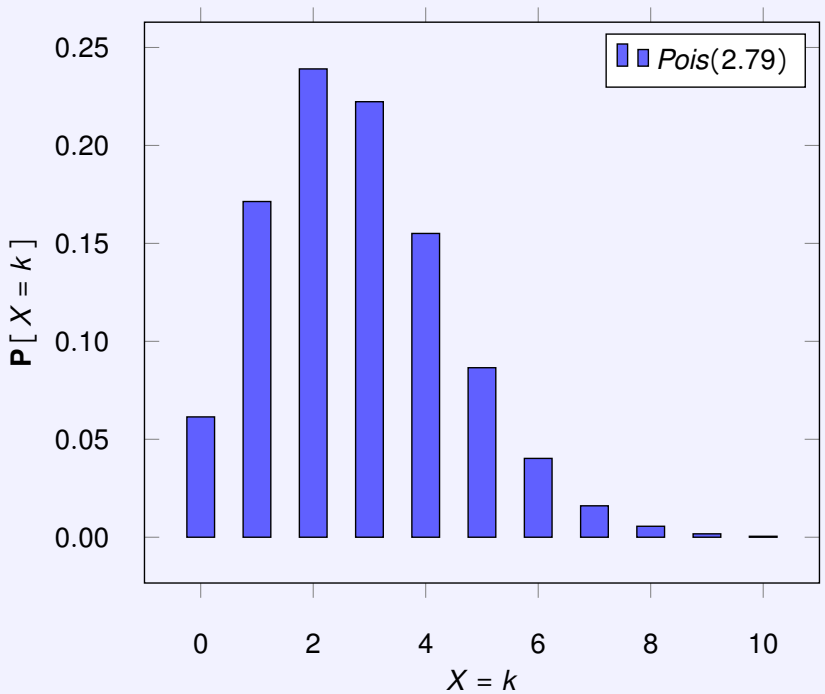

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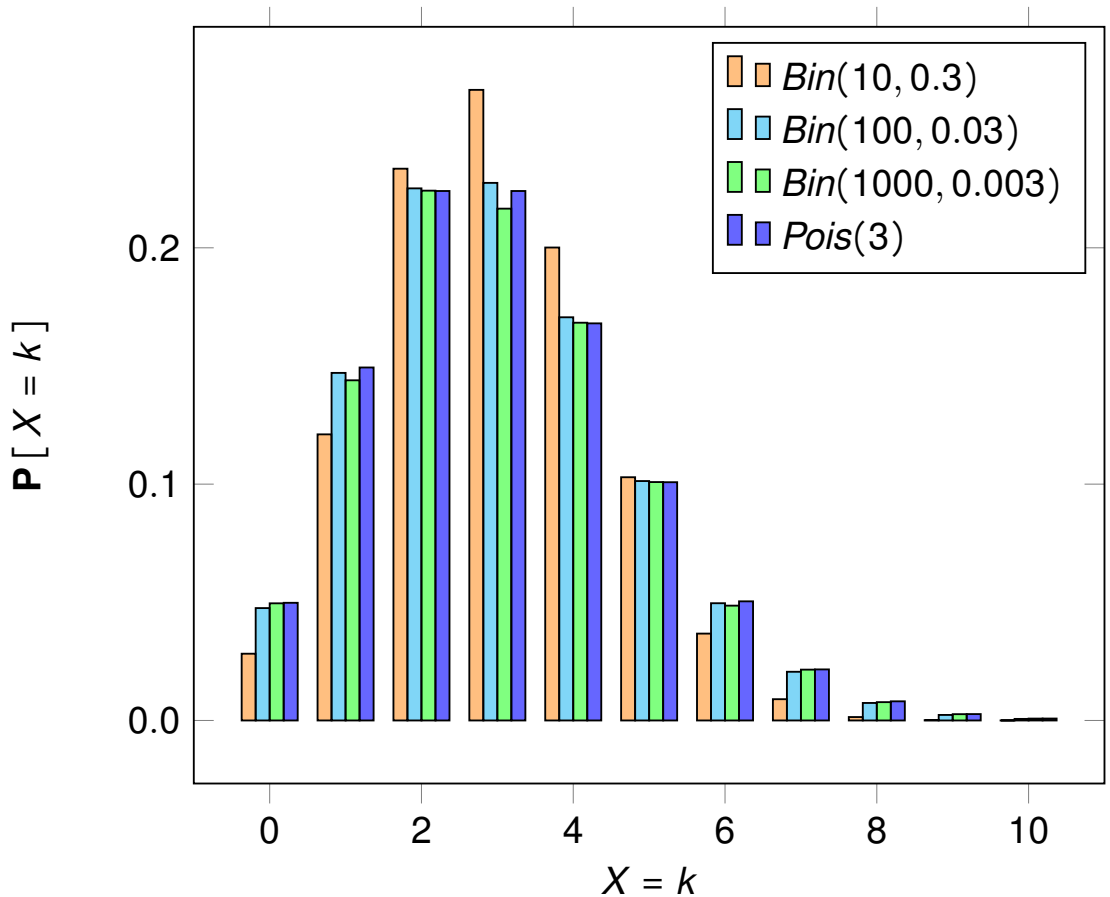

Poisson expectation

$$\text{PMF: } = k \in \{0, 1, 2, \ldots \infty\}; \mathbf{P}[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}$$

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Poisson variance

$$\mathbf{E}\left[X^2 \right] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} =$$

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$$\mathbf{V}[X] = \mathbf{E}\left[X^2 \right] - (\mathbf{E}[X])^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$$

Poisson variance

$$\mathbf{E}\left[X^2 \right] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \text{ (let } i = k-1)$$

$$= \lambda \sum_{i=0}^{\infty} (i+1) \frac{\lambda^i}{i!} e^{-\lambda} = \lambda \left(\underbrace{\sum_{i=0}^{\infty} i \frac{\lambda^i}{i!} e^{-\lambda}}_{\text{same as before}} + \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda}}_{\text{sum of PMFs}=1} \right) =$$

$$= \lambda(\lambda + 1) \qquad \text{thus}$$

$$\mathbf{V}[X] = \mathbf{E}\left[X^2 \right] - (\mathbf{E}[X])^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$$

$$\mathbf{E}\left[X^k \right] = \lambda \mathbf{E}\left[(X+1)^{k-1} \right] \quad \longleftarrow \text{ useful generalisation}$$

Bernoulli, Poisson, and random processes

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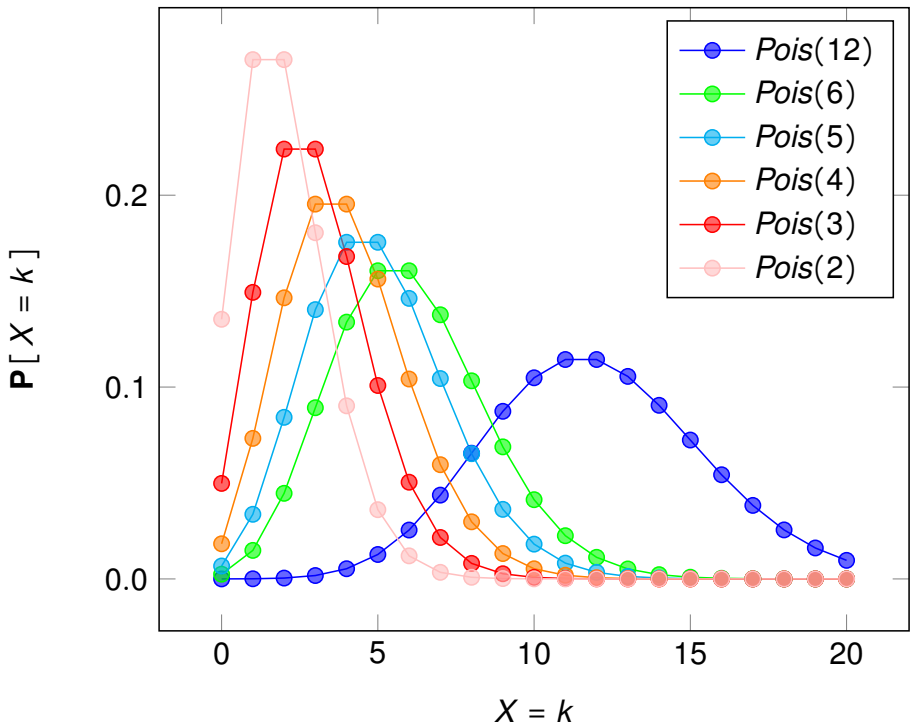

Outline

Geometric

Geometric discrete random variable

X is a geometric RV if X is a number of independent Bernoulli trials until the first success, and p is the probability of success on each Bernoulli trial.

$$\mathbf{X} \sim \mathbf{Geo}(\mathbf{p})$$

Range: $\{1, 2, \ldots\}$

PMF: $\mathbf{P}[X = n] = (1-p)^{n-1} p$

Expectation: $\mathbf{E}[X] = \frac{1}{p}$

Variance: $\mathbf{V}[X] = \frac{1-p}{p^2}$

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Examples: tossing a coin ($\mathbf{P}[\, \textit{head} \,] = p$) until first heads appears, generating bits with $\mathbf{P}[\, \textit{bit} = 1 \,] = p$ until first 1 is generated.

Geometric RV properties

PMF (E_i is the event that the i-th trial succeeds):

$$\mathbf{P}[\, X = n \,] = \mathbf{P}\left[\, E_1^c E_2^c \ldots E_{n-1}^c E_n \,\right] =$$

CDF ($\mathbf{P}[\, X > n \,]$ is the probability that at least the first n trials fail):

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Die example

Example

You roll a fair 6-sided die until it comes up with #6. What is the probability that it will take 3 rolls?

Answer

Die example

Example

You roll a fair 6-sided die until it comes up with #6. What is the probability that it will take 3 rolls?

Answer

Let X be a RV for # of rolls. Probability for any # on die is $\frac{1}{6}$.

Define RVs: $X \sim Geo(\frac{1}{6})$, want $\mathbf{P}[X = 3]$.

Solve:

Outline

Negative binomial

Negative binomial discrete random variable

X is a negative binomial RV if X is the number of independent Bernoulli trials until r successes and p is the probability of success on each trial.

$$\mathbf{X} \sim \mathbf{NegBin}(\mathbf{r}, \mathbf{p})$$

Range: $\{r, r+1, \ldots\}$

PMF: $\mathbf{P}[X = n] = \binom{n-1}{r-1}(1-p)^{n-r}p^r$

Expectation: $\mathbf{E}[X] = \frac{r}{p}$

Variance: $\mathbf{V}[X] = \frac{r(1-p)}{p^2}$

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Examples: tossing a coin until r-th heads appears, generating bits until the first r 1's are generated.

Note: $Geo(p) = NegBin(1, p)$.

NegBin example

Example (not real life!)

A PhD student is expected to publish 2 papers to graduate. A conference accepts each paper randomly and independently with probability $p = 0.25$. On average, how many papers will the student need to submit to a conference in order to graduate?

Answer

Adding NegBin example

Example

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of Z.

Answer

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- Consider the sequence of independent events tossing a coin with $\mathbf{P}[\ heads\ ] = p$.

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- Let X be a RV for # of coin tosses until m heads are observed. Thus $X \sim NegBin(m, p)$.

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- Looking at it from the beginning we tossed independently the coin until we observed $m + n$ heads, thus $Z = X + Y$ and thus $Z \sim NegBin(m + n, p)$.

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- Looking at it from the beginning we tossed independently the coin until we observed $m + n$ heads, thus $Z = X + Y$ and thus $Z \sim NegBin(m + n, p)$.
- Note: if $X_1, X_2, \ldots X_m$ are m independent $Geo(p)$ RVs, then the RV $X = X_1 + X_2 + \cdots + X_m$ has $NegBin(m, p)$ distribution.

Outline

Hypergeometric

Hypergeometric discrete random variable

X is a hypergeometric RV that samples n objects, without replacement, with i successes (random draw for which the object drawn has a specified feature), from a finite population of size N that contains exactly m objects with that feature.

$$\mathbf{X} \sim \mathbf{Hyp}(\mathbf{N}, \mathbf{n}, \mathbf{m})$$

Range: $\{0, 1, \ldots, n\}$

PMF: $\mathbf{P}[X = i] = \dfrac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$

Expectation: $\mathbf{E}[X] = n\dfrac{m}{N}$

Variance: $\mathbf{V}[X] = n\dfrac{m}{N}\left(1 - \dfrac{m}{N}\right)\left(1 - \dfrac{n-1}{N-1}\right)$

Example: an urn has N balls of which m are white and $N - m$ are black; we take a random sample **without replacement** of size n and measure X: # of white balls in the sample.

Survey sampling

Example

A street has 40 houses of which 5 houses are inhabited by families with an income below the poverty line. In a survey, 7 houses are sampled at random from this street. What is the probability that: (a) none of the 5 families with income below poverty line are sampled? (b) 4 of them are sampled? (c) no more than 2 are sampled? (d) at least 3 are sampled?

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Answer

Let X: # of families sampled which are below the poverty line.

$X \sim Hyp(N = 40, n = 7, m = 5)$.

Summary of discrete RV

	$Ber(p)$	$Bin(n,p)$	$Pois(\lambda)$	$Geo(p)$	$NegBin(r,p)$	$Hyp(N,n,m)$
PMF	$\mathbf{P}[X=1]=p$	$\mathbf{P}[X=k]=\binom{n}{k}p^k(1-p)^{n-k}$	$\mathbf{P}[X=k]=\frac{\lambda^k}{k!}e^{-\lambda}$	$\mathbf{P}[X=n]=(1-p)^{n-1}p$	$\mathbf{P}[X=n]=\binom{n-1}{r-1}(1-p)^{n-r}p^r$	$\mathbf{P}[X=i]=\frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$
$\mathbf{E}[X]$	p	np	λ	$\frac{1}{p}$	$\frac{r}{p}$	$n\frac{m}{N}$
$\mathbf{V}[X]$	$p(1-p)$	$np(1-p)$	λ	$\frac{1-p}{p^2}$	$\frac{r(1-p)}{p^2}$	$n\frac{m}{N}\left(1-\frac{m}{N}\right)\left(1-\frac{n-1}{N-1}\right)$
Descr.	1 experiment with prob p of success	n independent trials with prob p of success	# successes over experiment duration, $\lambda = np$ rate of success	# independent trials until first success	# independent trials until r successes	# successes of drawing item with a feature (without replacement) in a sample of size n from a population of size N with m items with the feature

Introduction to Probability

Lecture 5+: Continuous random variables

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

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CAMBRIDGE

Outline

From discrete to continuous RV

- So far, all RV were discrete: can only take on integer values.
- If RV need to take on values in the real number domain ($\mathbb{R}$), then continuous random variable.
- Examples of continuous RV: Uniform RV, Exponential RV, Normal RV.

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- Continuous RV are just like discrete RV, except that every **sum** becomes an **integral**.
- Example of possible values of continuous RV X:

$$(0,1) = \{x \in \mathbb{R}; 0 < x < 1\}$$
$$[0,1] = \{x \in \mathbb{R}; 0 \leq x \leq 1\}$$
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$$(-\infty, \infty) = \text{all real numbers}$$

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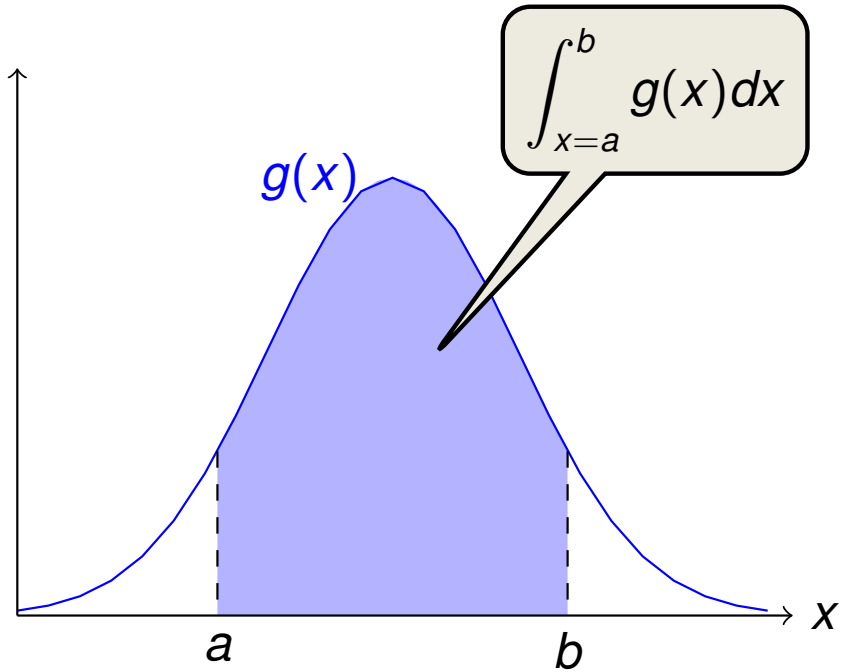
- Examples:
  - X: price of a stock
  - X: time that a machine works before breakdown
  - X: error in an experimental measurement

Integrals revision



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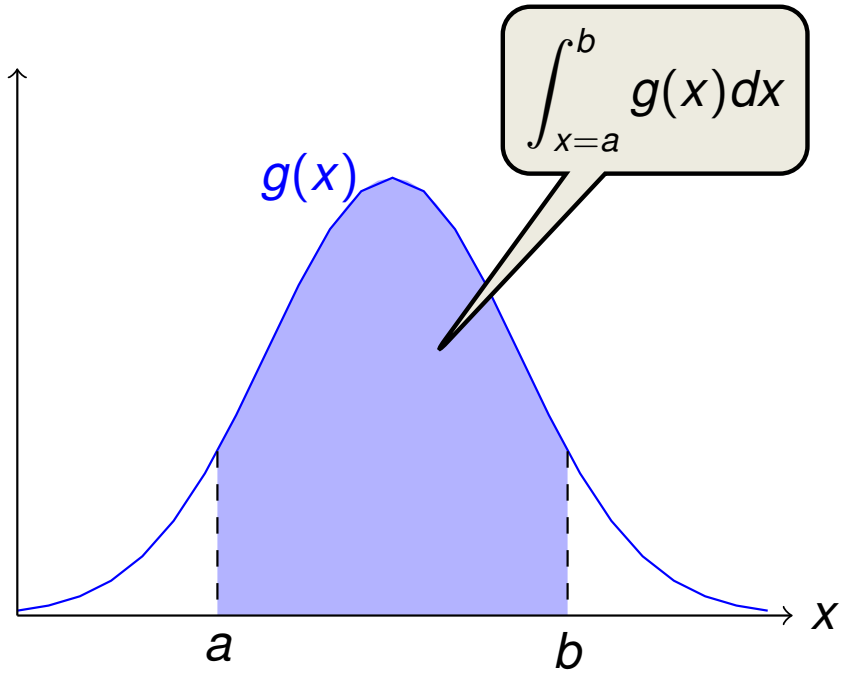
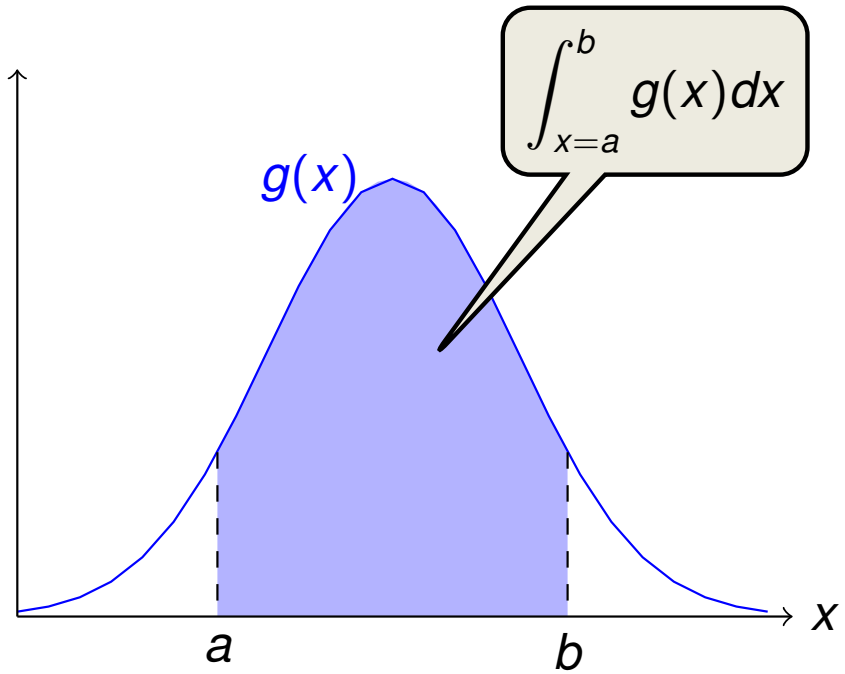


Integral = area under a curve = $\int_{x=a}^{b} g(x)dx = G(x)\Big|_{a}^{b} = G(b) - G(a)$

where $G(x)$ is the antiderivative for g(x).

Integrals revision



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Some examples:

$$\int_a^b x^2 dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3 - a^3}{3} \qquad \int a\,dx = ax + C$$

$$\int \frac{1}{x} dx = ln|x| + C \qquad \int e^x dx = e^x + C$$

Continuous paradigm

- The most important property of discrete RV was probability mass function (PMF) denoting the probability of the RV taking on a certain value.
- But in the continuous world this is impossible:
  What is the probability that a newborn child weighs exactly 3.215438765432532 kg? **NONE**
- Real values are defined with infinite precision, thus the probability that a RV takes on a specific value is not meaningful when the RV is continuous.
- We need a function that says how likely is it that a RV takes on a particular value relative to other values that it could take on: probability density function.

Definition of continuous RV

Continuous random variable

A random variable X is continuous if there is a probability density function (PDF), $f(x) \geq 0$ such that for $-\infty < x < \infty$:

$$\mathbf{P}[a \leq X \leq b] = \int_a^b f(x)dx$$

To preserve the axioms that guarantee that $\mathbf{P}[a \leq X \leq b]$ is a probability, the following properties must hold:

$$0 \leq \mathbf{P}[a \leq X \leq b] \leq 1$$

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- Note: we also write $f(x)$ as $f_X(x)$.
- In continuous world, every RV has a PDF: its relative value wrt to other possible values.
- Integrate $f(x)$ to get probabilities.

Comparing PMF and PDF

Discrete random variable X	Continuous random variable X
Probability mass function (PMF):	Probability density function (PDF):
$p(x)$	$f(x)$
Compute probability:	Compute probability:
$\mathbf{P}[X = x] = p(x)$	
$\mathbf{P}[a \leq X \leq b] = \sum_{x=a}^{b} p(x)$	$\mathbf{P}[a \leq X \leq b] = \int_{x=a}^{b} f(x)dx$

Both are measures of how likely is X to take on a value.

Computing probability example

Example

Let X be a continuous RV with PDF:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

What is $\mathbf{P}[X \geq 1]$?

Answer

Computing probability example

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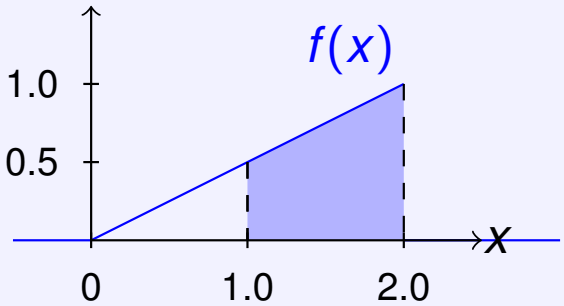

PDF properties

- $f(x)$ is NOT a probability, it is probability density:

$$\mathbf{P}[X = a] = \int_a^a f(x)dx = 0 \neq f(a)$$

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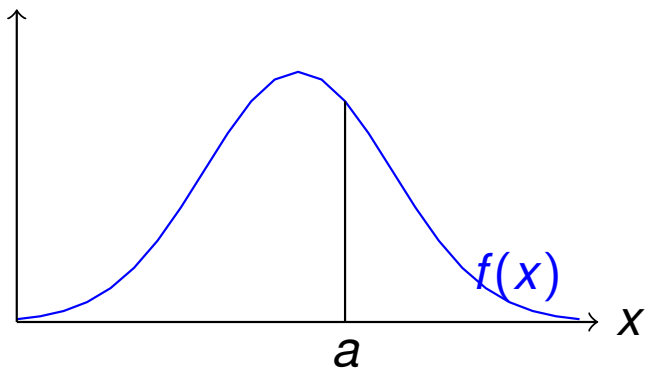

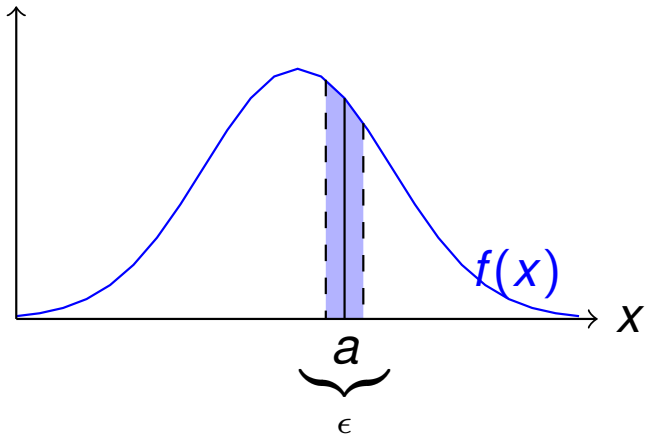

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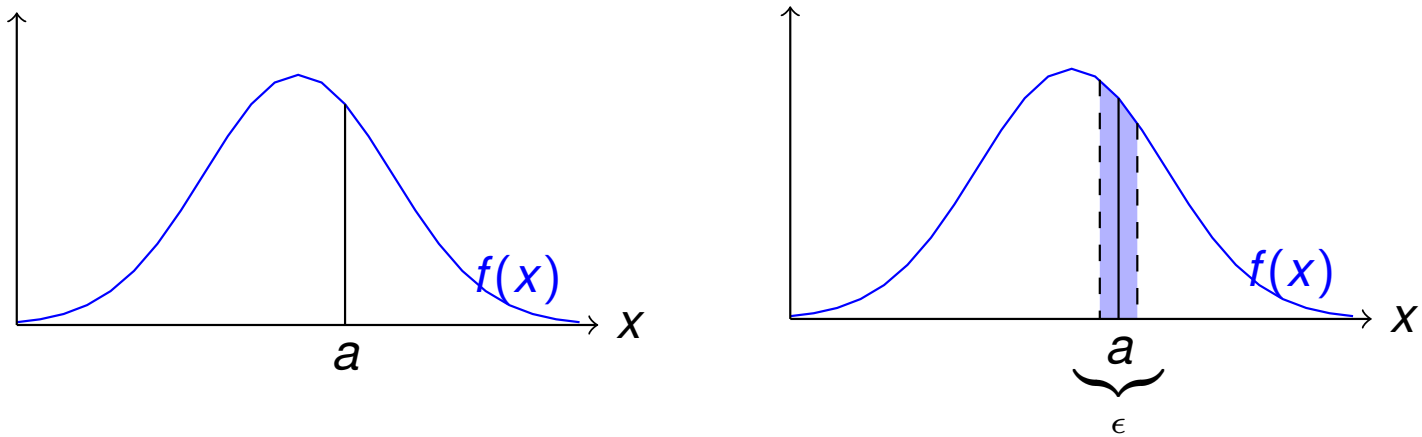

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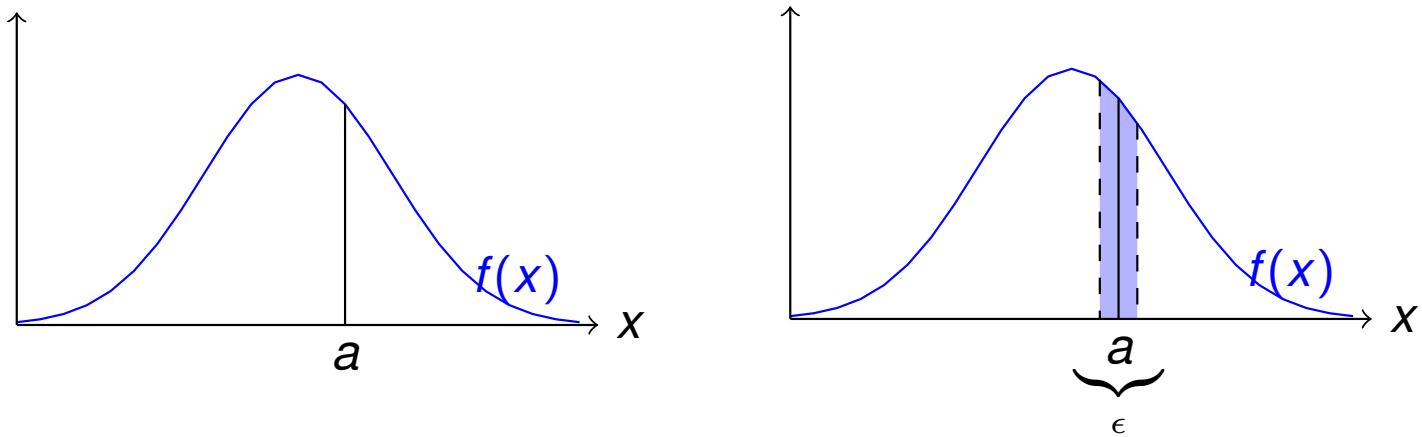

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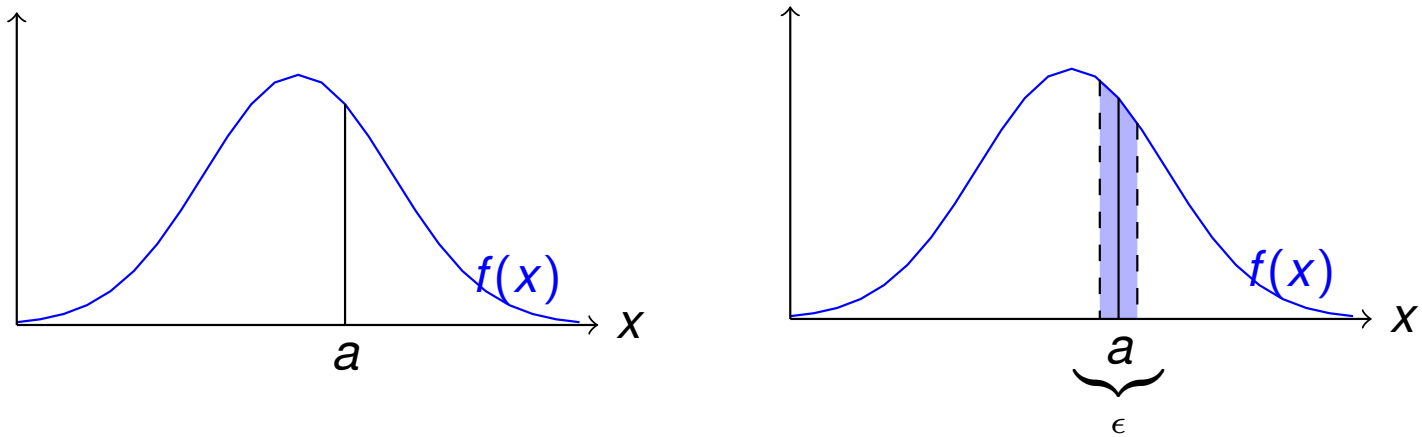

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PDF and probability example

Example

Let X be a continuous RV with PDF:

$$f(x) = \begin{cases} C(4x - 2x^2) & \text{when } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of the constant C? What is $\mathbf{P}[X > 1]$?

Answer

PDF and probability example

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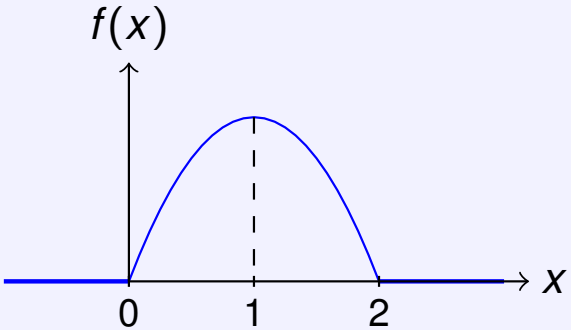

C is a normalisation constant. We know that PDF must sum to 1:

PDF and probability example cont.

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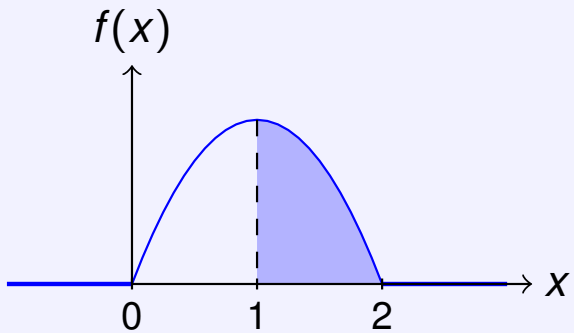

PDF and probability example cont.

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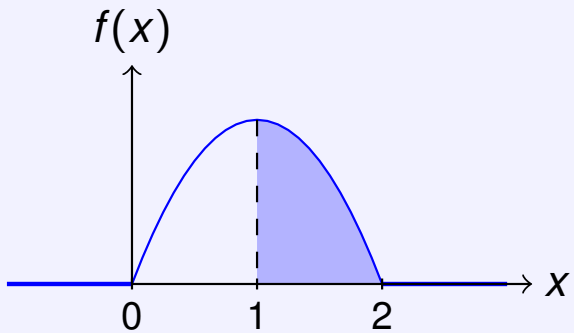

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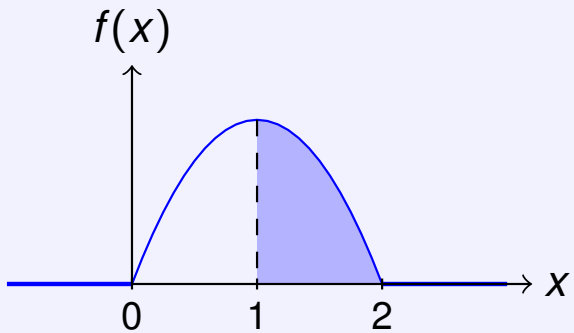

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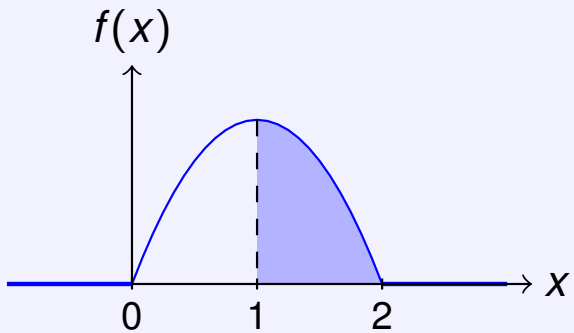

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$$= \frac{3}{8}\left(\left(8 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right)\right) = \frac{1}{2}$$

Outline

Cumulative distribution function

- Since PDF is not a probability, we need to solve an integral every single time we want to calculate a probability.
- To save effort, cumulative distribution function (CDF) computes this: $F(a) = F_X(a) = \mathbf{P}[X \leq a]$ where $-\infty < a < \infty$.
- Recall: CDF for *discrete* RV is $F(a) = \sum_{\text{all } x \leq a} p(x)$

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Cumulative distribution function for a continuous RV

For a continuous random variable X with PDF $f(x)$, the cumulative distribution function (CDF) is:

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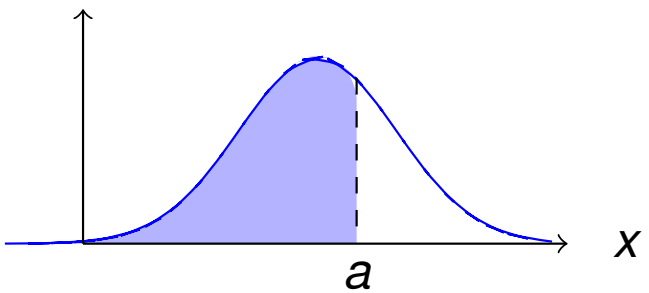

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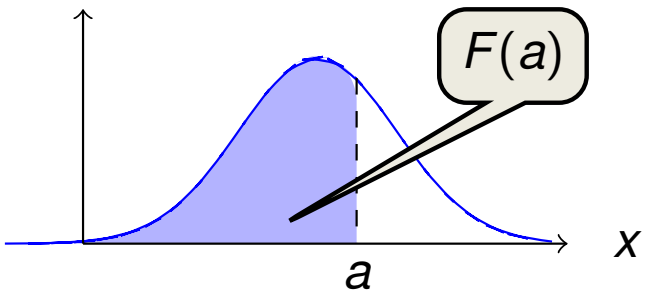

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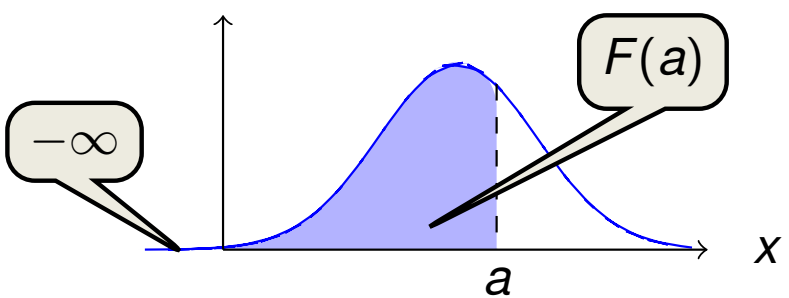
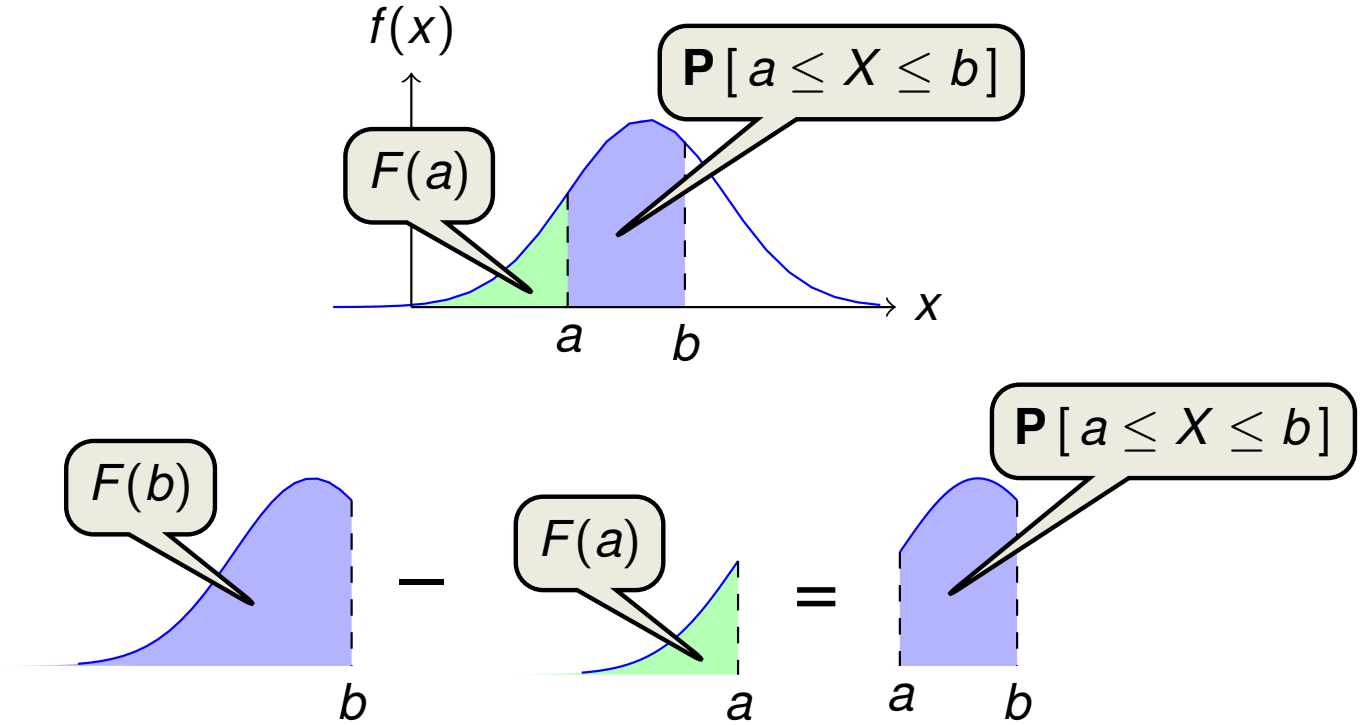


CDF properties

- While PDF is not a probability, CDF is.
- If you learn to use CDFs, you can avoid integrating the PDF.
- It is a matter of convention that CDF is probability that a RV takes on a value less than (or equal to) the input value as opposed to greater than.
- Useful examples of using CDF:

Probability question	**Solution**	**Explanation**
$\mathbf{P}[X \leq a]$	$F(a)$	Definition of CDF
$\mathbf{P}[X < a]$	$F(a)$	Note that $\mathbf{P}[X = a] = 0$
$\mathbf{P}[X > a]$	$1 - F(a)$	$\mathbf{P}[X \leq a] + \mathbf{P}[X > a] = 1$
$\mathbf{P}[a < X < b]$	$F(b) - F(a)$	$F(a) + \mathbf{P}[a < X < b] = F(b)$

Computing CDF



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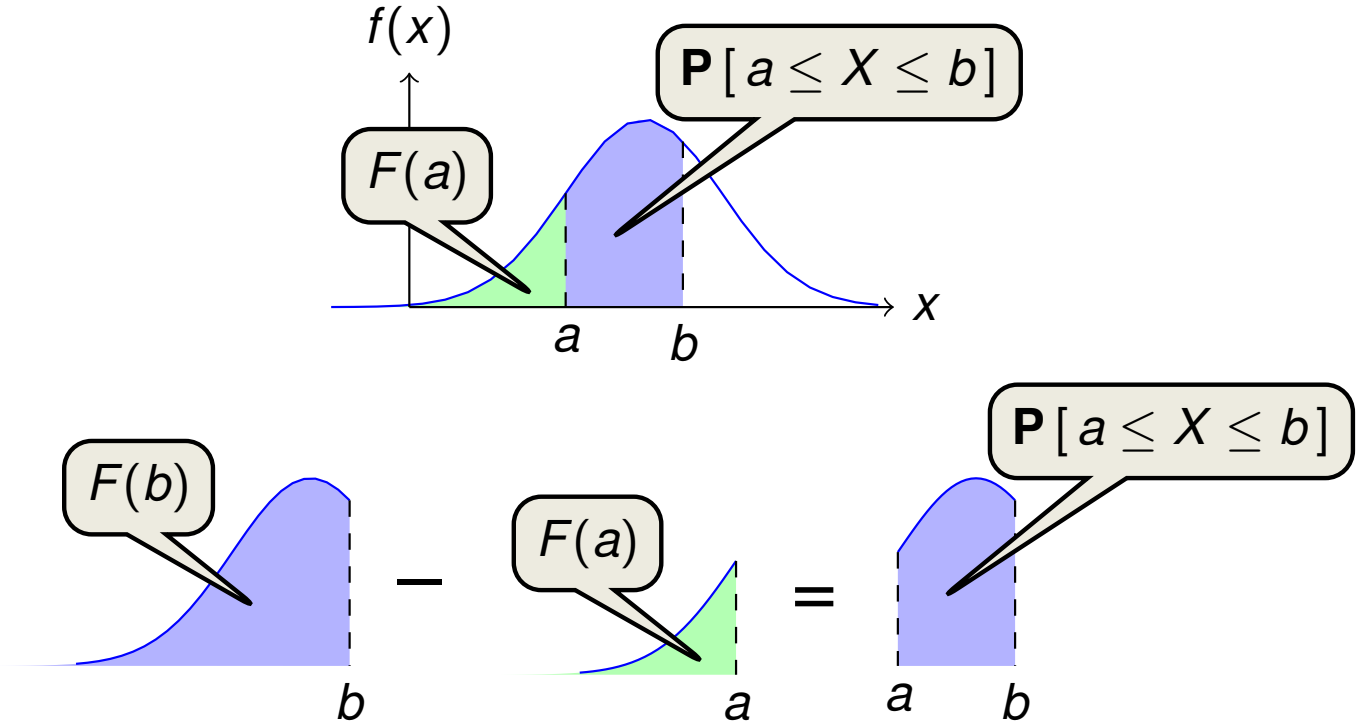

$$F(b) - F(a) = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$$

$$= \left(\int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx\right) - \int_{-\infty}^{a} f(x)dx$$

$$= \int_{a}^{b} f(x)dx = \mathbf{P}[a < X < b] = \mathbf{P}[a \leq X \leq b]$$

Expectation and variance for continuous RV

Discrete RV X

$$\mathbf{E}[X] = \sum_x xp(x)$$

$$\mathbf{E}[g(X)] = \sum_x g(x)p(x)$$

Continuous RV X

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} xf(x)dx$$

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Both continuous and discrete RVs

$$\mathbf{E}[aX + b] = a\mathbf{E}[X] + b \qquad \text{Linearity of expectation}$$

$$\mathbf{V}[X] = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X]^2) \qquad \text{Properties of}$$

$$\mathbf{V}[aX + b] = a^2\mathbf{V}[X] \qquad \text{variance}$$

Outline

Uniform continuous RV

Uniform continuous random variable

A uniform continuous random variable X is defined as follows:

$$\mathbf{X} \sim \mathbf{Uni}(\alpha, \beta)$$

Range: $[\alpha, \beta]$, sometimes (α, β)

PDF: $$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{when } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

Expectation: $$\mathbf{E}[X] = \frac{\alpha + \beta}{2}$$

Variance: $$\mathbf{V}[X] = \frac{(\beta - \alpha)^2}{12}$$

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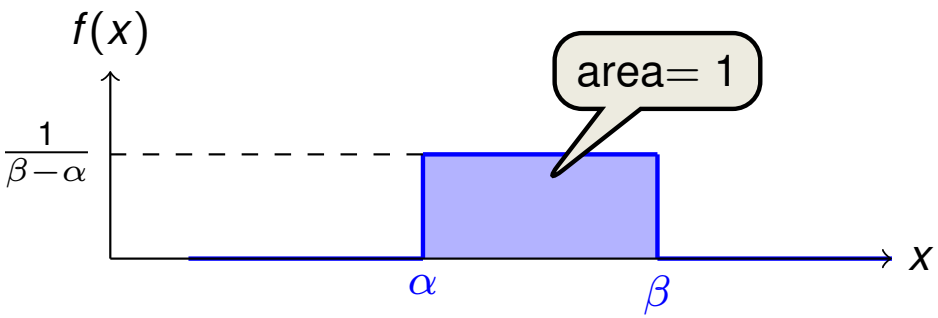

- Notice that the density $\frac{1}{\beta - \alpha}$ is exactly the same regardless of the value of x. This makes it uniform.
- The PDF is $\frac{1}{\beta - \alpha}$ since it is a constant such that the integral over all possible inputs evaluates to 1.

Public transport example

Example

The University bus arrives at the Computer Lab bus stop at 7:00, 7:15 and so on at 15 minute intervals. You arrive at the bus stop a time uniformly distributed in the interval between 1pm and 1:30pm. What is the probability that you wait less than 5 minutes for the bus?

Answer

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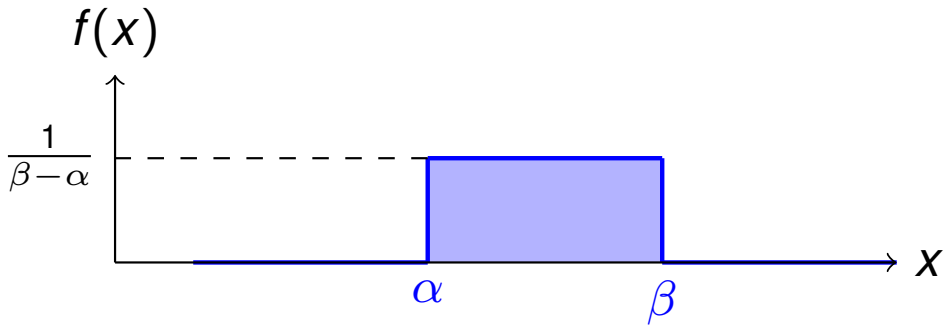
Answer

Let X be a RV for the time you arrive after 1pm to the bus stop.

Define RVs: $X \sim Uni(0, 30)$

Solve:

Expectation for Uniform RV – convince yourself



$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta - \alpha}\, dx$$

$$= \frac{1}{\beta - \alpha} \frac{1}{2} x^2 \Big|_{\alpha}^{\beta} = \frac{1}{\beta - \alpha} \frac{1}{2} (\beta^2 - \alpha^2)$$

$$= \frac{1}{2} \frac{(\beta + \alpha)(\beta - \alpha)}{\beta - \alpha} = \frac{\alpha + \beta}{2}$$

Outline

Exponential continuous RV

Exponential continuous random variable

An exponential random variable X represents the time until an event (first success) occurs. It is parametrised by $\lambda > 0$, the constant rate at which the event occurs.

$$\mathbf{X} \sim \mathbf{Exp}(\lambda)$$

Range: $[0, \infty)$

PDF: $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Expectation: $\mathbf{E}[X] = \frac{1}{\lambda}$ (time)

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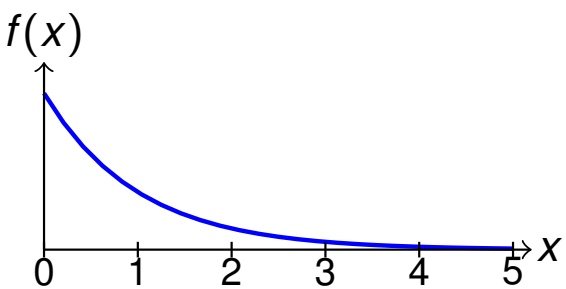

- Examples: time until next earthquake, time for request to reach web server, time until end of mobile phone contract.
- Note that λ is the same as the one in the Poisson RV.
- Poisson RV counts # of events that occur in a fixed interval, exponential RV measures the amount of time until the next event occurs.

Pandemic example

Example

Major pandemics occur once every 100 years. What is the probability of a major pandemic in the next 5 years? What is the standard deviation of years until the next pandemic?

Answer

Pandemic example

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Major pandemics occur once every 100 years. What is the probability of a major pandemic in the next 5 years? What is the standard deviation of years until the next pandemic?

Answer

Let X be a RV for the time when the next pandemic happens.
Let a unit of time be 1 year.

Define RVs: $X \sim Exp(\lambda), \mathbf{E}[X] = \frac{1}{\lambda} = 100$, thus $\lambda = \frac{1}{100} = 0.01$
$X \sim Exp(\lambda = 0.01)$.

Solve: Compute $\mathbf{P}[X < 5], \mathbf{SD}[X]$.

CDF of Exponential RV – convince yourself

CDF for Exponential RV

If X is an exponential continuous random variable, $X \sim Exp(\lambda)$, then its cumulative distribution function CDF (where $x \geq 0$) is

$$F(x) = 1 - e^{-\lambda x}$$

Proof:

$$\begin{aligned} F(x) = \mathbf{P}[X \leq x] &= \int_0^x \lambda e^{-\lambda x} dx \\ &= \lambda \frac{1}{-\lambda} e^{-\lambda x} \Big|_0^x \\ &= -1(e^{-\lambda x} - e^{-\lambda 0}) \\ &= 1 - e^{-\lambda x} \end{aligned}$$

Outline

Normal continuous RV

Normal continuous random variable

A normal random variable X, parametrised over mean μ and variance σ^2 is defined as

$$\mathbf{X} \sim \mathcal{N}(\mu, \sigma^2)$$

Range:	$(-\infty, \infty)$
PDF:	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
Expectation:	$\mathbf{E}[X] = \mu$
Variance:	$\mathbf{V}[X] = \sigma^2$

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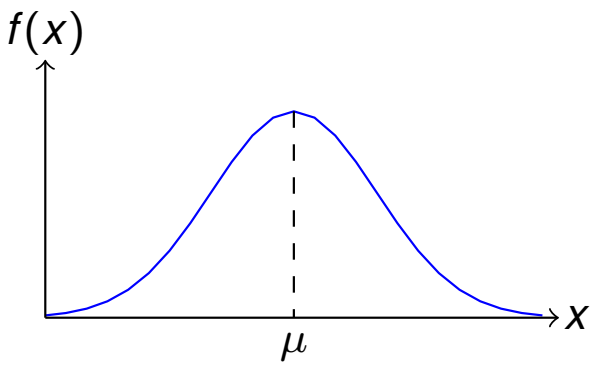

- The most important random variable type, AKA Gaussian RV and Bell curve.
- Generated from summing independent RV, thus occurs often in nature (cf. Central Limit Theorem in Lecture 8).
- Used to model entropic (conservative) distribution of data with mean and variance.

Normal RV paradigm

Goal: translate problem statement into a RV – model real life situation with probability distributions (e.g., height distribution in a class).

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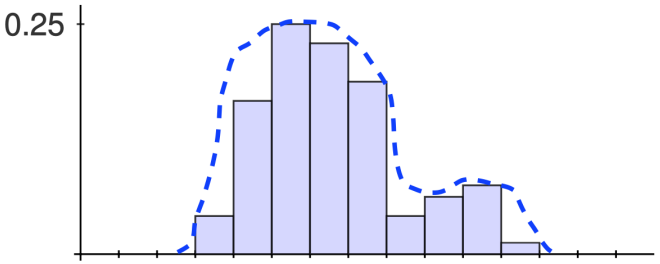
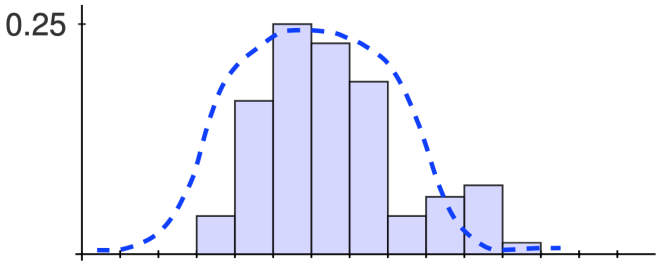


Perfect fit!
But what about another class?
Overfit?



Same mean and variance!
Generalises well.

Normal RV paradigm

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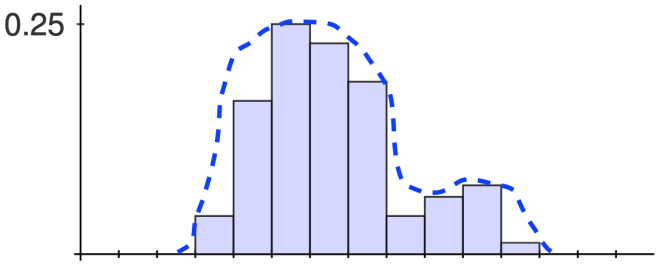

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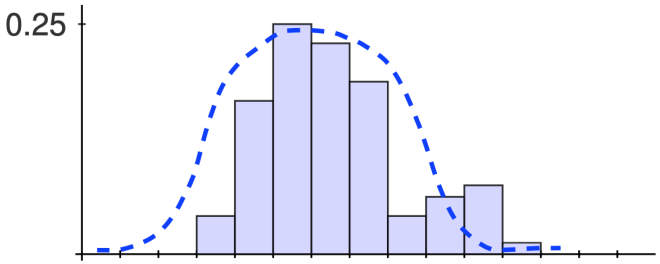

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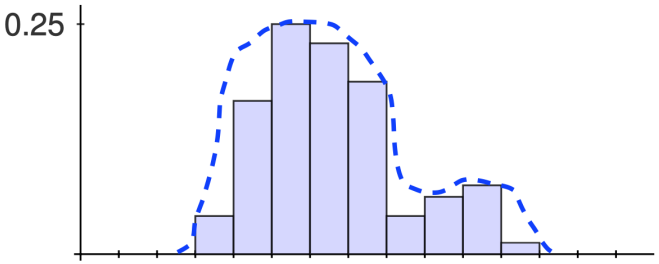

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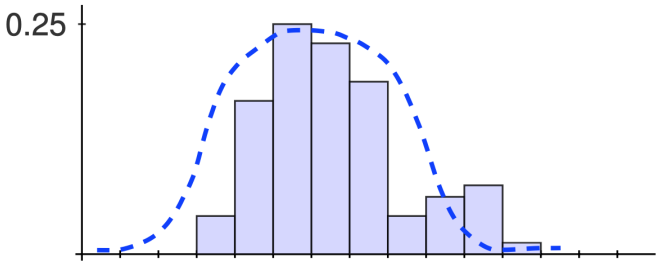

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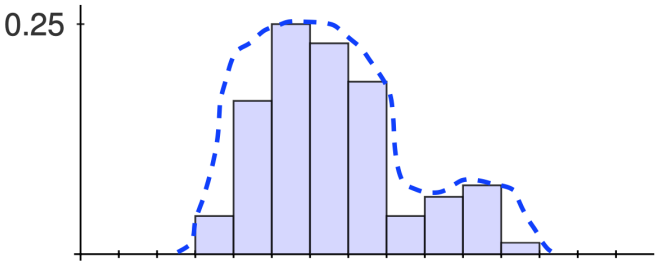

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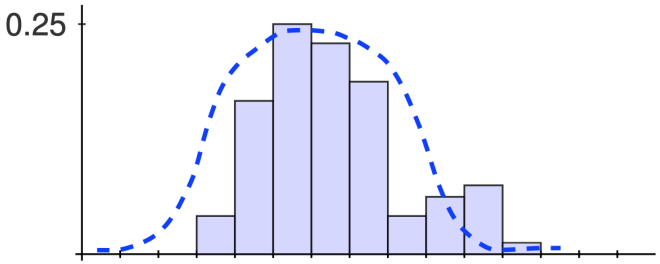

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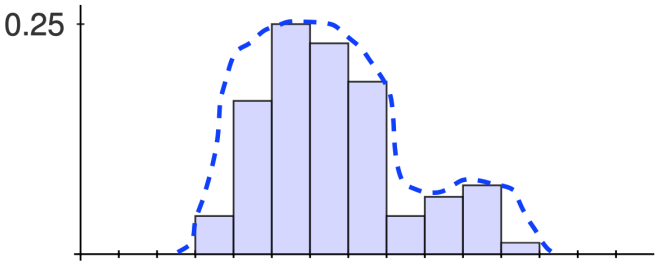

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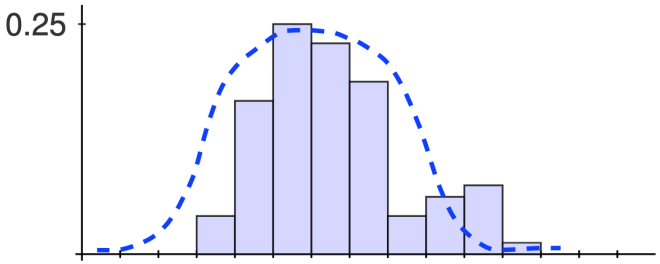

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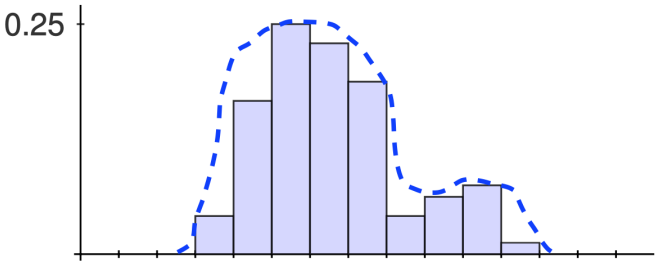

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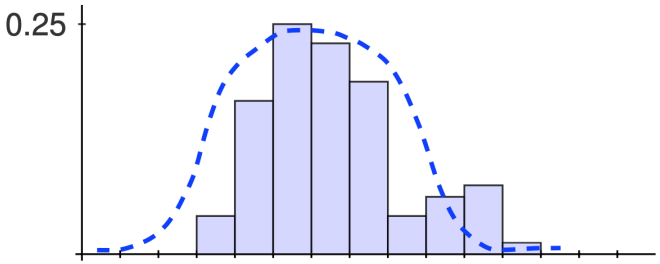

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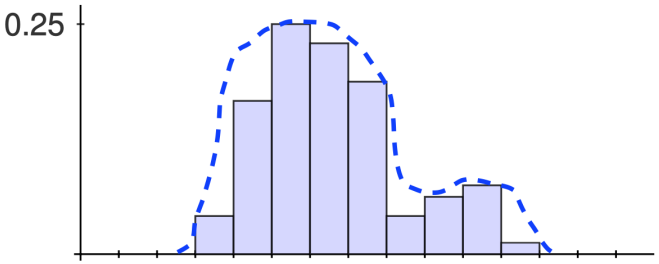

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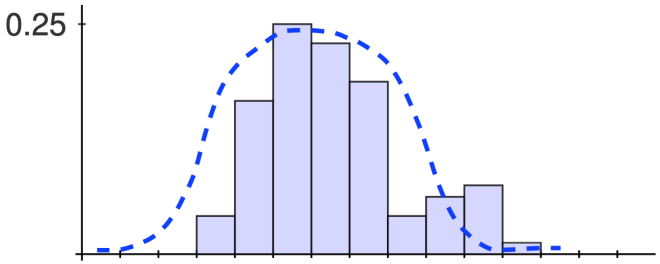

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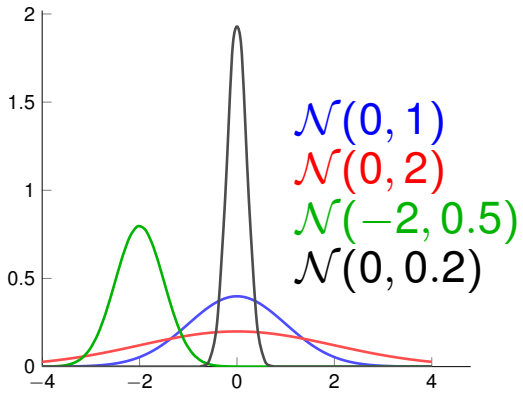

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Cannot be solved analytically!

That is, no closed form for the integral of the Normal PDF. (But...)

Properties for Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $\mathbf{P}[X \leq x] = F(x)$.

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Proof outline:

- $\mathbf{E}[Y] = \mathbf{E}[aX + b] = a\mathbf{E}[X] + b = a\mu + b$ (linearity of expectation)
- $\mathbf{V}[Y] = \mathbf{V}[aX + b] = a^2\mathbf{V}[X] = a^2\sigma^2$
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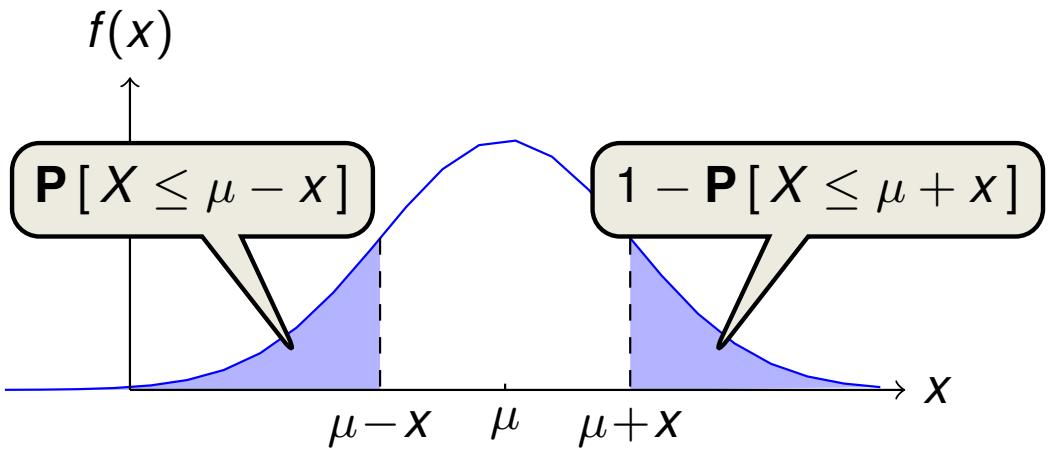
- The PDF of a Normal RV is symmetric about the mean μ.

$$F(\mu - x) = 1 - F(\mu + x)$$



Computing probabilities with Normal RV

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CDF of the Standard Normal, Z

Z: Standard Normal RV

Standard Normal random variable Z

The Standard Normal continuous random variable Z is defined as

$$\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathbf{1})$$

Expectation: $\mathbf{E}[Z] = \mu = 0$ (zero mean)

Variance: $\mathbf{V}[Z] = \sigma^2 = 1$ (unit variance)

- Not a new distribution: a special case of the Normal ($\mathcal{N}(\mu, \sigma^2) = \mu + \sigma\mathcal{N}(0, 1)$).
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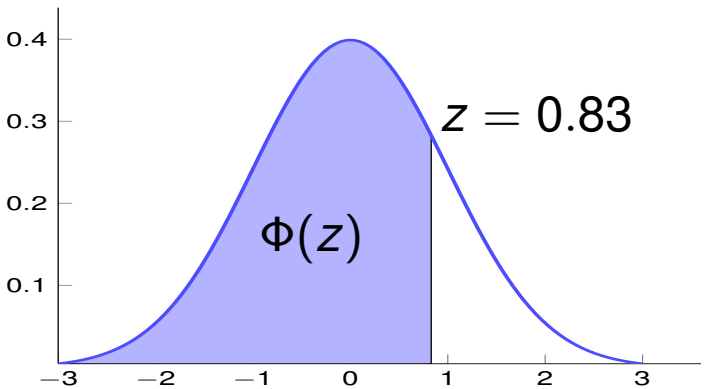

$\mathbf{P}[Z \leq 0.83] = \Phi(0.83) =$

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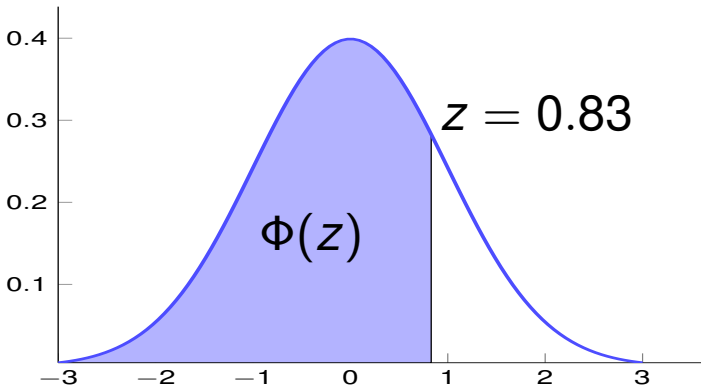

$$\mathbf{P}[Z \leq 0.83] = \Phi(0.83) = 0.7967$$

Table A.3 Standard Normal Curve Areas (*cont.*) $\Phi(z) = P(Z \leq z)$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633

Walking example revisited

Example

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Answer

$X \sim \mathcal{N}(\mu = 10, \sigma^2 = 2)$
(But $\mathbf{P}[X \geq 12] = \int_{12}^{\infty} f(x)dx$ has no analytic solution.)

1. Compute $z = \frac{(x-\mu)}{\sigma}$:

2. Look up $\Phi(z)$ in table:

Introduction to Probability

Lecture 6: Marginals and Joint Distributions

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

Motivation

Experiments often involve several random variables, and some of them may influence each other.

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To this end, we will introduce:

- Joint/Marginal distribution of two (or more) variables
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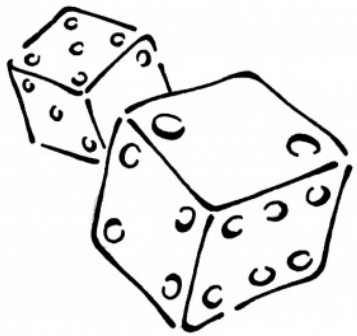
For simplicity, we will mainly focus on discrete random variables.

Warm-Up Exercise



Example

Let $X_1, X_2 \in \{1, 2, \ldots, 6\}$ be two independent rolls of an unbiased die. Let $S := X_1 + X_2$ and $M := \max\{X_1, X_2\}$. List the elements of the event $\{S = 7, M \leq 5\}$ and deduce the probability.

Answer

Joint Probability

Joint Probability Mass Function

The joint probability mass function of two discrete random variables X and Y is the function $p : \mathbb{R}^2 \to [0, 1]$, defined by:

$$p_{X,Y}(a, b) = \mathbf{P}[X = a, Y = b] \qquad \text{for } -\infty < a, b < \infty.$$

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Marginal Distribution

Given a joint distribution $F_{X,Y}$ of two random variables X, Y, one obtains the marginal distribution of X for any a as follows:

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Joint Distribution contains (much) more information than the two marginals!

Discrete Example 1

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Answer

	b					
a	1	2	3	4	5	6
2	$1/36$	0	0	0	0	0
3	0	$2/36$	0	0	0	0
4	0	$1/36$	$2/36$	0	0	0
5	0	0	$2/36$	$2/36$	0	0
6	0	0	$1/36$	$2/36$	$2/36$	0
7	0	0	0	$2/36$	$2/36$	$2/36$
8	0	0	0	$1/36$	$2/36$	$2/36$
9	0	0	0	0	$2/36$	$2/36$
10	0	0	0	0	$1/36$	$2/36$
11	0	0	0	0	0	$2/36$
12	0	0	0	0	0	$1/36$

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	b						
a	1	2	3	4	5	6	$p_S(a)$
2	1/36	0	0	0	0	0	1/36
3	0	2/36	0	0	0	0	2/36
4	0	1/36	2/36	0	0	0	3/36
5	0	0	2/36	2/36	0	0	4/36
6	0	0	1/36	2/36	2/36	0	5/36
7	0	0	0	2/36	2/36	2/36	6/36
8	0	0	0	1/36	2/36	2/36	5/36
9	0	0	0	0	2/36	2/36	4/36
10	0	0	0	0	1/36	2/36	3/36
11	0	0	0	0	0	2/36	2/36
12	0	0	0	0	0	1/36	1/36
$p_M(b)$	1/36	3/36	5/36	7/36	9/36	11/36	1

Discrete Example 2

Example

Suppose an urn contains balls numbered $1, 2, \ldots, N$. We draw $1 \leq n \leq N$ balls uniformly and without replacement from the urn. Let $X_i \in \{1, 2, \ldots, N\}$ be the number of the ball drawn in the i-th step. What is the marginal distribution of X_i?

Answer

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Answer

We first compute the joint distribution. For distinct $a_1, a_2, \ldots, a_n$,

Fix i and consider the marginal distribution of X_i:

Joint Distributions of Continuous Variables

Definition

Random variables X and Y have a joint continuous distribution if for some function $f : \mathbb{R}^2 \to \mathbb{R}$ and for all numbers $a_1 \leq b_1$ and $a_2 \leq b_2$,

$$\mathbf{P}\left[\, a_1 \leq X \leq b_1, a_2 \leq Y \leq b_2 \,\right] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y)\, dx\, dy.$$

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As in one-dimensional case we switch from F to f by differentiating (or integrating):

$$F(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y) dx\, dy \qquad \text{and} \qquad f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

Example of a Joint Distribution of Continuous Random Variables

- Consider the density:

$$f(x, y) = \frac{30}{\pi} \cdot e^{-50x^2 - 50y^2 + 80xy},$$

where $-\infty < x, y < \infty$.

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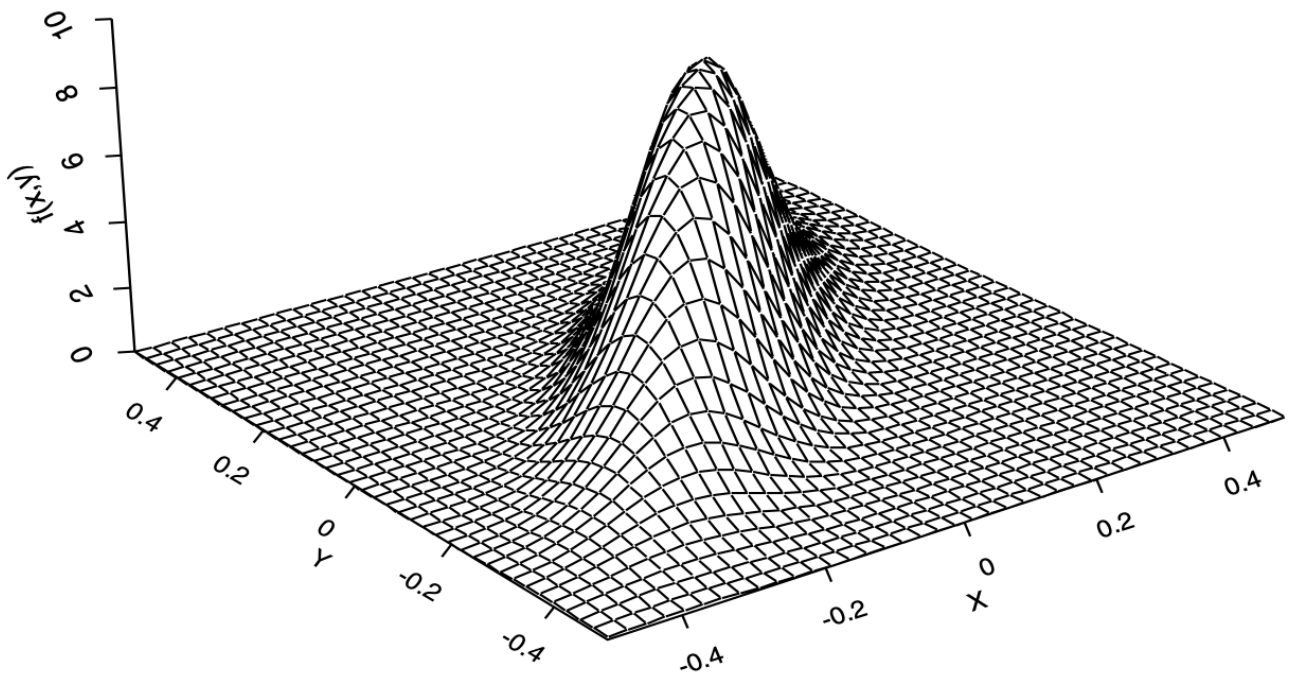

Source: Modern Introduction to Statistics

Dealing with Continuous Variables

Example (1/2)

Suppose that the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & \text{for } 0 < x < \infty, 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Compute (i) $\mathbf{P}[X > 1, Y < 1]$ and (ii) $\mathbf{P}[X < Y]$.

Answer

(i) We first compute:

$$\mathbf{P}[X > 1, Y < 1] = \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}\,dxdy$$

Dealing with Continuous Variables (cont.)

Example (2/2)

Suppose that the joint probability density of X and Y is given by

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Compute (i) $\mathbf{P}[X > 1, Y < 1]$ and (ii) $\mathbf{P}[X < Y]$.

Answer

(ii) We have:

$$\mathbf{P}[X < Y] = \int_0^\infty \int_0^y 2e^{-x}e^{-2y}\,dxdy$$

$$= \frac{1}{3}.$$

Introduction to Probability

Lecture 7: Independence, Covariance and Correlation

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

UNIVERSITY OF CAMBRIDGE

Independence of Random Variables

Definition of Independence

Two random variables X and Y are independent if for all values a, b:

$$\mathbf{P}[X \leq a, Y \leq b] = \mathbf{P}[X \leq a] \cdot \mathbf{P}[Y \leq b].$$

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Using the joint probability distribution, the above is equivalent to for all a, b,

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All these definitions extend in the natural way to more than two variables!

Factorisation

Factorisation

The definition of independence of X and Y implies the following factorisation formula: for any "suitable" sets A and B,

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Example

Let X and Y be two independent variables. Let $I = (a, b]$ be any interval and define $U := \mathbf{1}_{X \in I}$ and $V := \mathbf{1}_{Y \in I}$. Prove U and V are independent.

Answer

Buffon's Needle Problem (1/2)

Georges-Louis Leclerc de Buffon 1707–1788 (Source Wikipedia)

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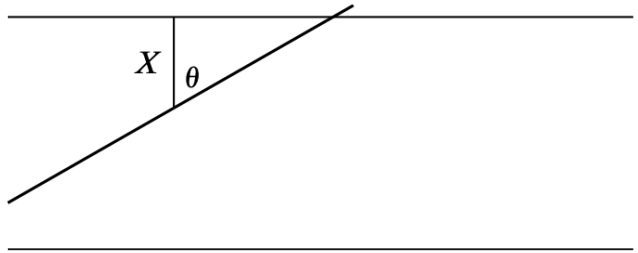

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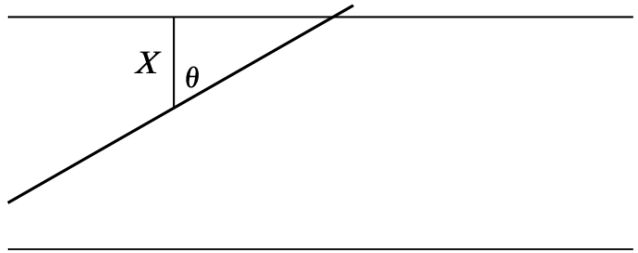

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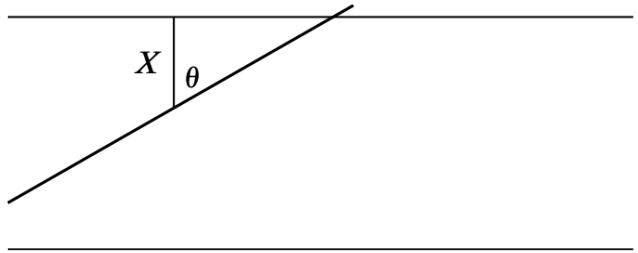

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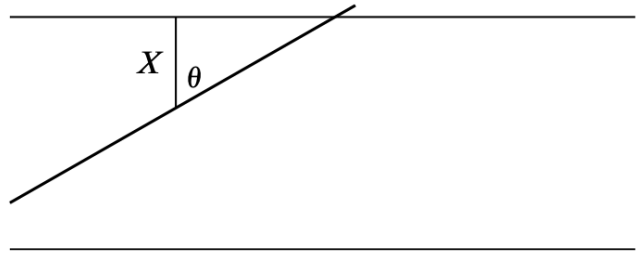

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Can be thought of as: 1. Sample the middle point of needle, 2. Sample the angle.

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Let us compute the probability that the line intersects:

$$\mathbf{P}\left[X < \frac{L}{2} \cdot \cos(\theta)\right]$$

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This gives us a method to estimate π!

Covariance

Definition of Covariance

Let X and Y be two random variables. The covariance is defined as:

$$\mathbf{Cov}\left[X, Y\right] = \mathbf{E}\left[\left(X - \mathbf{E}\left[X\right]\right) \cdot \left(Y - \mathbf{E}\left[Y\right]\right)\right].$$

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- Two variables X, Y with $\mathbf{Cov}[X, Y] < 0$ are negatively correlated.
- Two variables X, Y with $\mathbf{Cov}[X, Y] = 0$ are uncorrelated.

Illustration of 3 Cases for Cov $[X, Y]$

500 outcomes of randomly generated pairs of RVs (X, Y) with different joint distributions

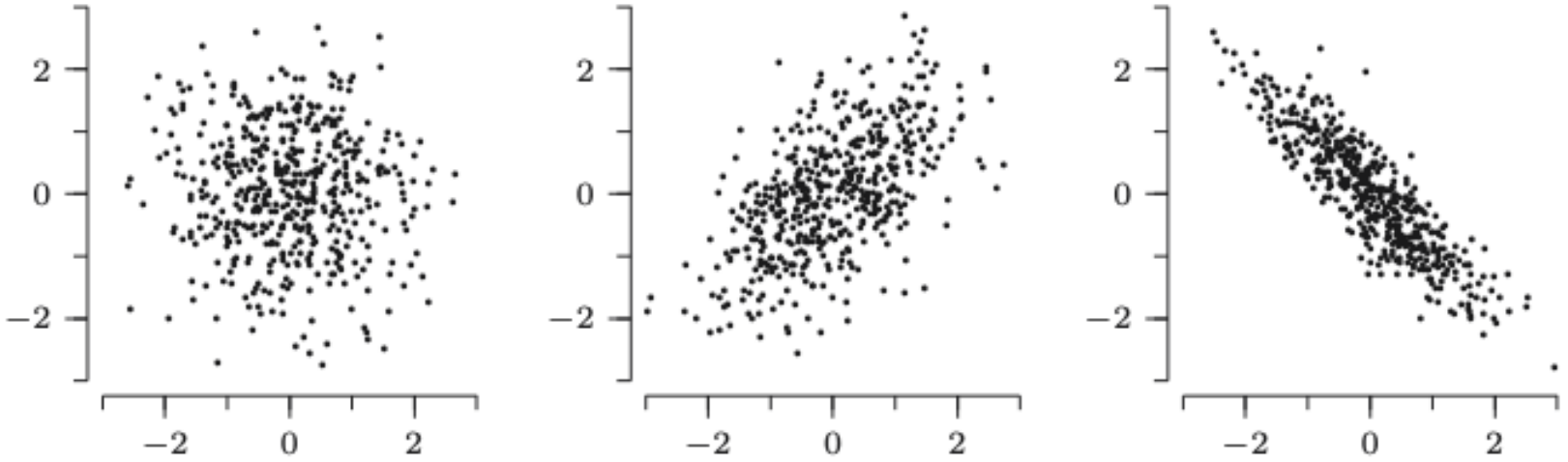

Fig. 10.1. Some scatterplots.
Source: Textbook by Dekking

1. What is the covariance (positive, negative, neutral)?

2. Where is the covariance the largest (in magnitude)?

Independence implies Uncorrelated

Example

Let X and Y be two independent random variables. Then X and Y are uncorrelated, i.e., $\mathbf{Cov}[X, Y] = 0$.

Answer

We give a proof for the discrete case:

Uncorrelated may not imply Independence

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- Further, $\mathbf{E}[X] = 0$ (and $\mathbf{E}[Y] = 1/3$), and hence:

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Uncorrelated may not imply Independence

Example

Find a (simple) example of two random variables X and Y which are uncorrelated but dependent.

Answer

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$$1 = \mathbf{P}[X \cdot Y = 0] > \mathbf{P}[X = 0] \cdot \mathbf{P}[Y = 0] = 2/9.$$

Variance of Sums and Covariances

Variance of Sum Formula

- For any two random variables X, Y,

$$\mathbf{V}[X+Y] = \mathbf{V}[X] + \mathbf{V}[Y] + 2 \cdot \mathbf{Cov}[X, Y].$$

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Generalisation of the case where X and Y are even independent!

- For any random variables $X_1, X_2, \ldots, X_n$:

$$\mathbf{V}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{V}[X_i] + 2 \cdot \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbf{Cov}[X_i, X_j].$$

Computing Variances of Sums of Uncorrelated Variables

Example

Recall the example where $X \in \{-1, 0, +1\}$ uniformly and $Y := \mathbf{1}_{X=0}$. Compute $\mathbf{V}[X + Y]$.

Answer

Correlation Coefficient: Normalising the Covariance

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Range of the Correlation Coefficient

Example

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Answer

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&= 2 \cdot (1 + \rho(X, Y)).
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$$

Introduction to Probability

Lecture 8: Basic Inequalities and Law of Large Numbers

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

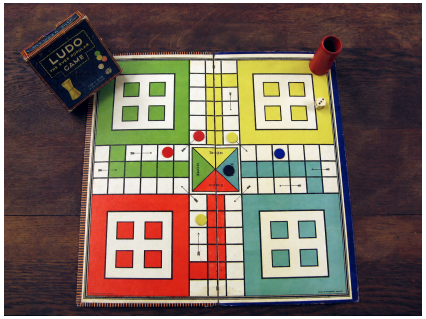
Easter 2026

Compiled: May 19, 2026 at 18:29

Outline

Board Games Involving Dice

- Games with One Dice: ⚀



Board Games Involving Dice

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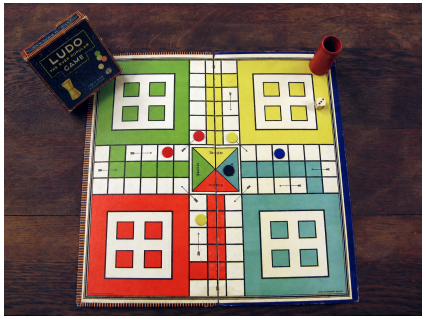

- Games with Two Dice: ⚁⚃

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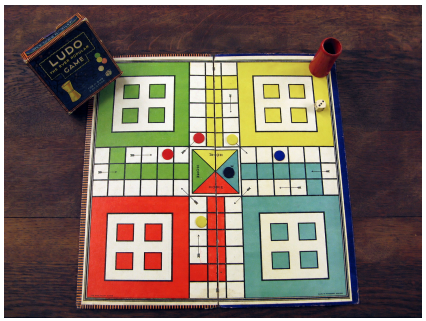
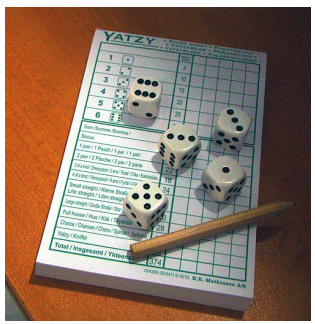


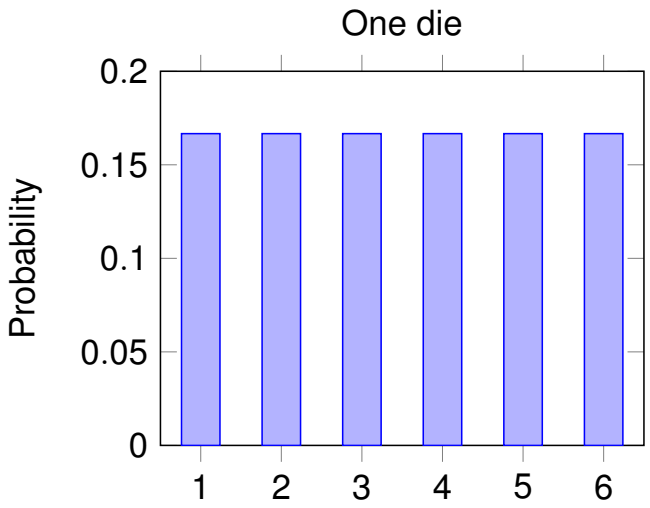
- Games with Two Dice: ⚁⚃

- Games with Five Dice: ⚁⚃⚄⚀⚄



Source: All images from Wikipedia.

Joint Distributions of Sums



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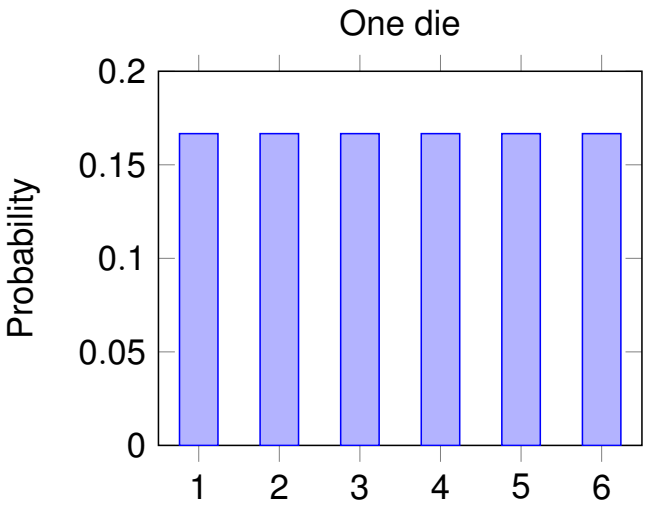

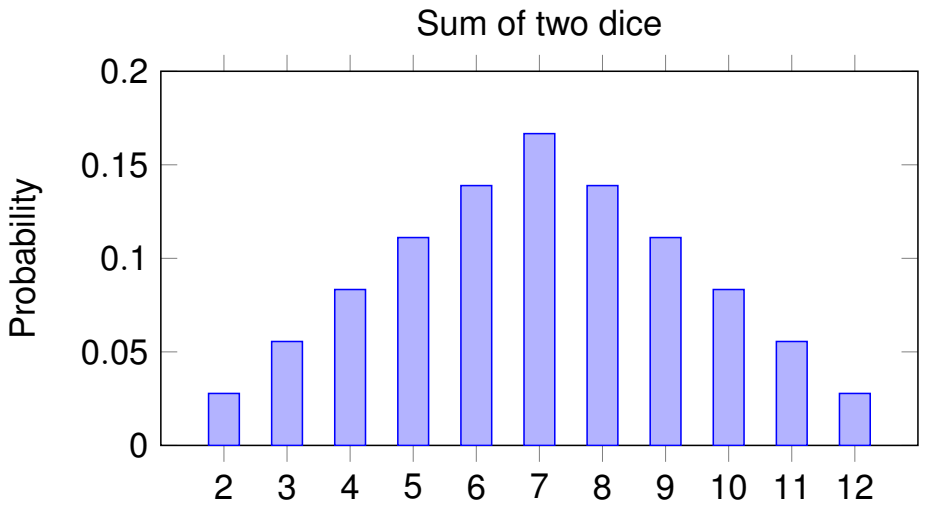

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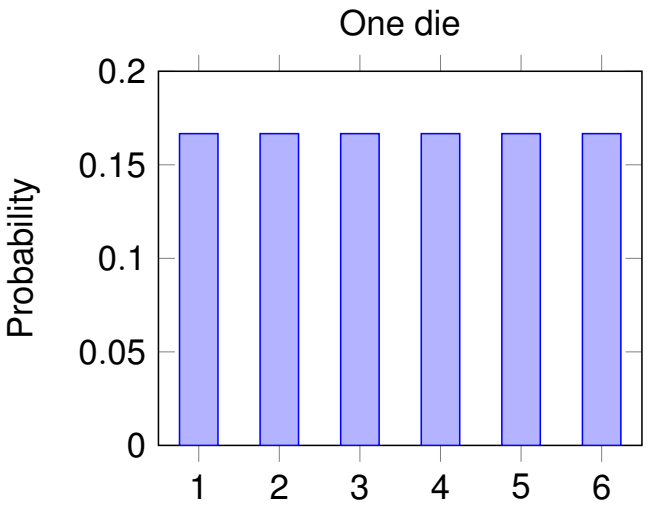
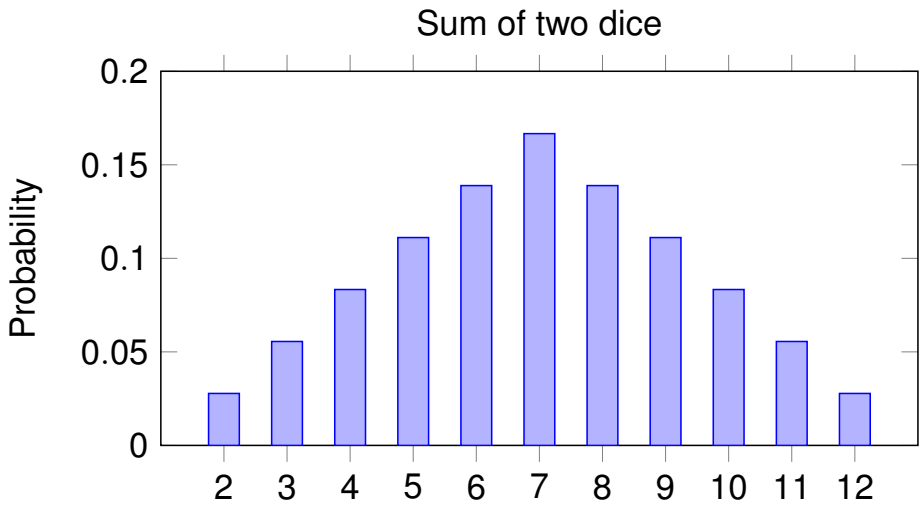
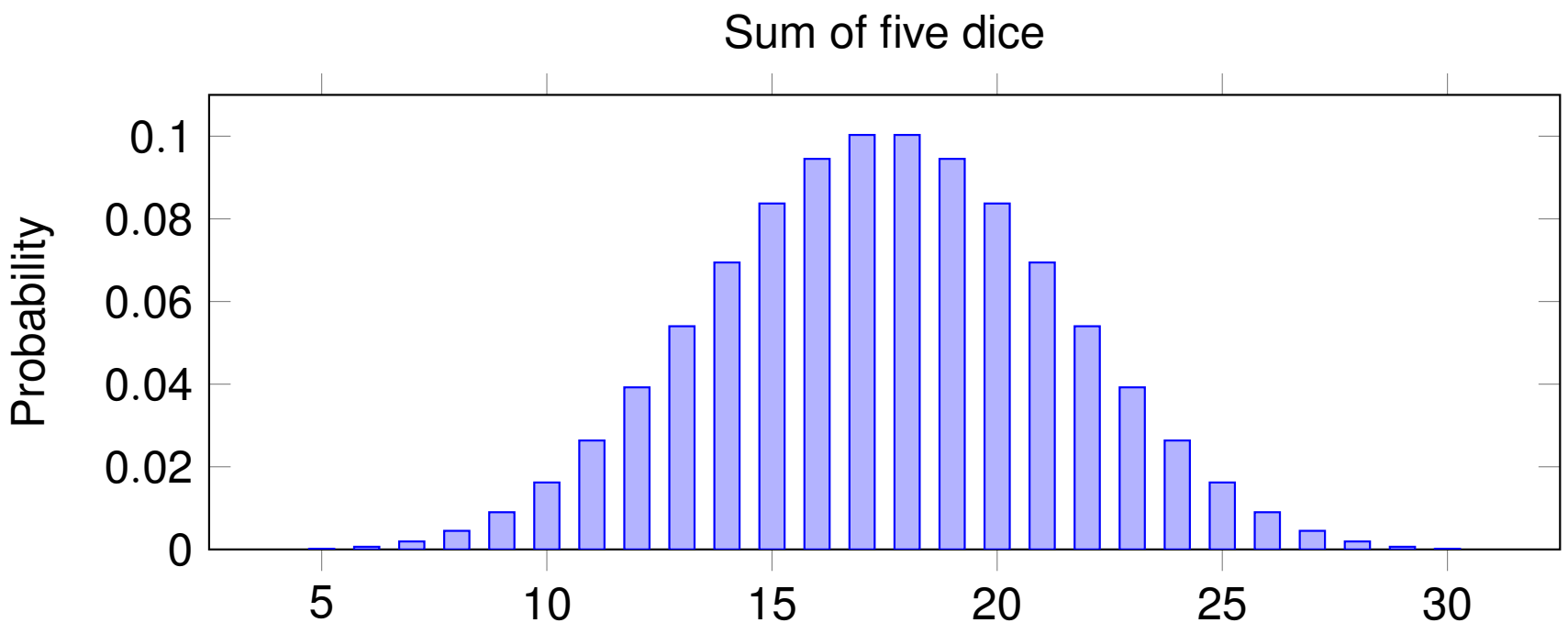
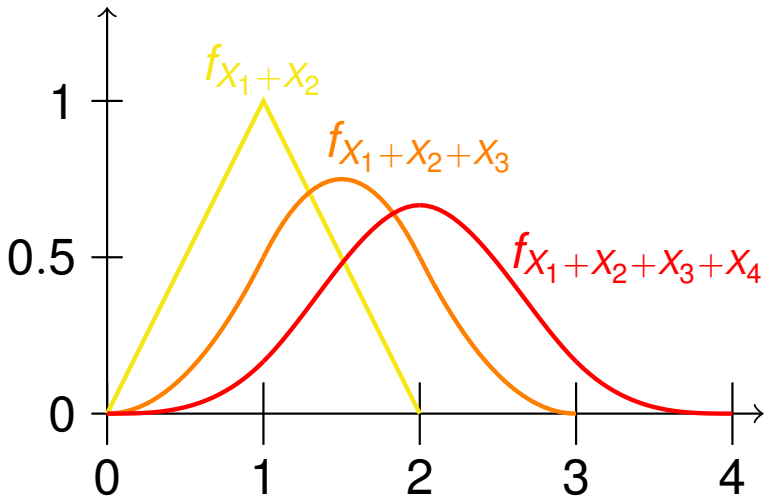


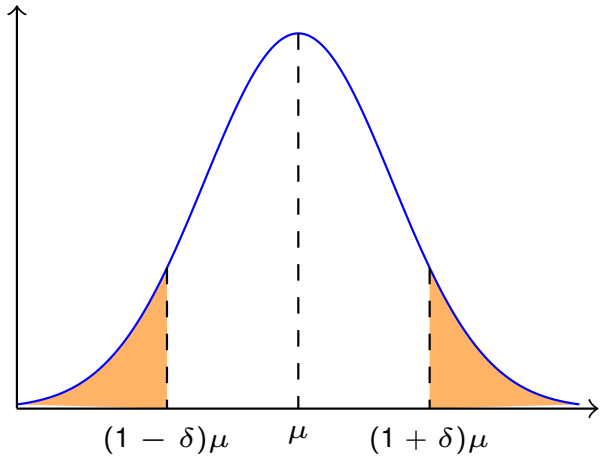




Motivation

We will study sums of independent and identically distributed variables. How does their distribution look like, and how well do they concentrate around the expectation?





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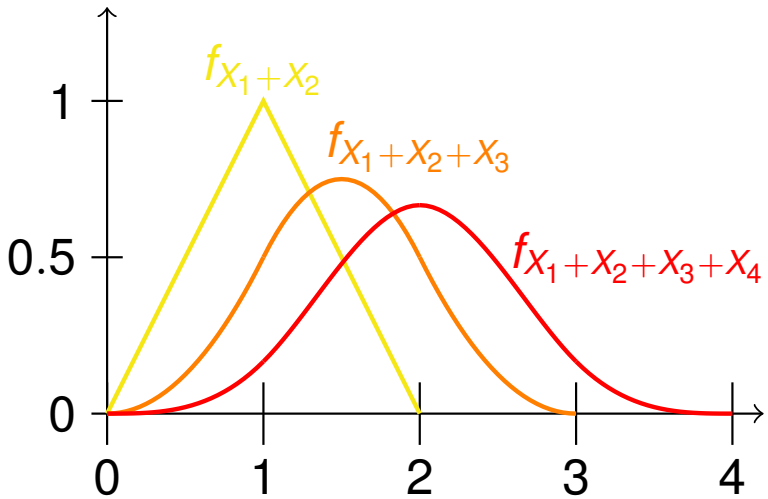

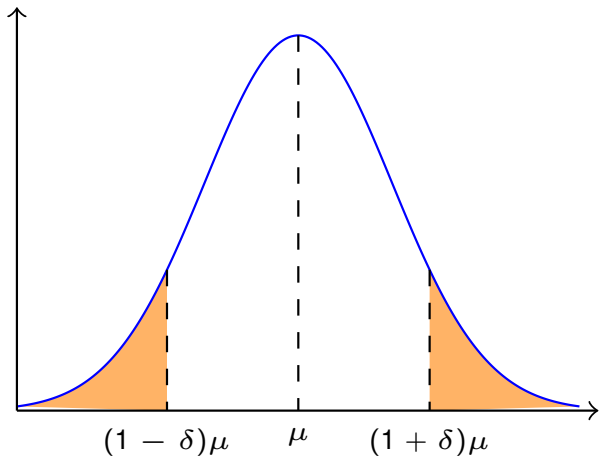

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2. Chebyshev's inequality
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4. Central Limit Theorem

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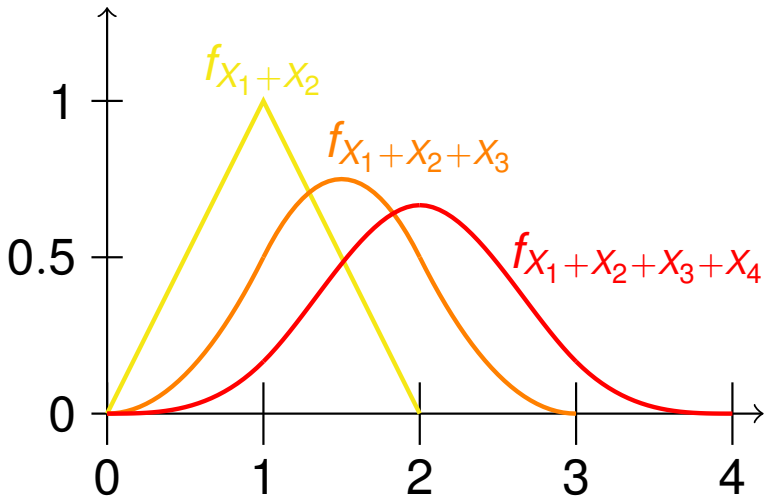

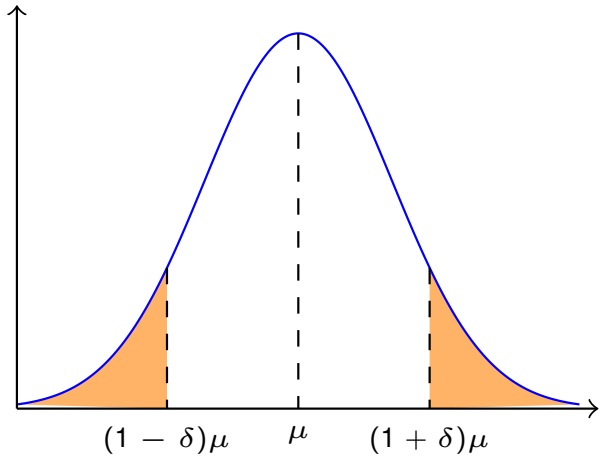

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2. Chebyshev's inequality
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Re-use concepts from previous lectures:

1. Independence (Random Var.) (Lec. 1, 7)
2. Expectation and Variance (Lec. 2, 3)
3. Normal Distribution (Lec. 5)
4. Sums of Random Variables (Lec. 6)

Outline

Markov's Inequality

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For any non-negative random variable X with finite $\mathbf{E}[X]$, it holds for any $a > 0$,

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A. Markov (1856-1922)

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- Advantage: Very basic inequality, we only need to know $\mathbf{E}[X]$
- Downside: For many distributions, the tail bound might be quite loose
- Proof is similar to the proof of Chebyshev's inequality (Exercise!)

Applying Markov's Inequality

Example 2

Consider throwing an unbiased, six-sided dice 120 times and let X denote the number of times we obtain a six.

1. Derive an upper bound on $\mathbf{P}[X \geq 30]$.
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Applying Markov's Inequality

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Consider throwing an unbiased, six-sided dice 120 times and let X denote the number of times we obtain a six.

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Answer

1. First compute $\mathbf{E}[X] = 1/6 \cdot 120 = 20$. Then by Markov:

$$\mathbf{P}[X \geq 30] \leq \frac{20}{30} = \frac{2}{3}.$$

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Both bounds, especially the second, are quite loose!

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- Chebyshev's inequality is also known as Second Moment Method

Derivation of Chebychev's inequality

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- Dividing both sides by a^2 yields the result.

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Example: Chebychev is (usually) much stronger than Markov

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Throw an unbiased coin n times and let X be the total number of heads. In an experiment, with n large, we would usually expect a number of heads that is close to the expectation. Can we justify that?

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Much better! (Inversely) Linear in n

Outline

Law of Large Numbers

The Weak Law of Large Numbers

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- There is also a strong law of large numbers:

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How does a “typical” realisation look like?

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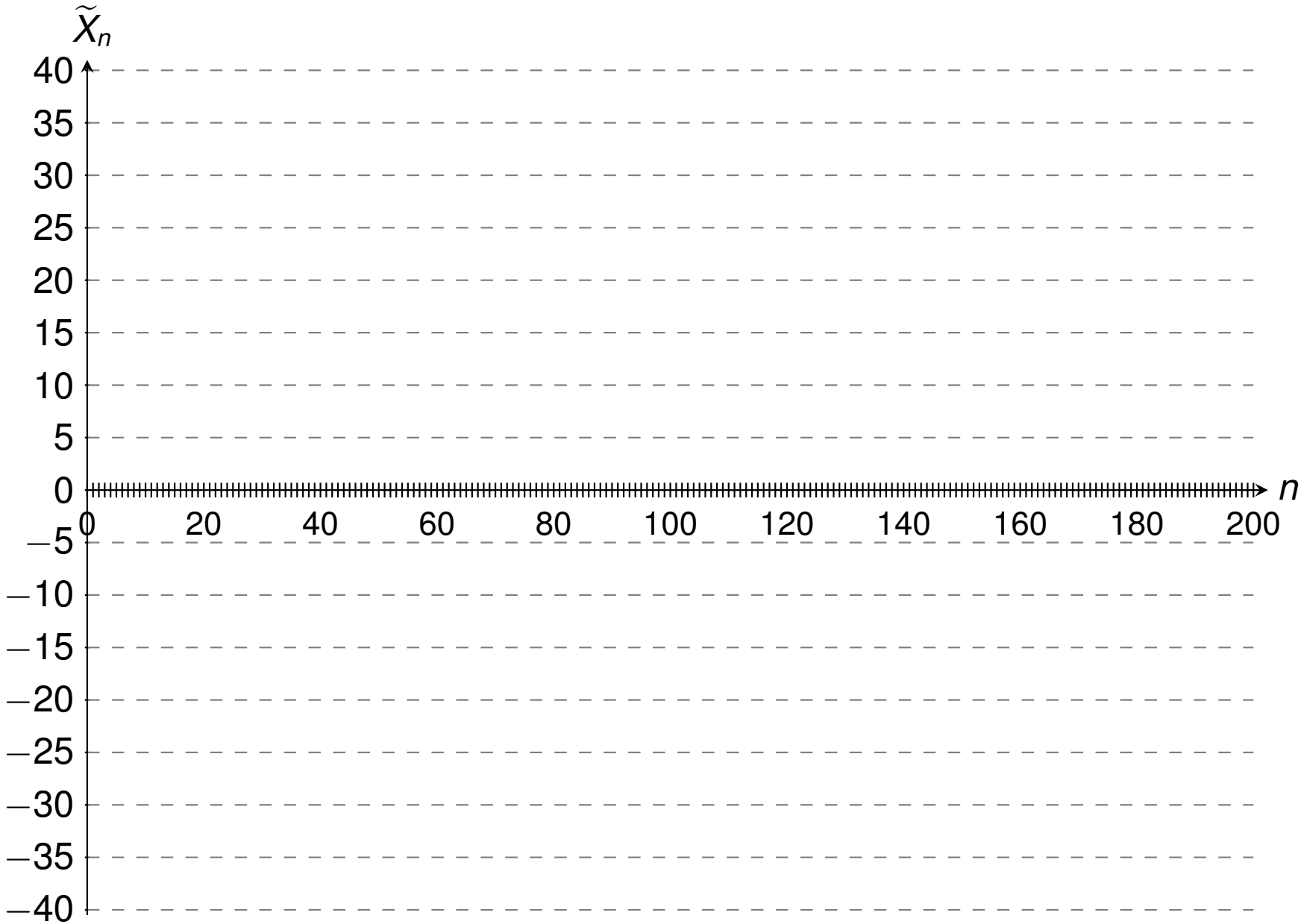

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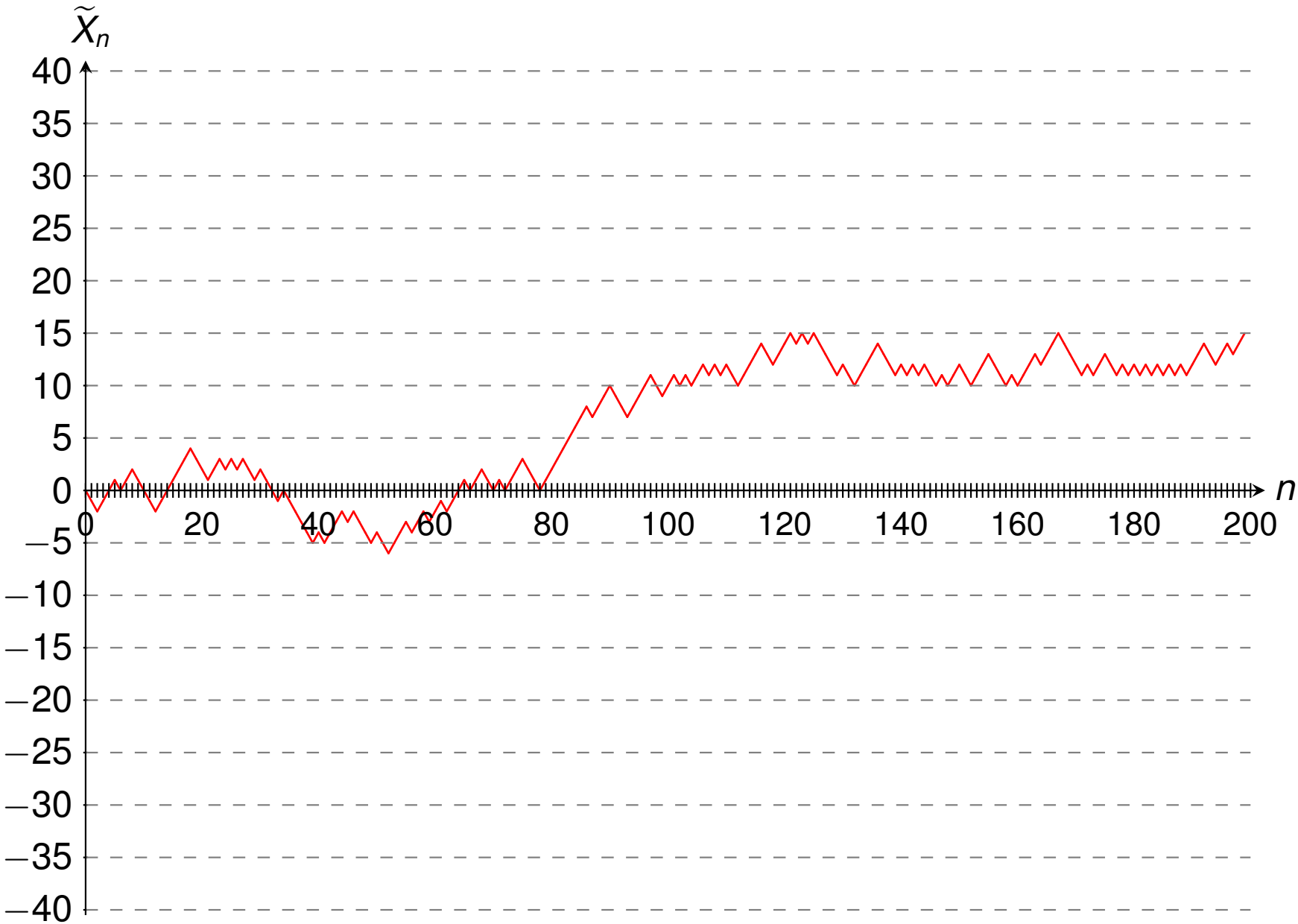

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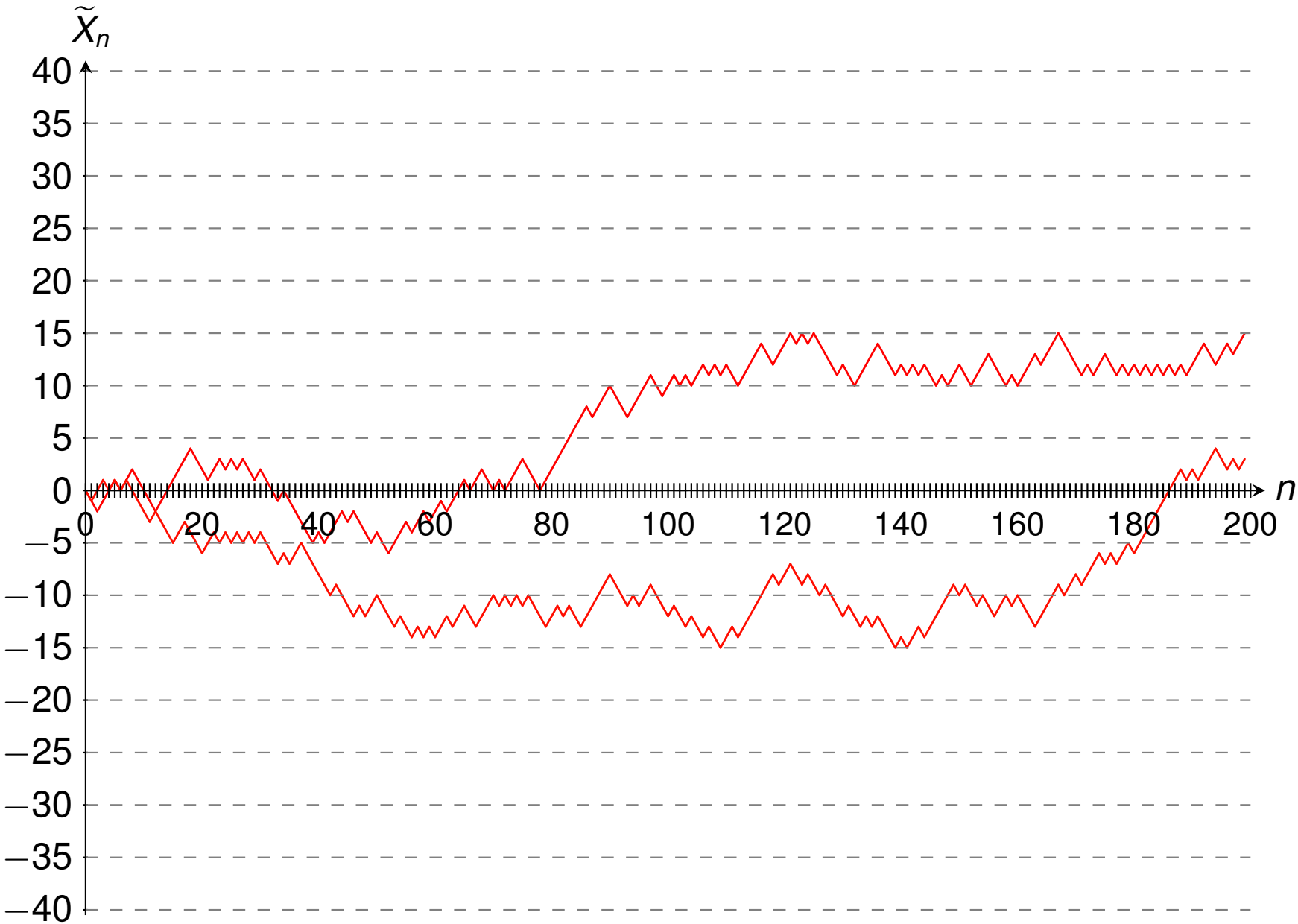

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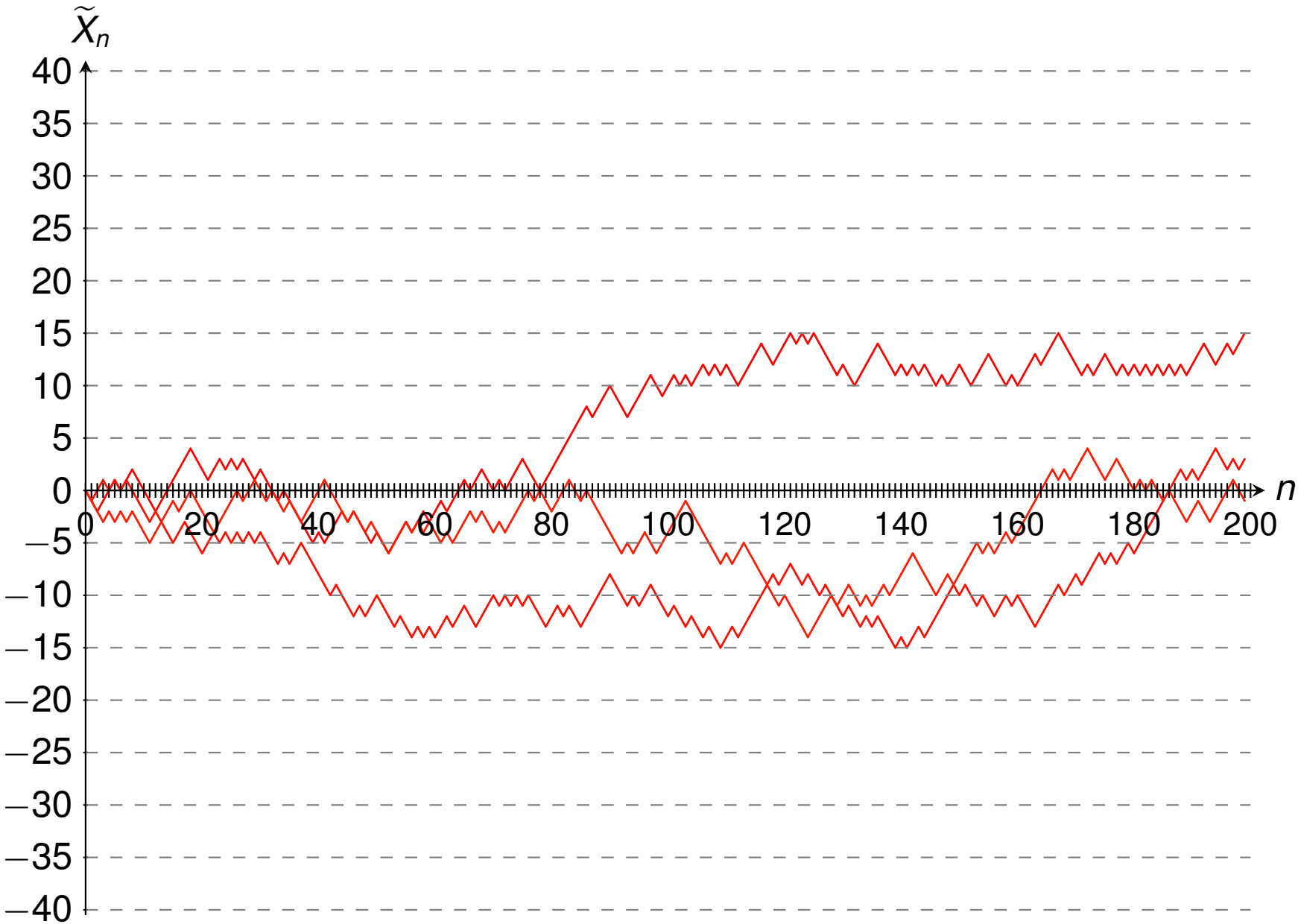

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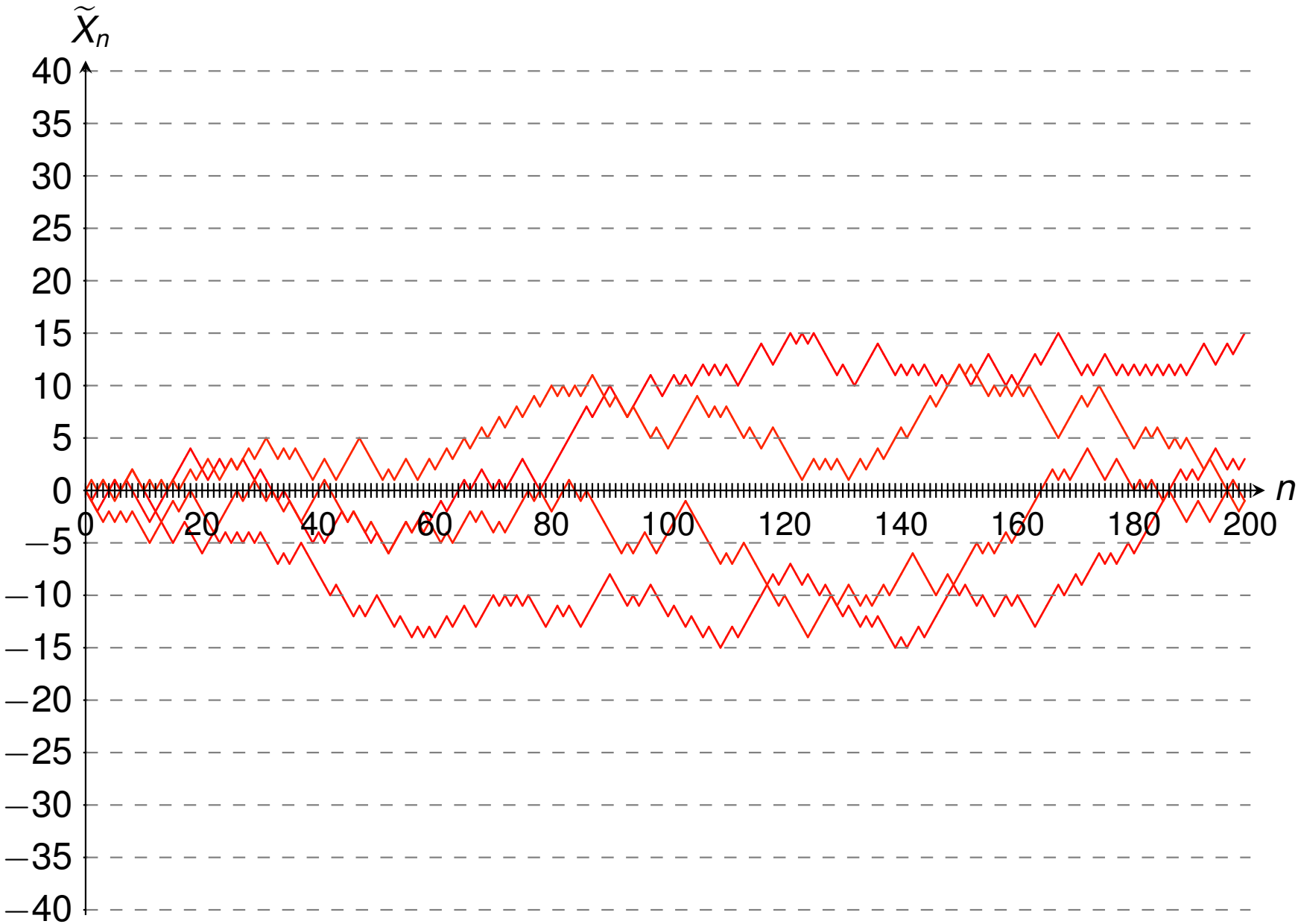

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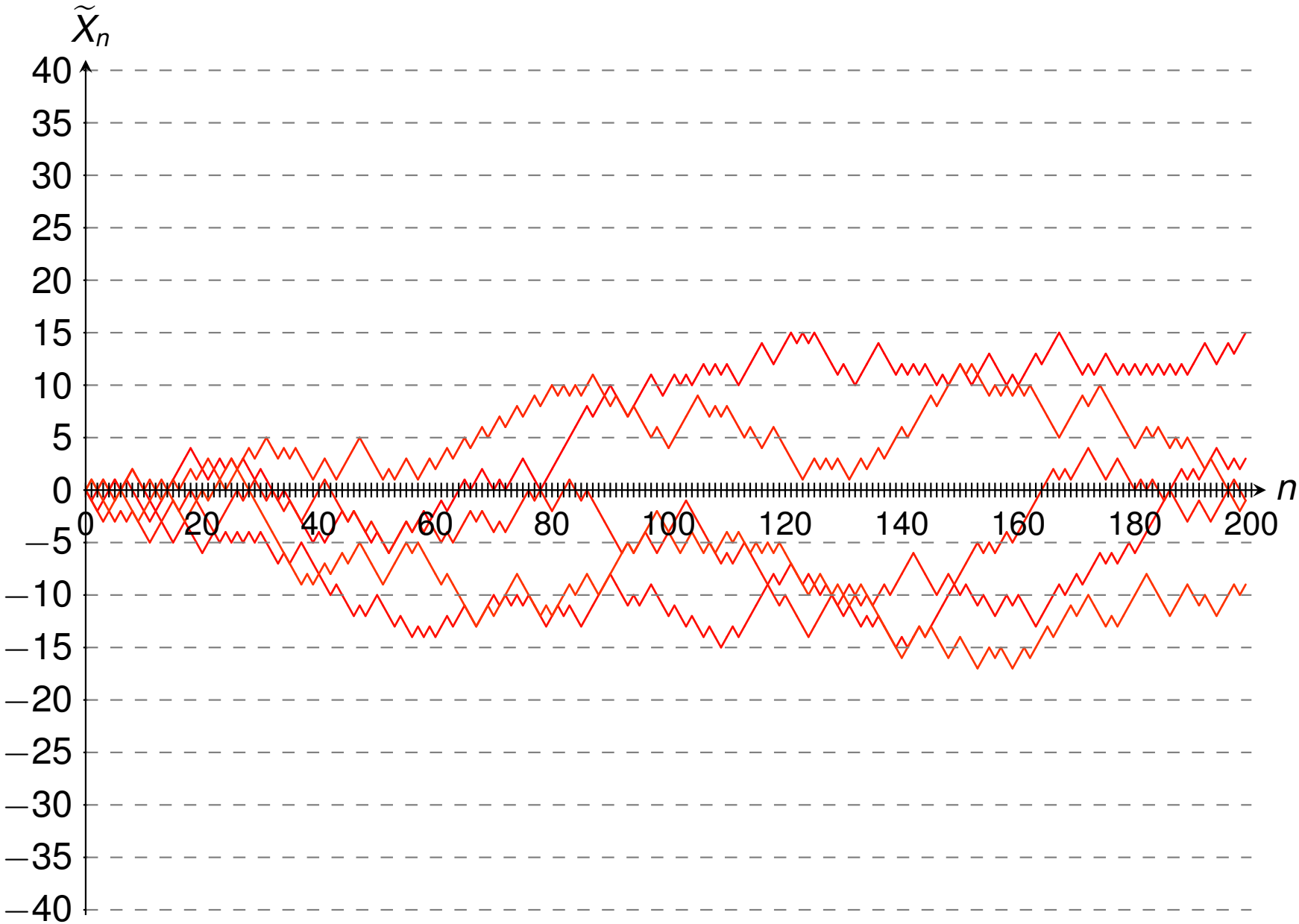

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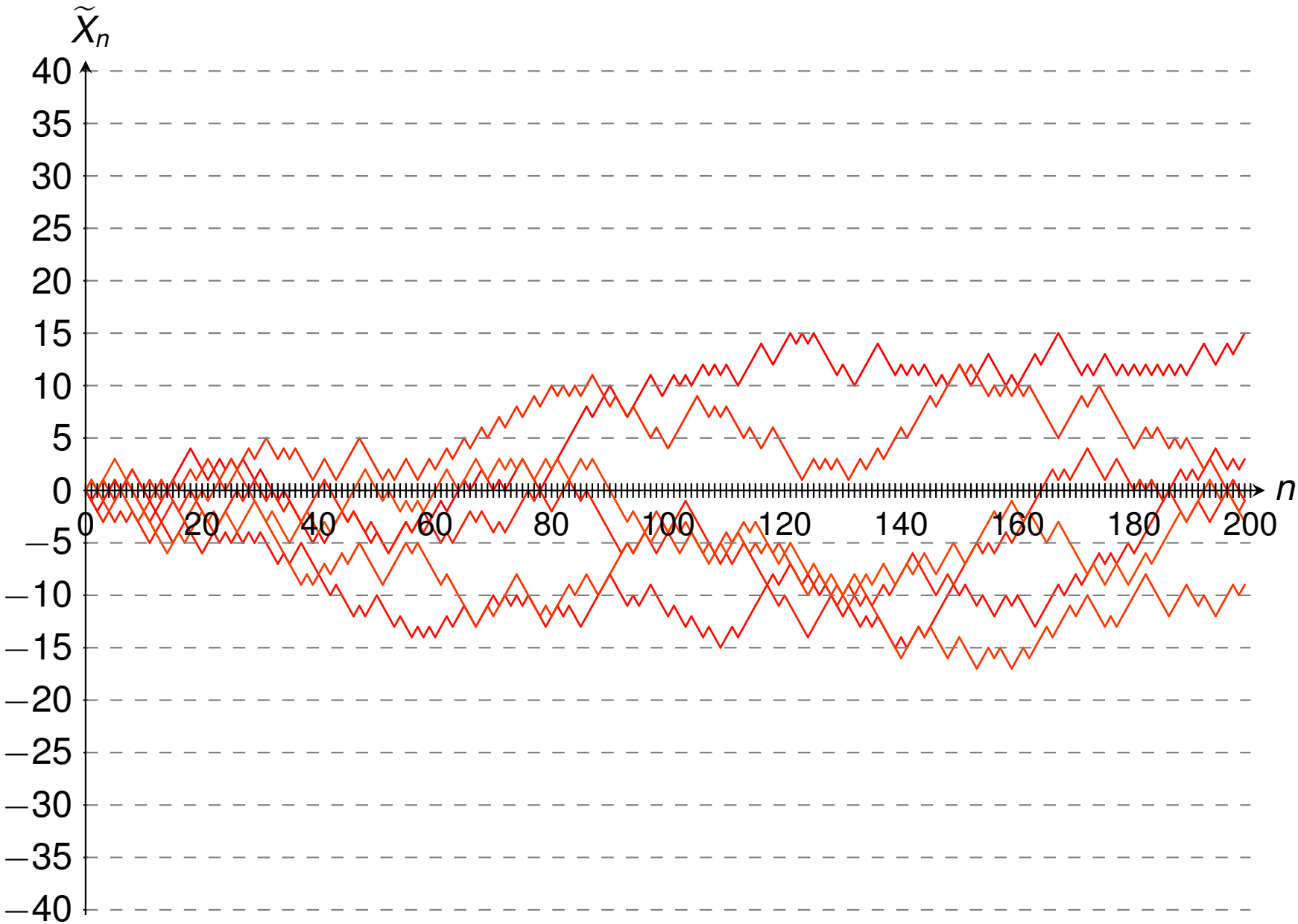

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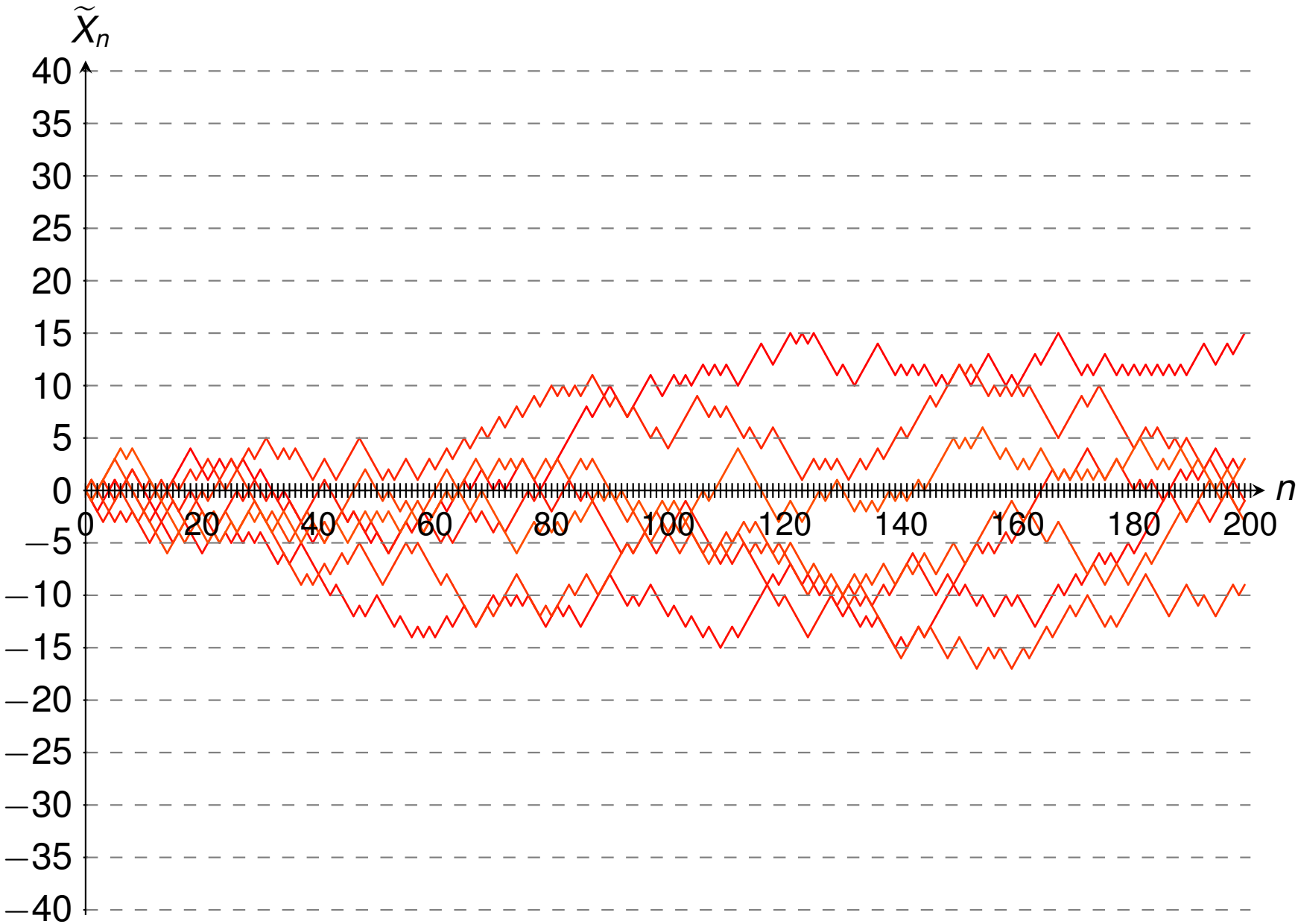

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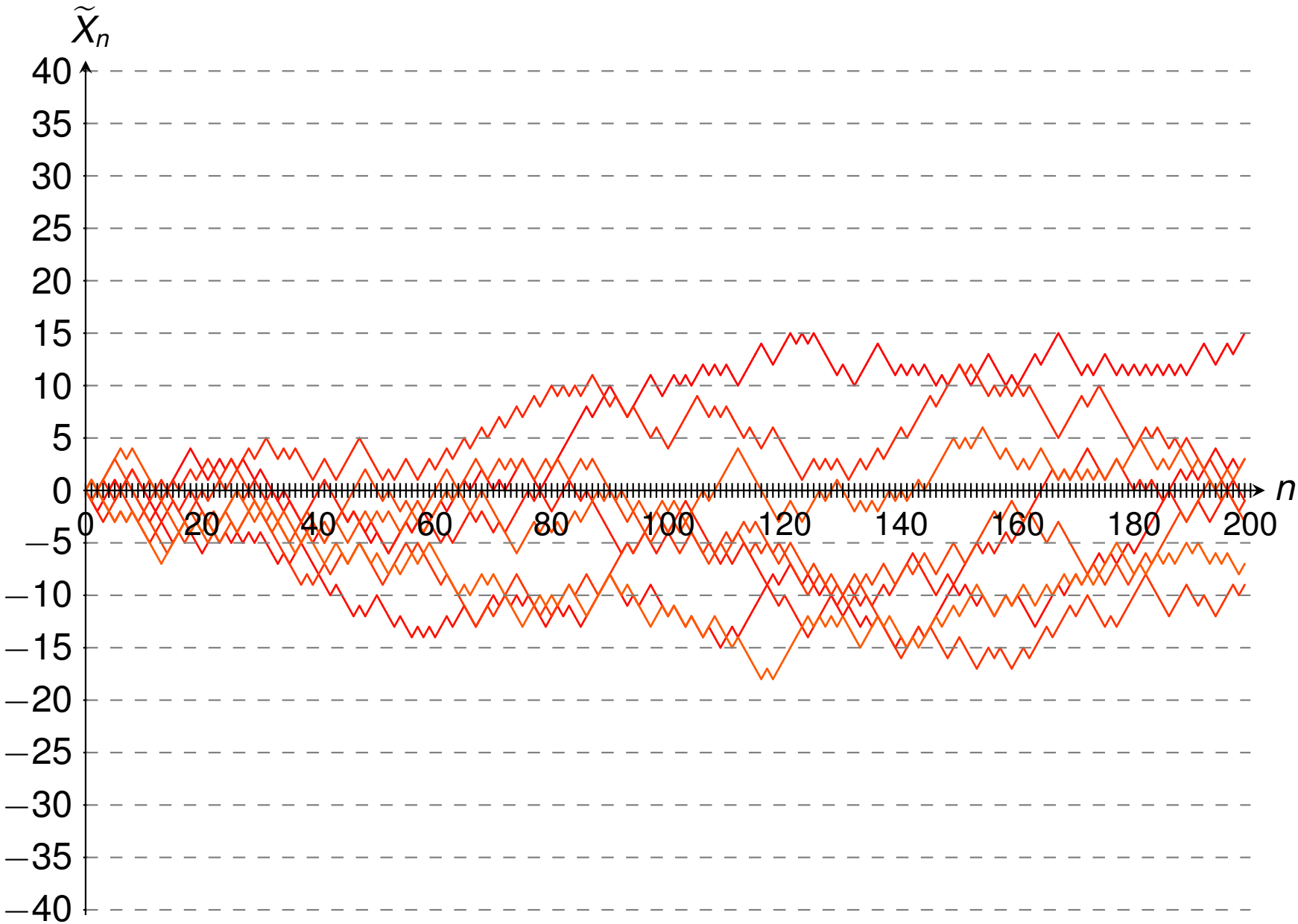

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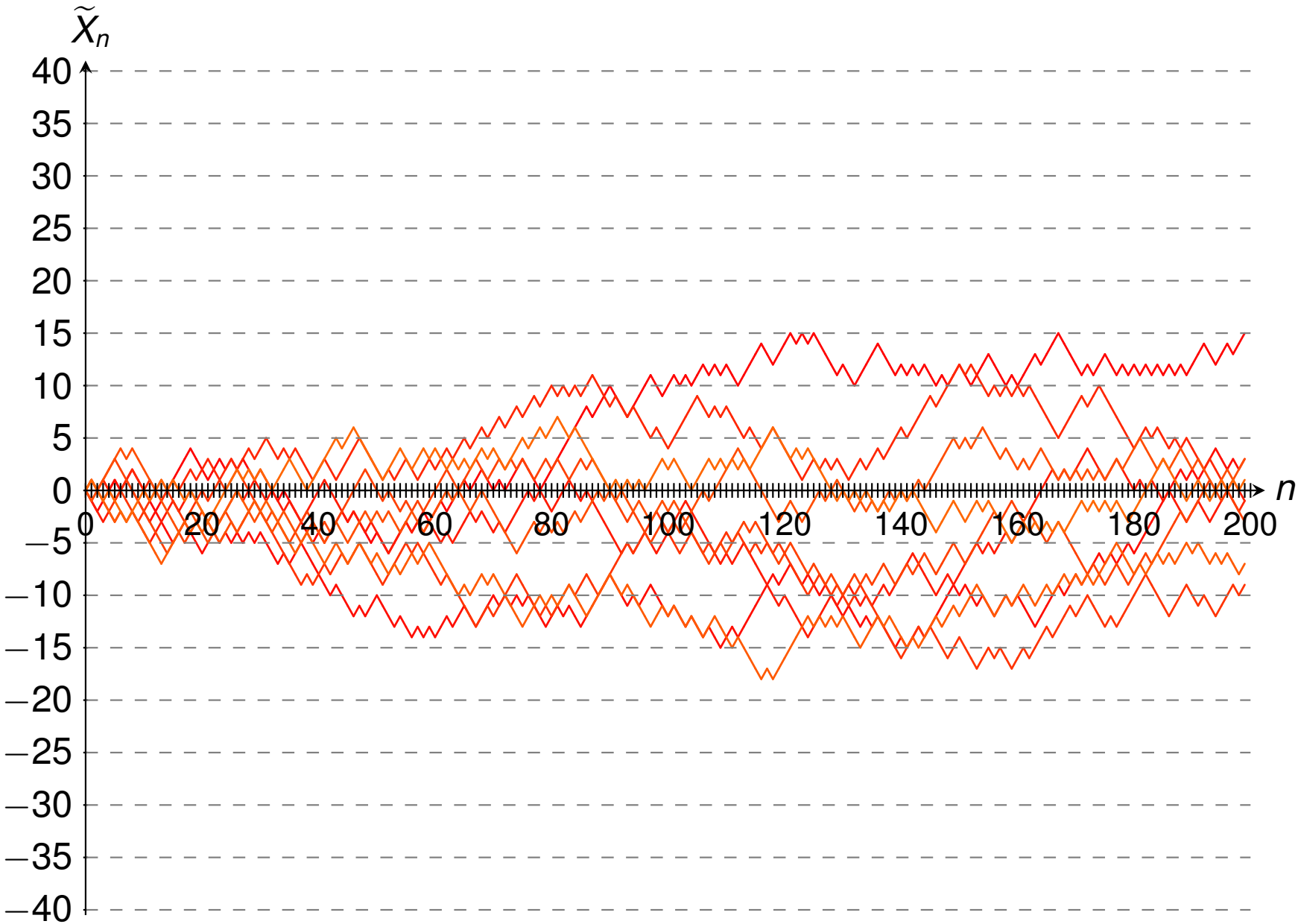

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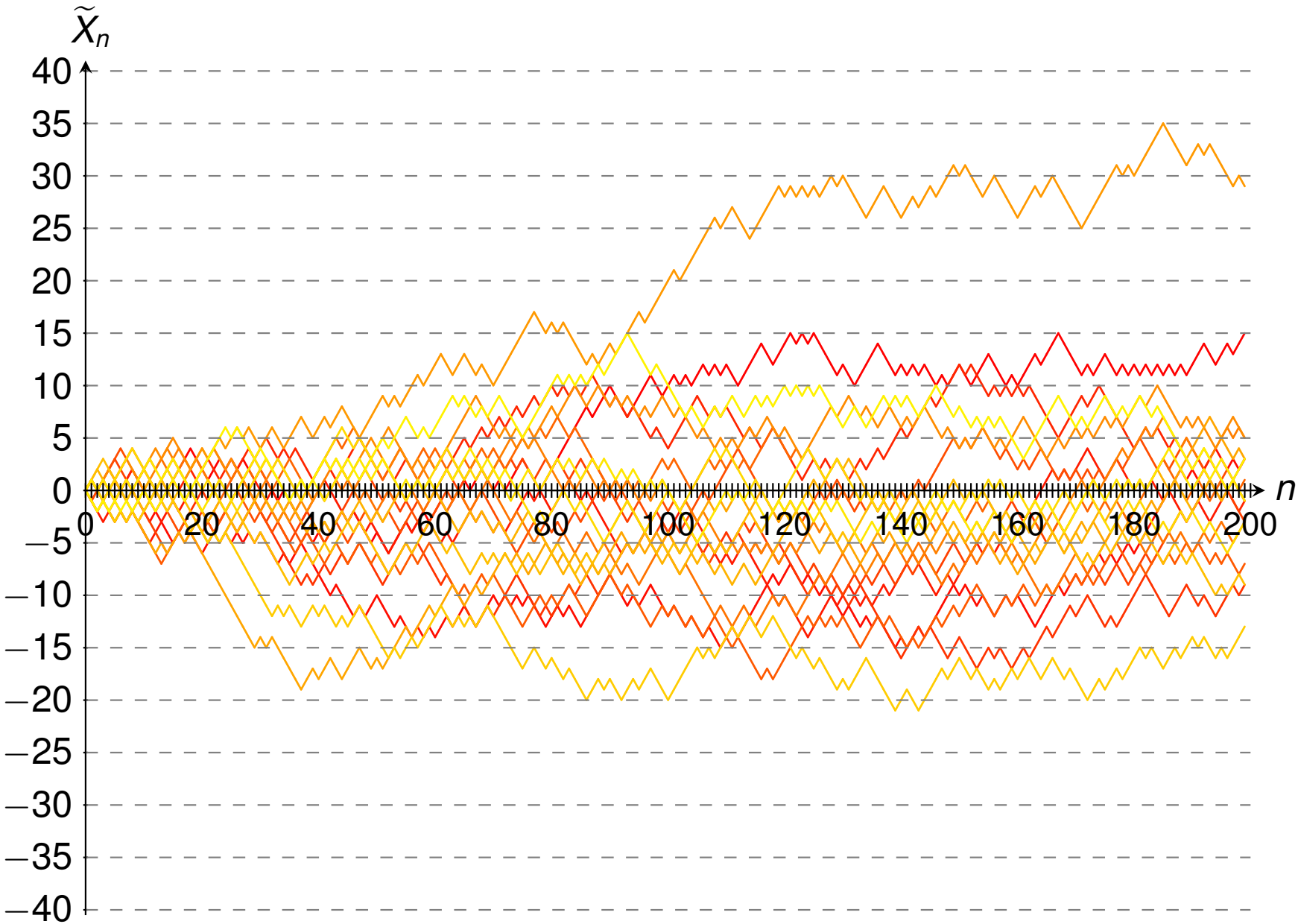

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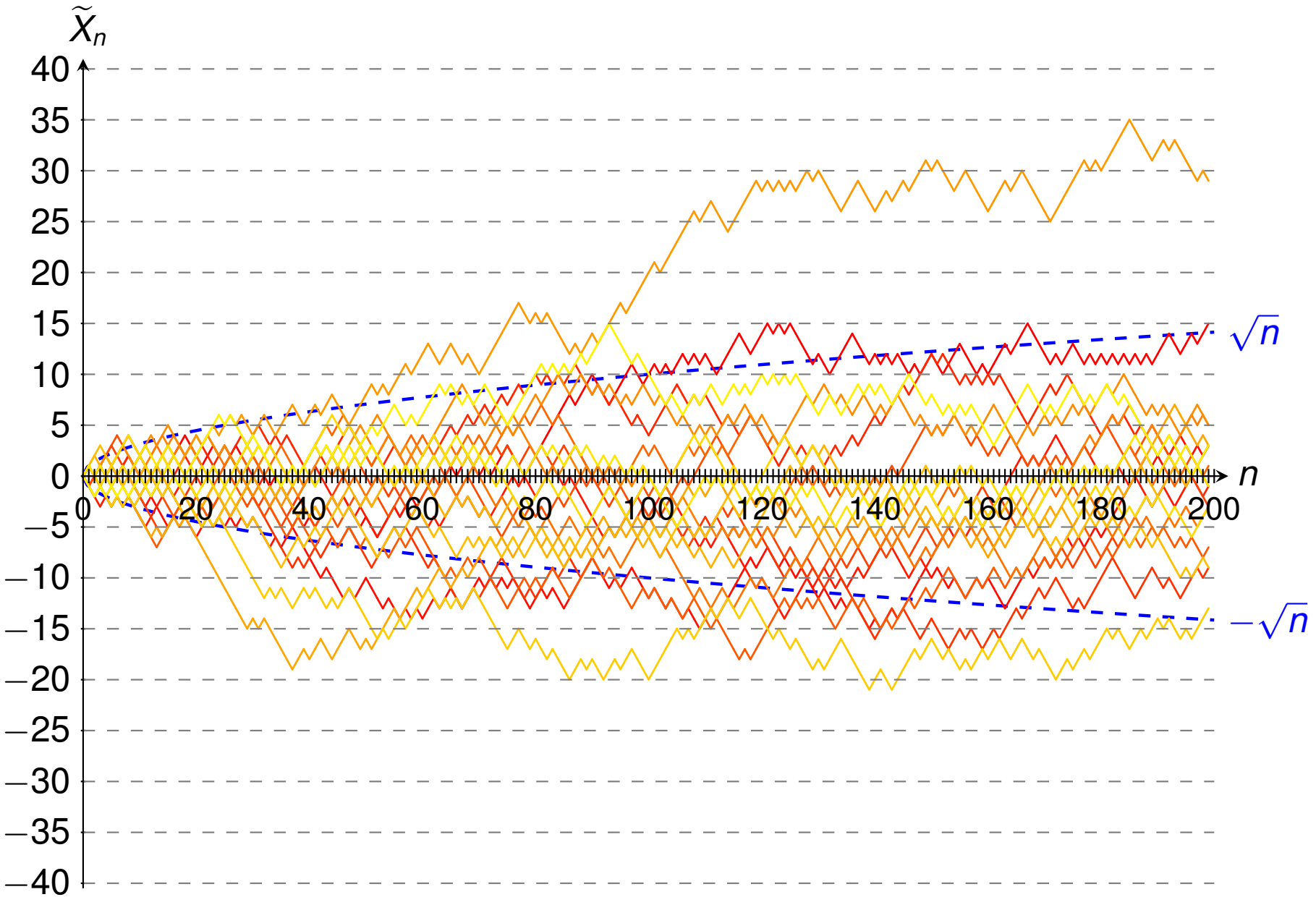
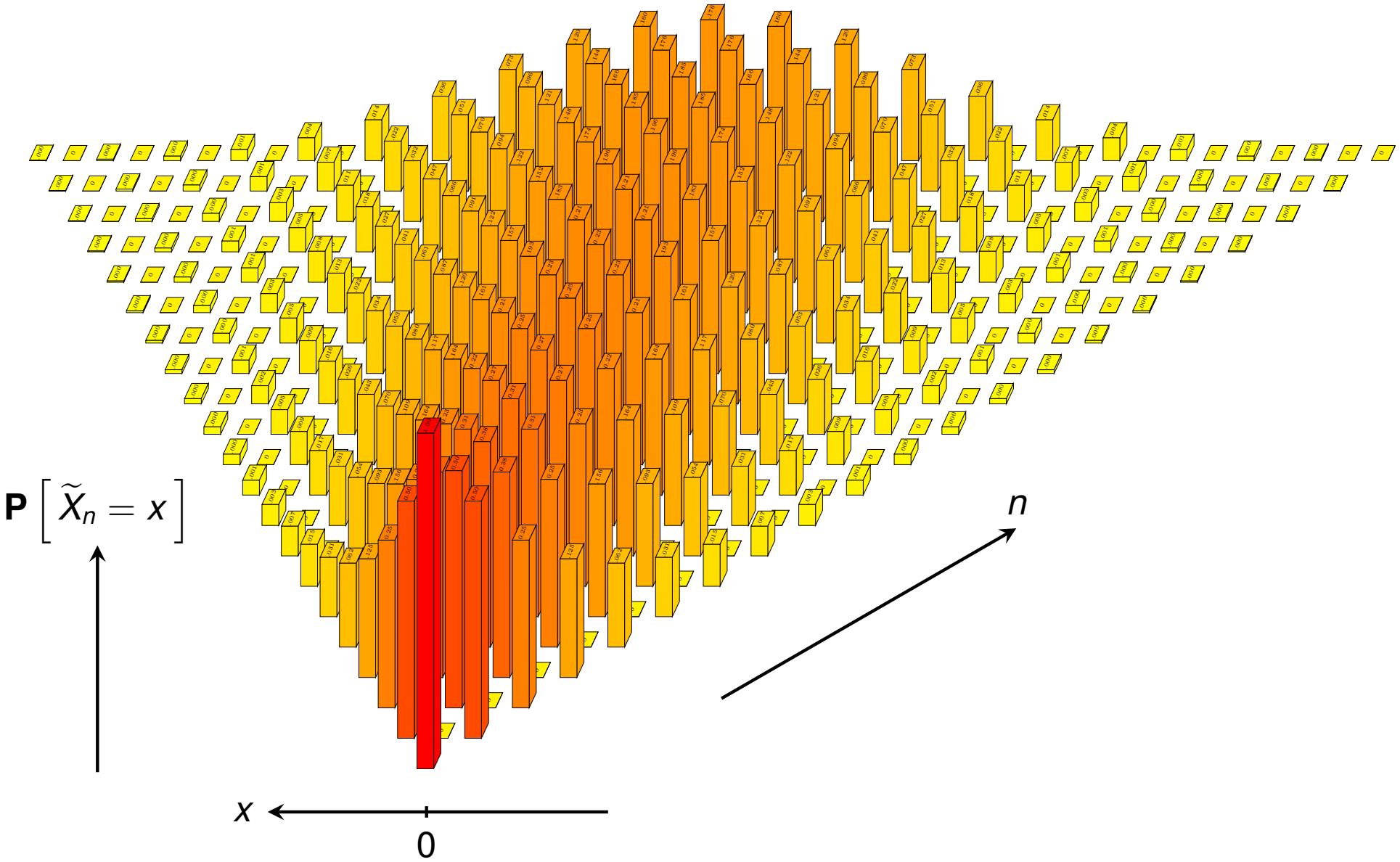
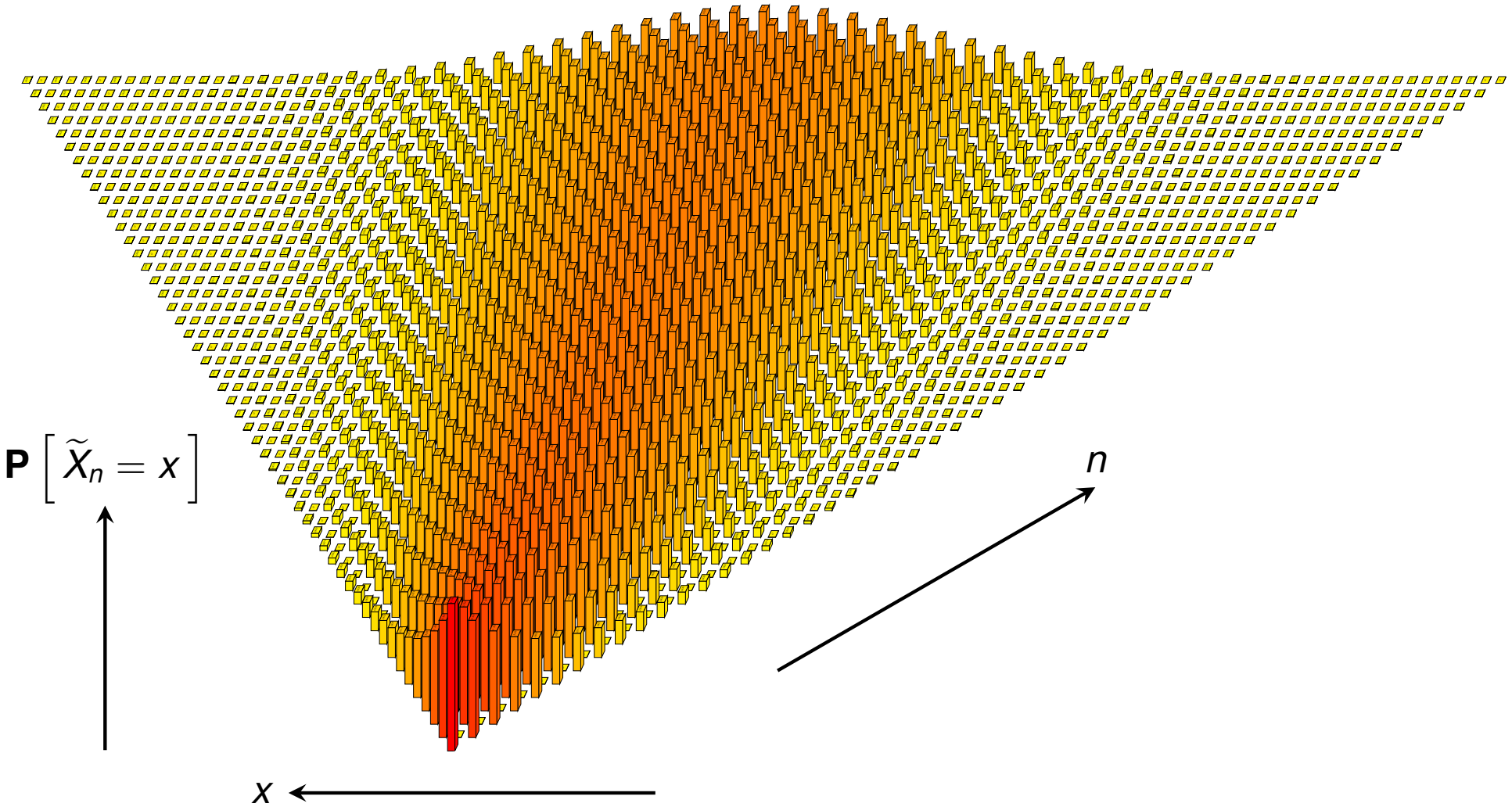


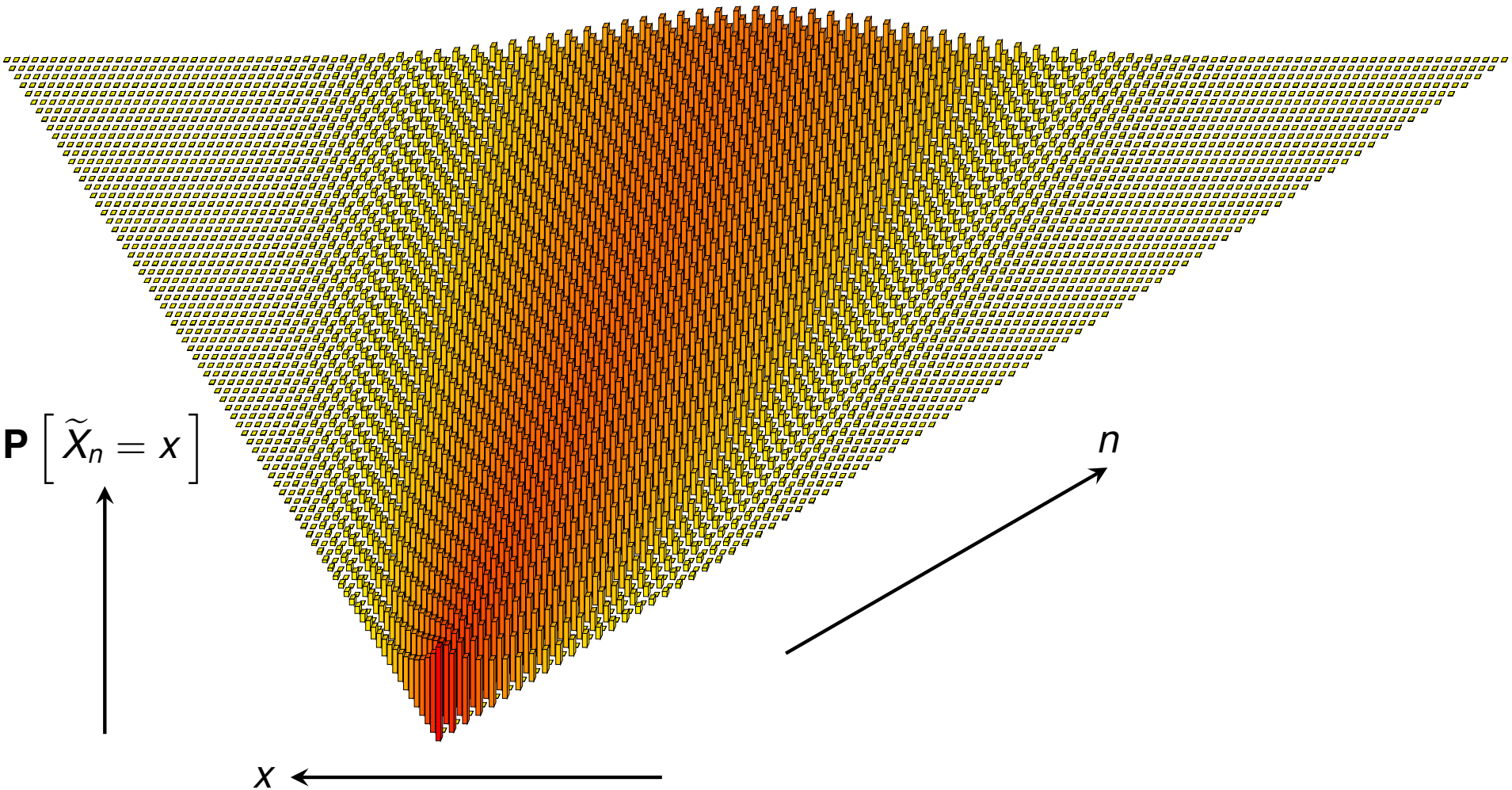
Plot of the Distributions for $n = 0, 1, \ldots, 20$



Plot of the Distributions for $n = 0, 1, \ldots, 50$



Plot of the Distributions for $n = 0, 1, \ldots, 80$



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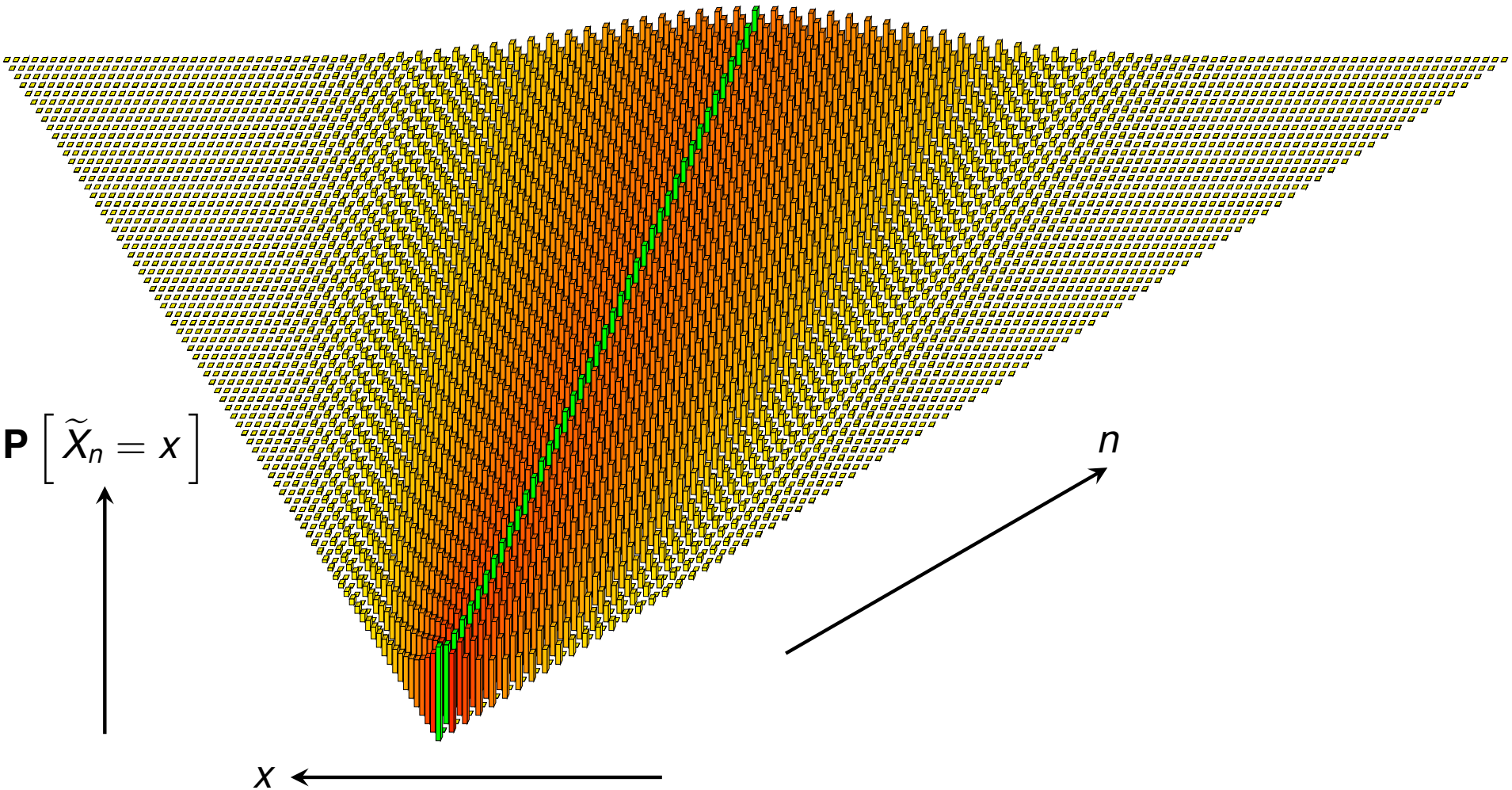
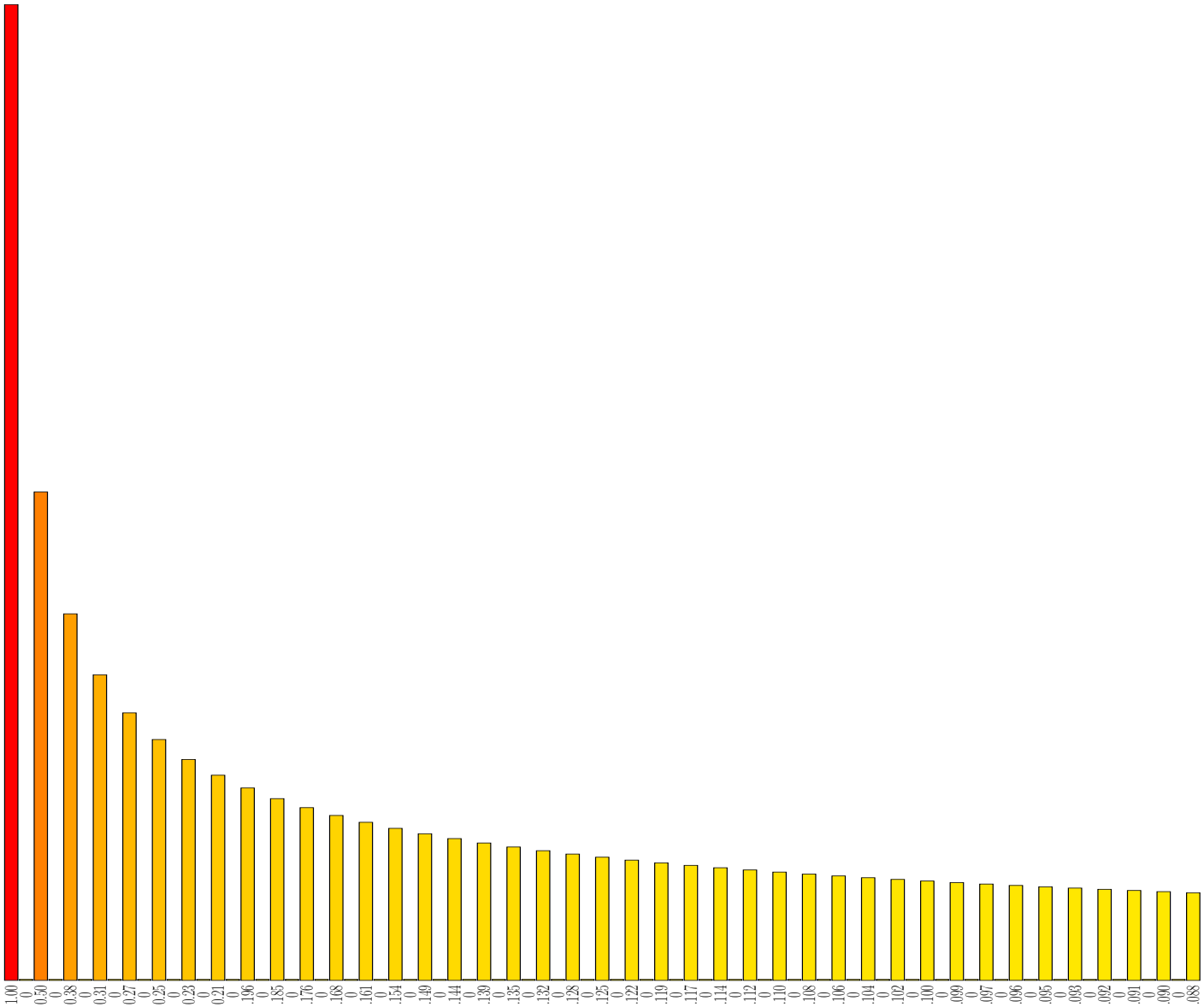


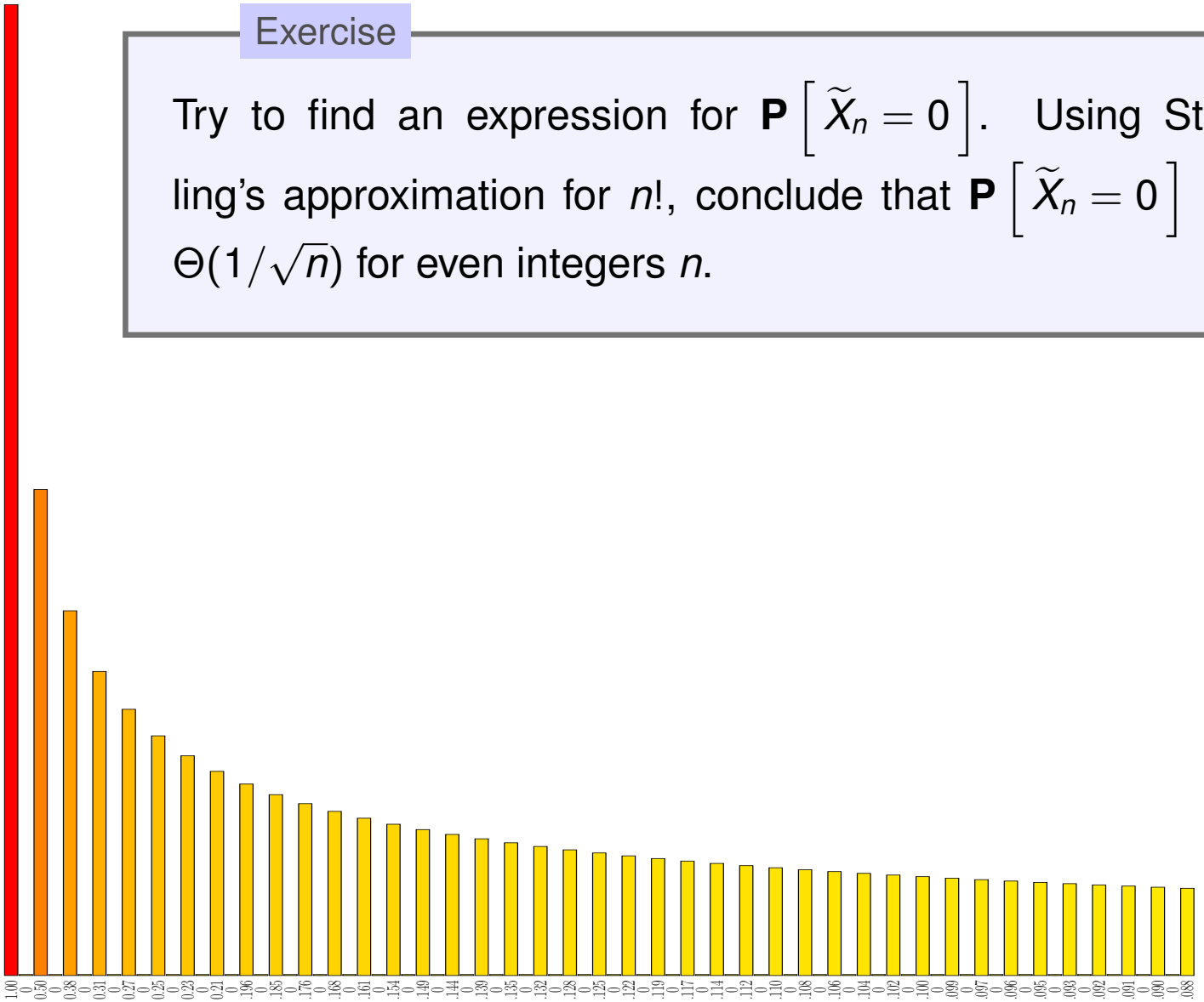
Interlude: Approximation of $\mathbf{P}[\,\widetilde{X}_n = 0\,]$



Interlude: Approximation of $\mathbf{P}[\widetilde{X}_n = 0]$

Exercise

Try to find an expression for $\mathbf{P}\left[\widetilde{X}_n = 0\right]$. Using Stirling's approximation for $n!$, conclude that $\mathbf{P}\left[\widetilde{X}_n = 0\right] = \Theta(1/\sqrt{n})$ for even integers n.



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- Let X_i be independent random variables taking values $\in \{-1, +1\}$ with probability $1/2$ each
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Consider now the average (sample mean): $\overline{X}_n := 1/n \cdot \sum_{i=1}^{n} X_i$.

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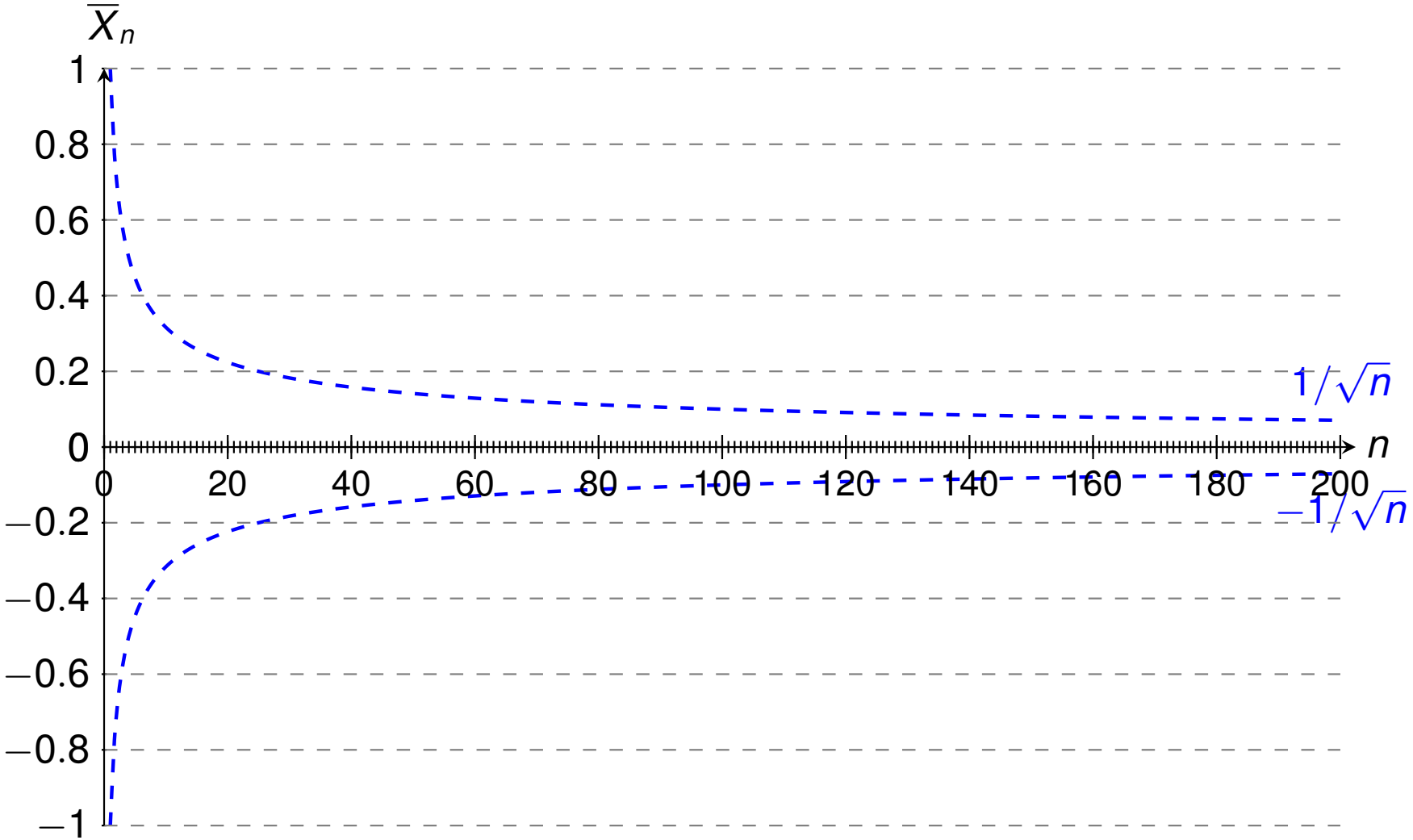

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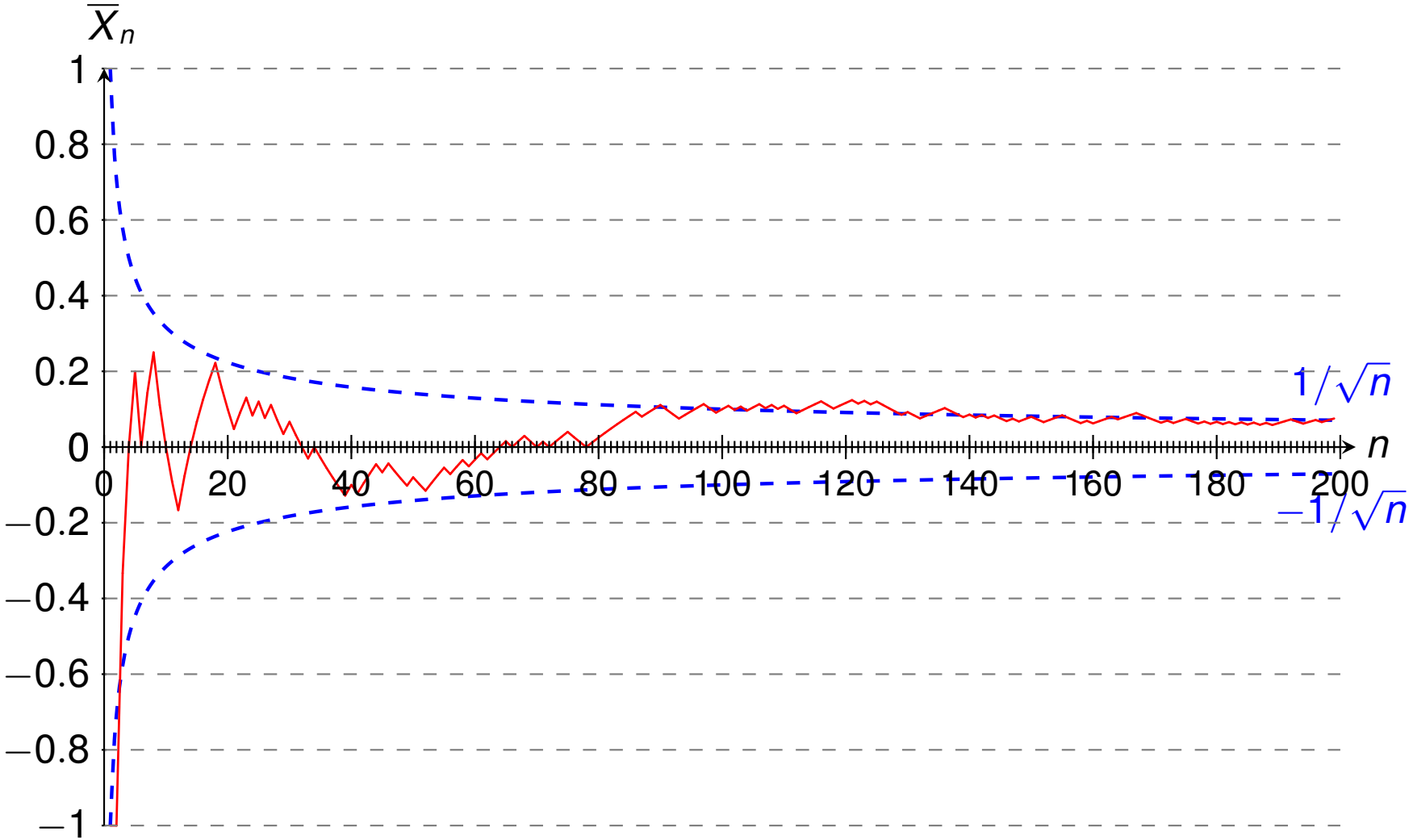

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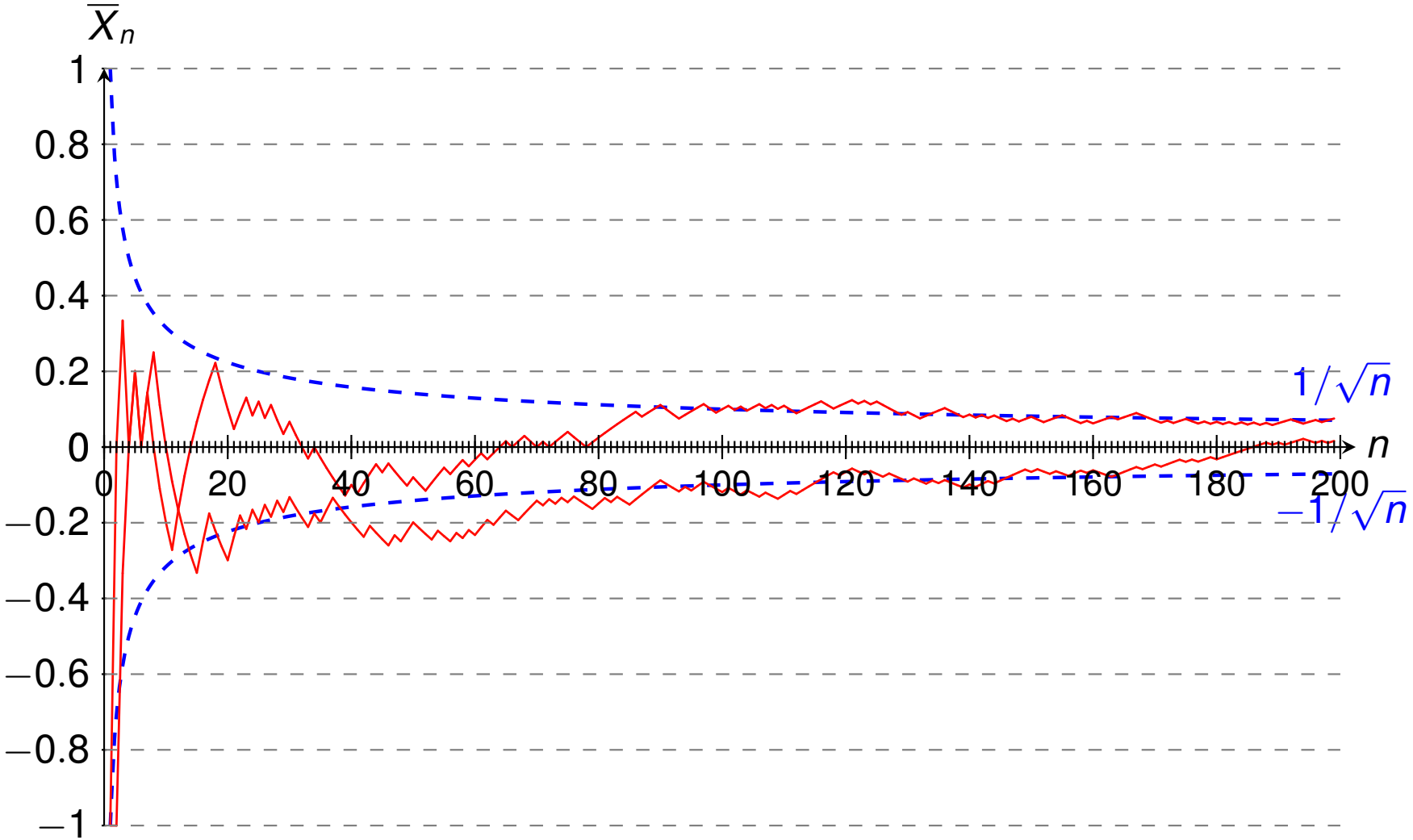

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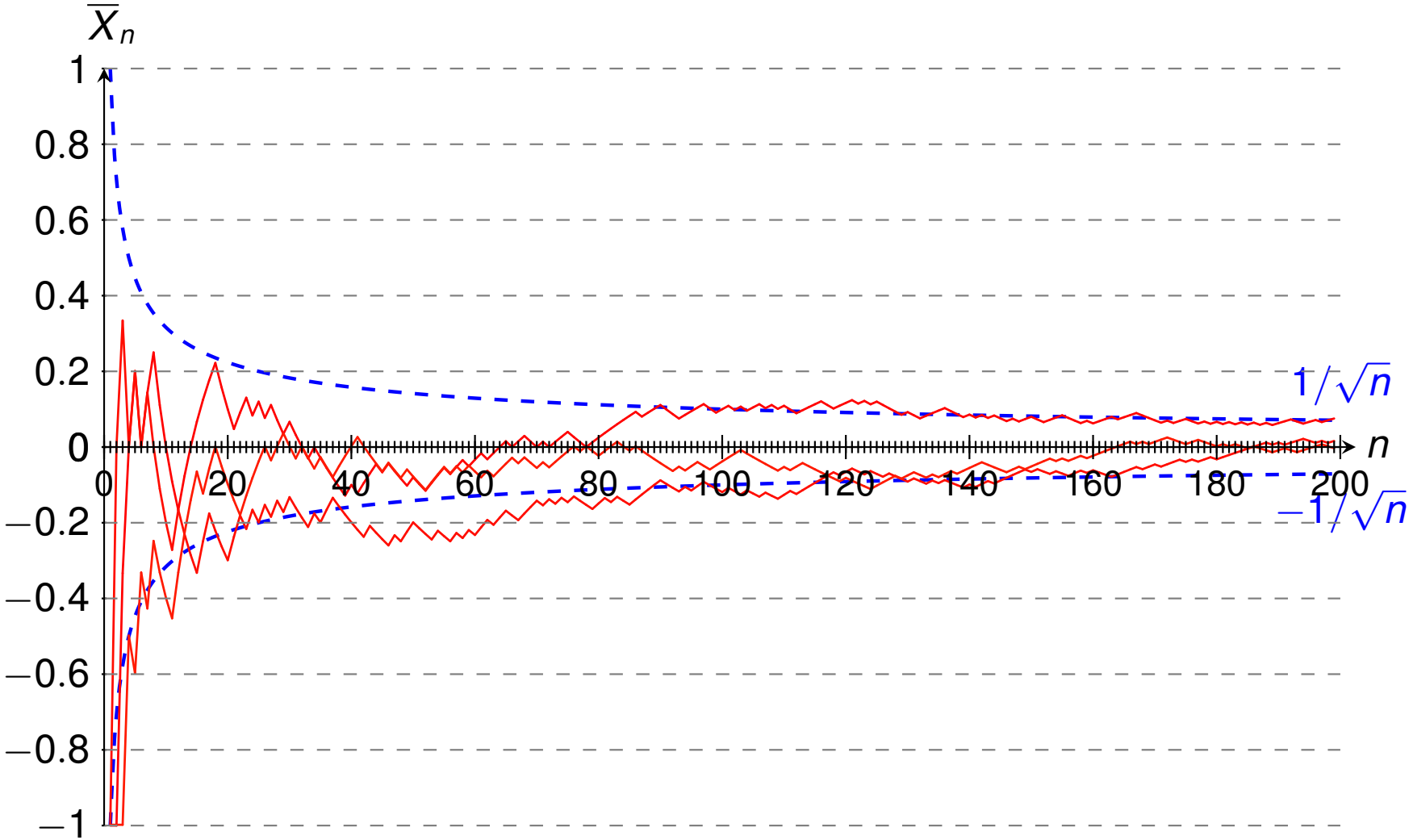

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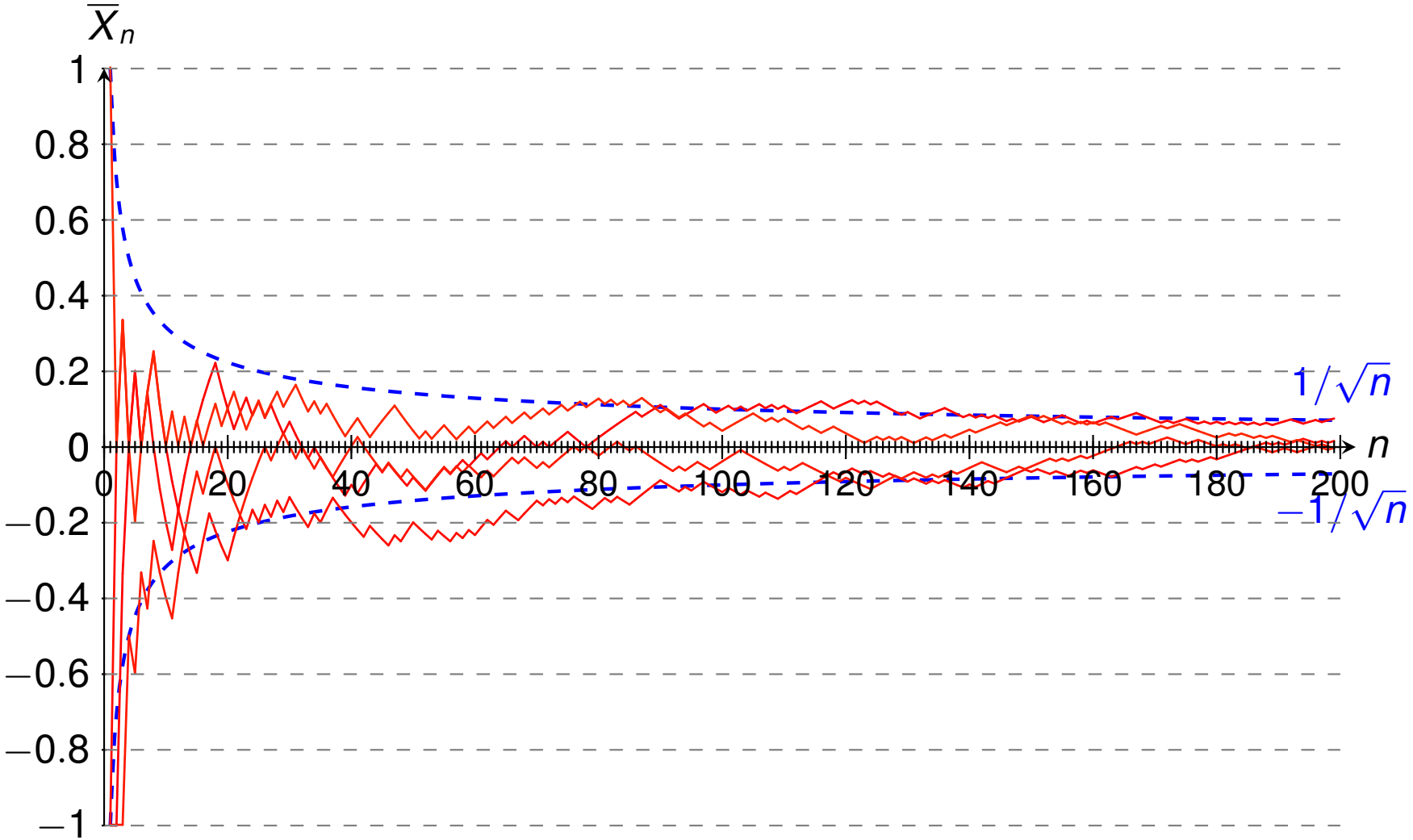

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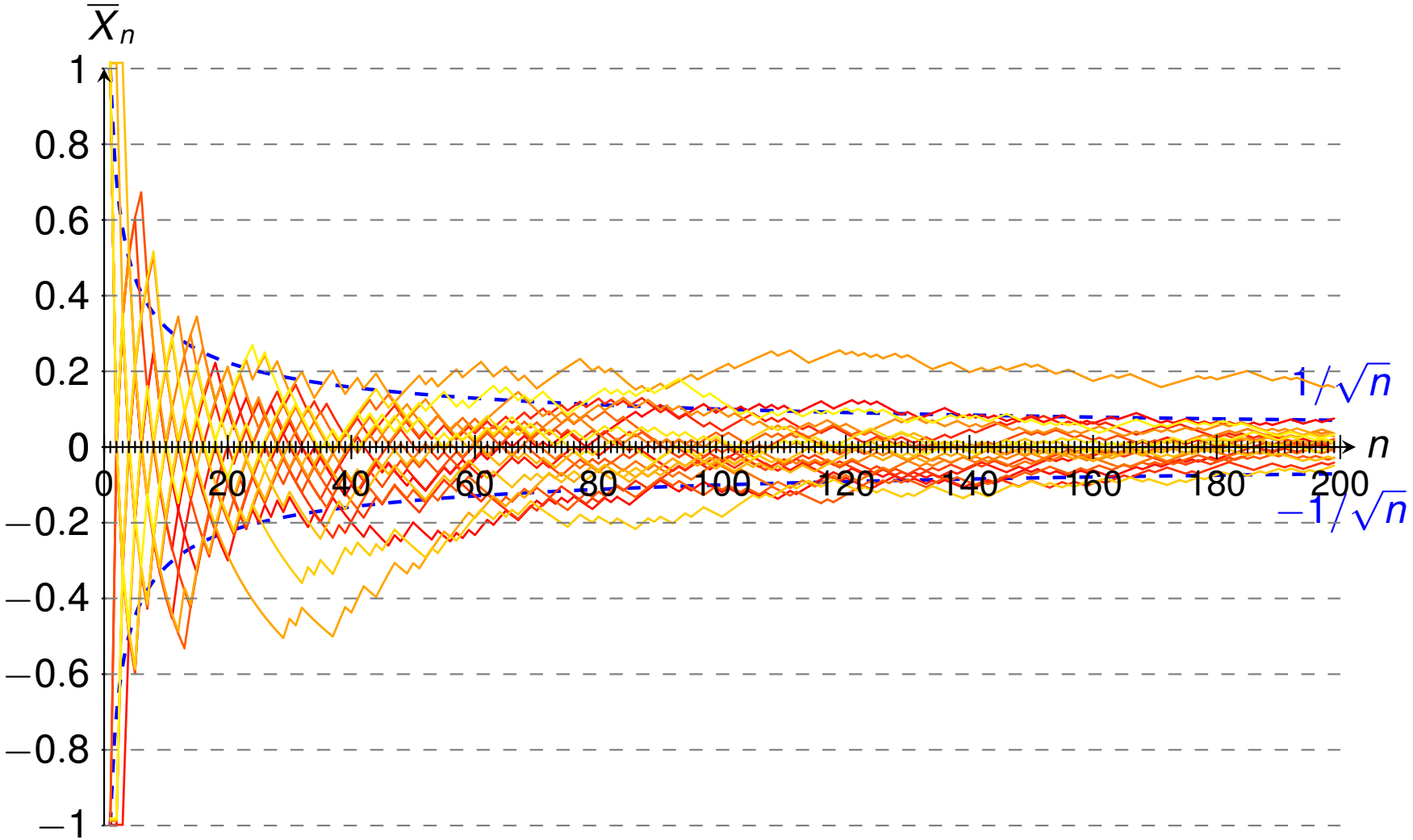

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Let $\overline{X}_n := 1/n \cdot \sum_{i=1}^{n} X_i$, where the X_i's are i.i.d. with finite expectation μ and finite variance σ^2. Then, for any $\epsilon > 0$,

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  (Let $\epsilon > 0$, $\delta > 0$. Pick $N = \frac{\sigma^2}{\epsilon^2 \cdot \delta}$. Then for any $n \geq N$, the probability above is smaller than δ.)

Inferring Probabilities of an Event

Example 4

Suppose that, instead of the expectation μ, we want to estimate the probability of an event, e.g.,

$$p := \mathbf{P}[X \in (a, b]], \text{ where } a < b.$$

How can we use the Law of Large Numbers?

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- The random variables $Y_1, Y_2, \ldots, Y_n$ are i.i.d., so we can apply the Law of Large Numbers to $\overline{Y}_n$.

Introduction to Probability

Lectures 9: Central Limit Theorem

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

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Outline

Weak Law of Large Numbers (4/4)

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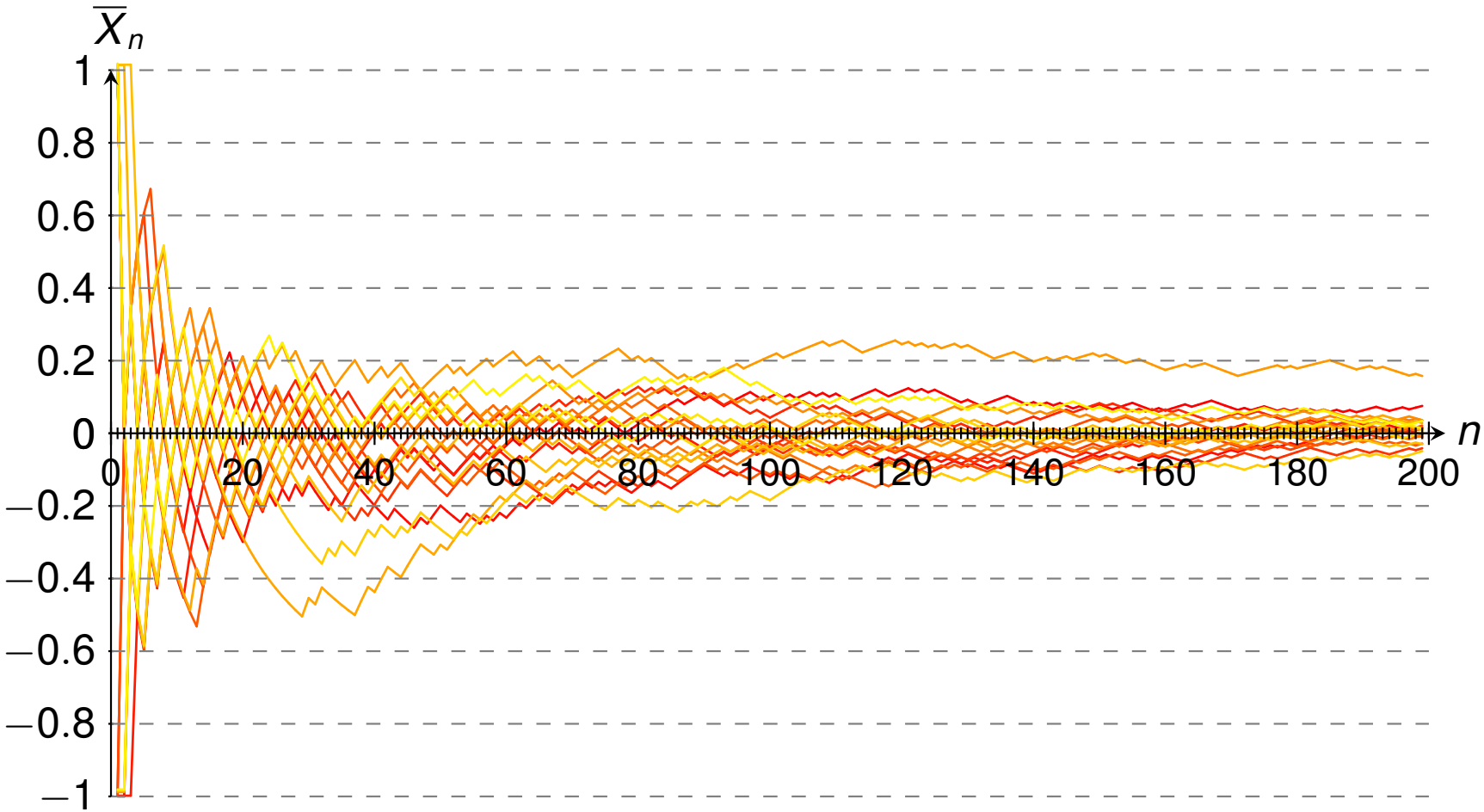

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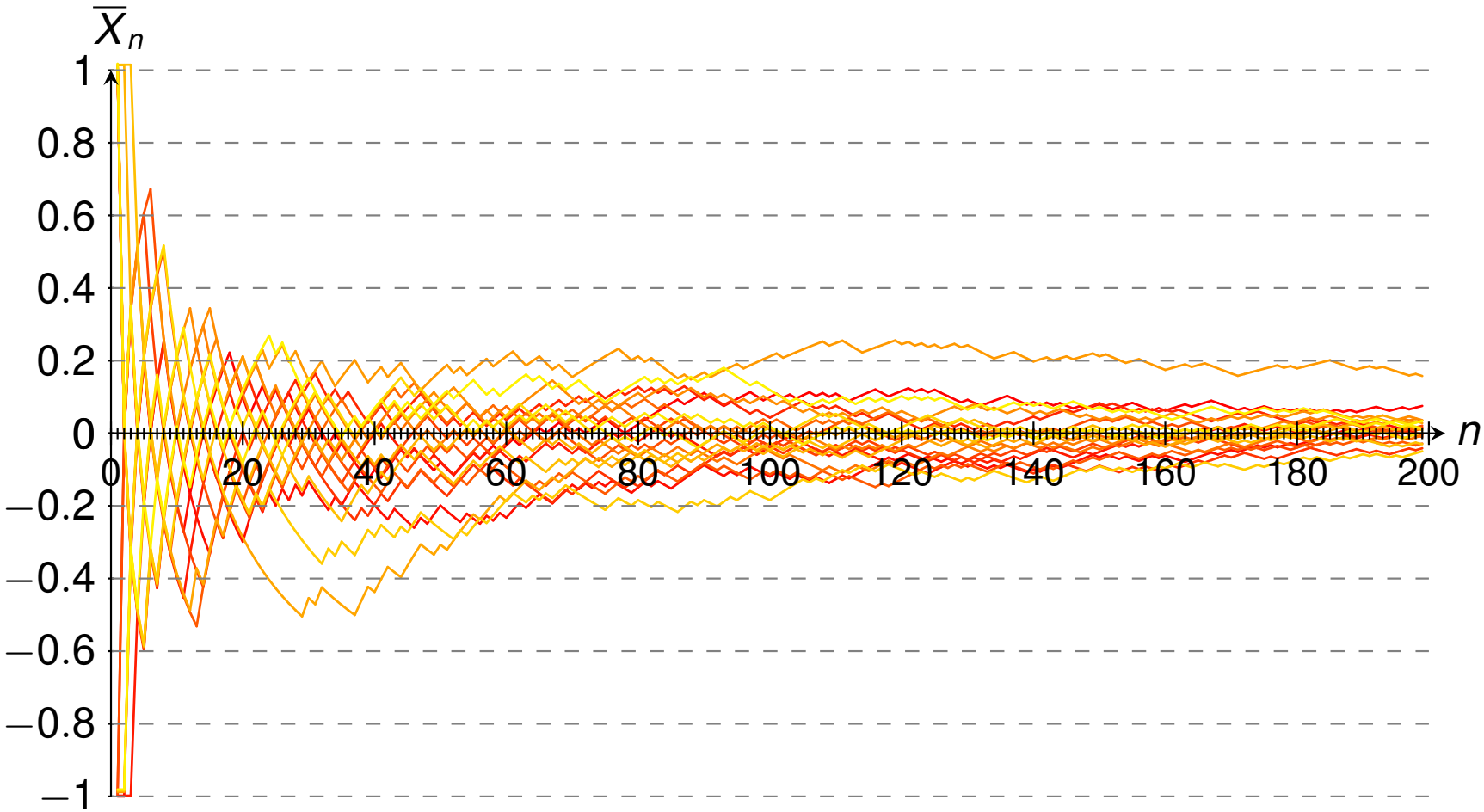

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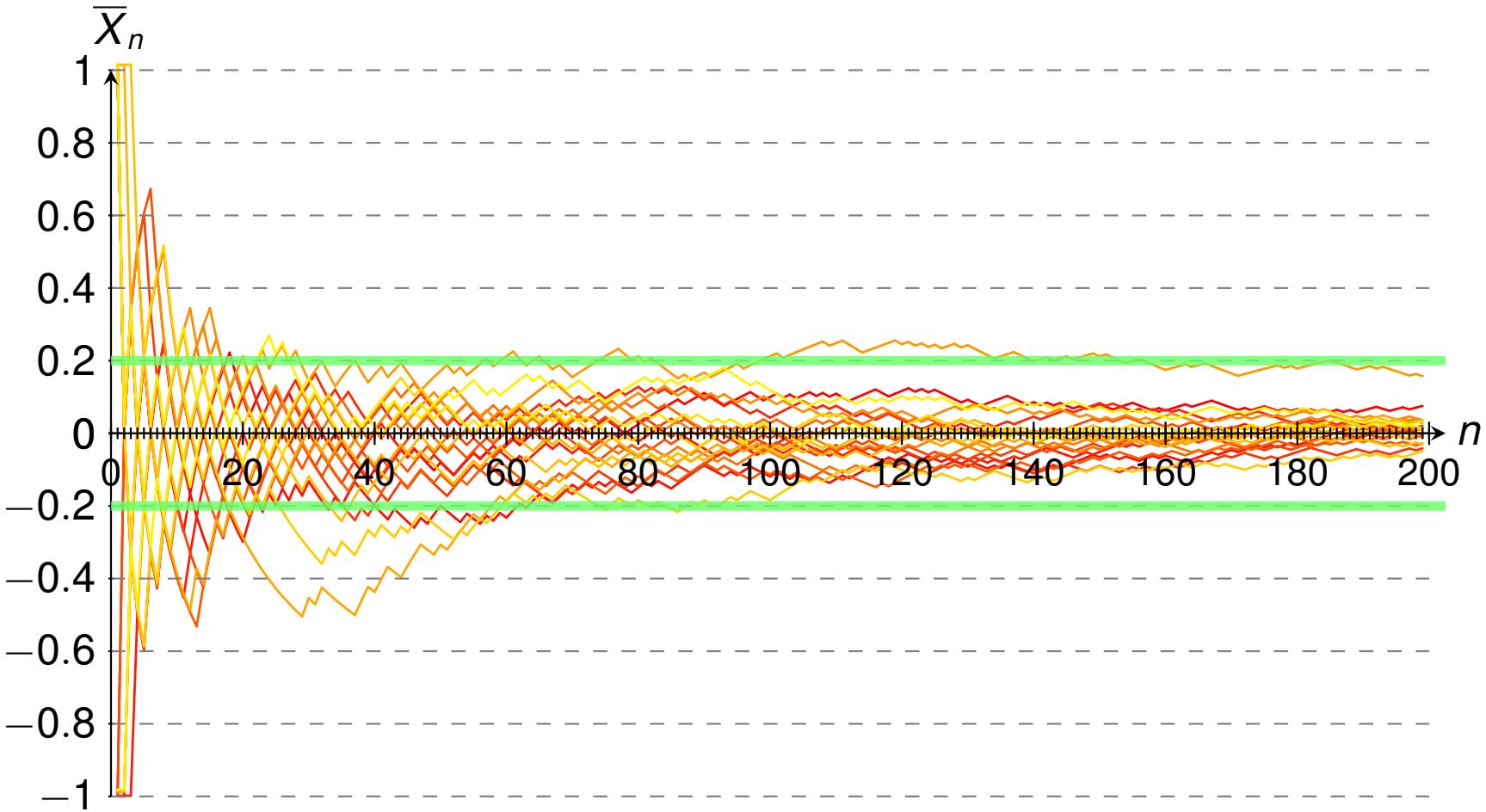

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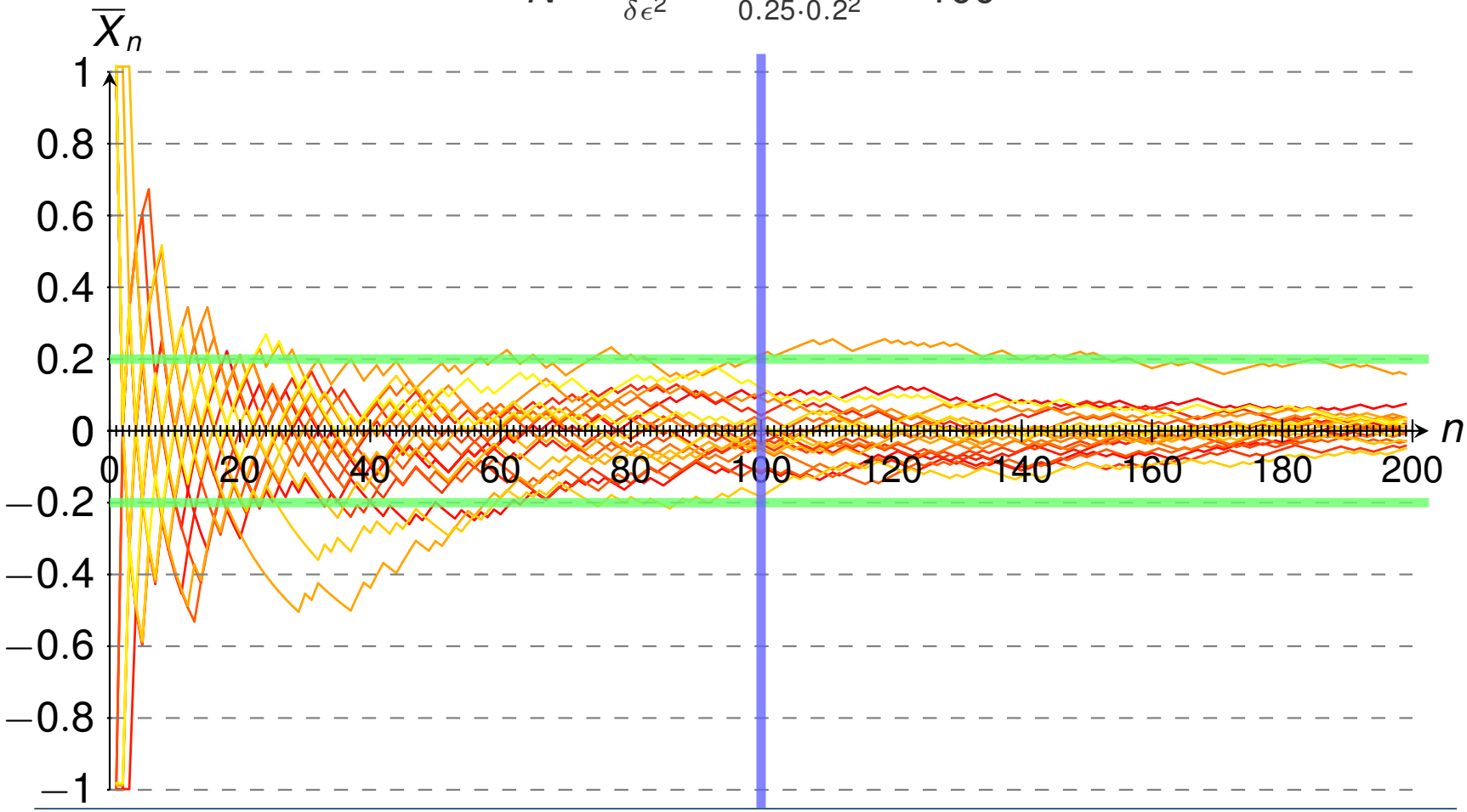

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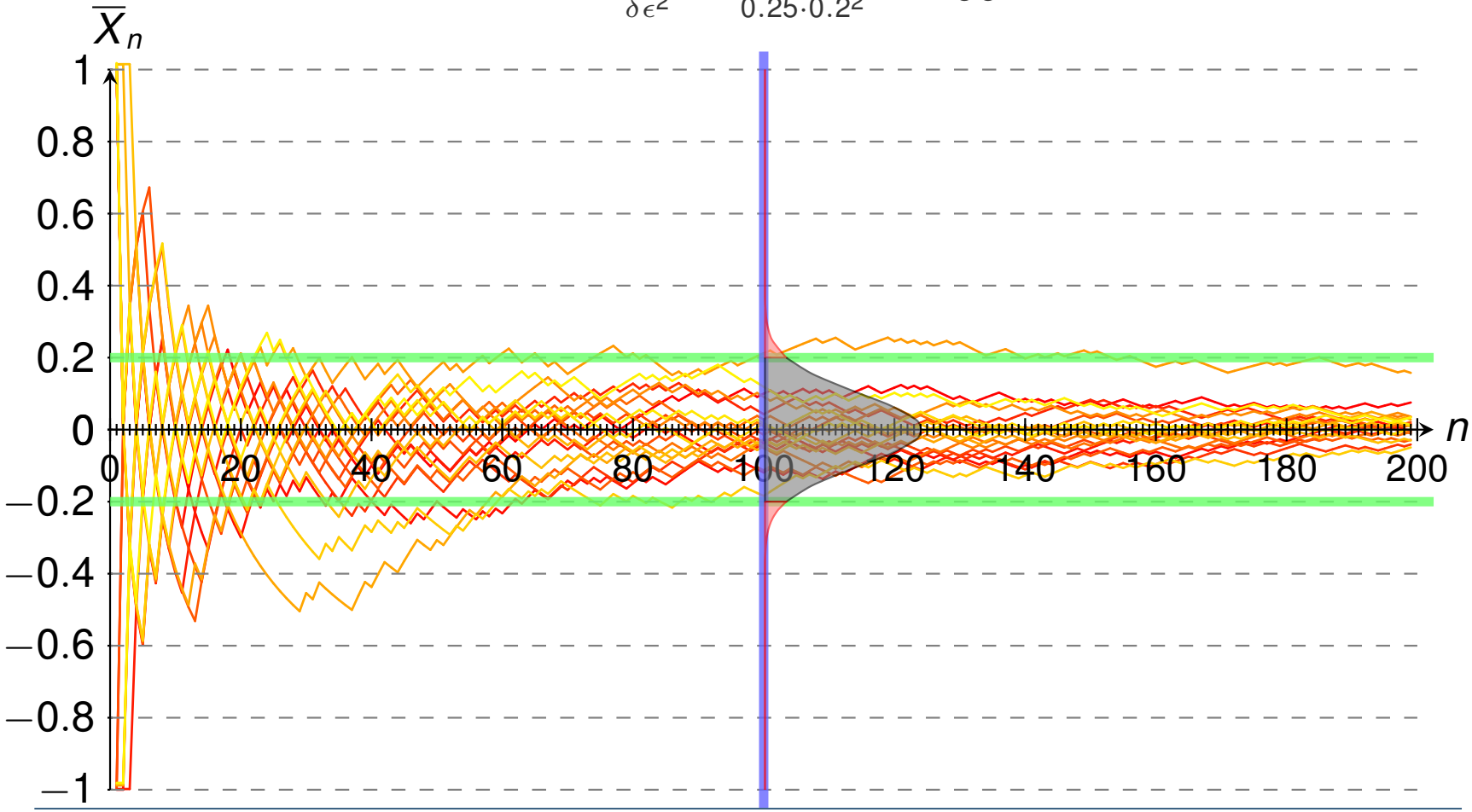

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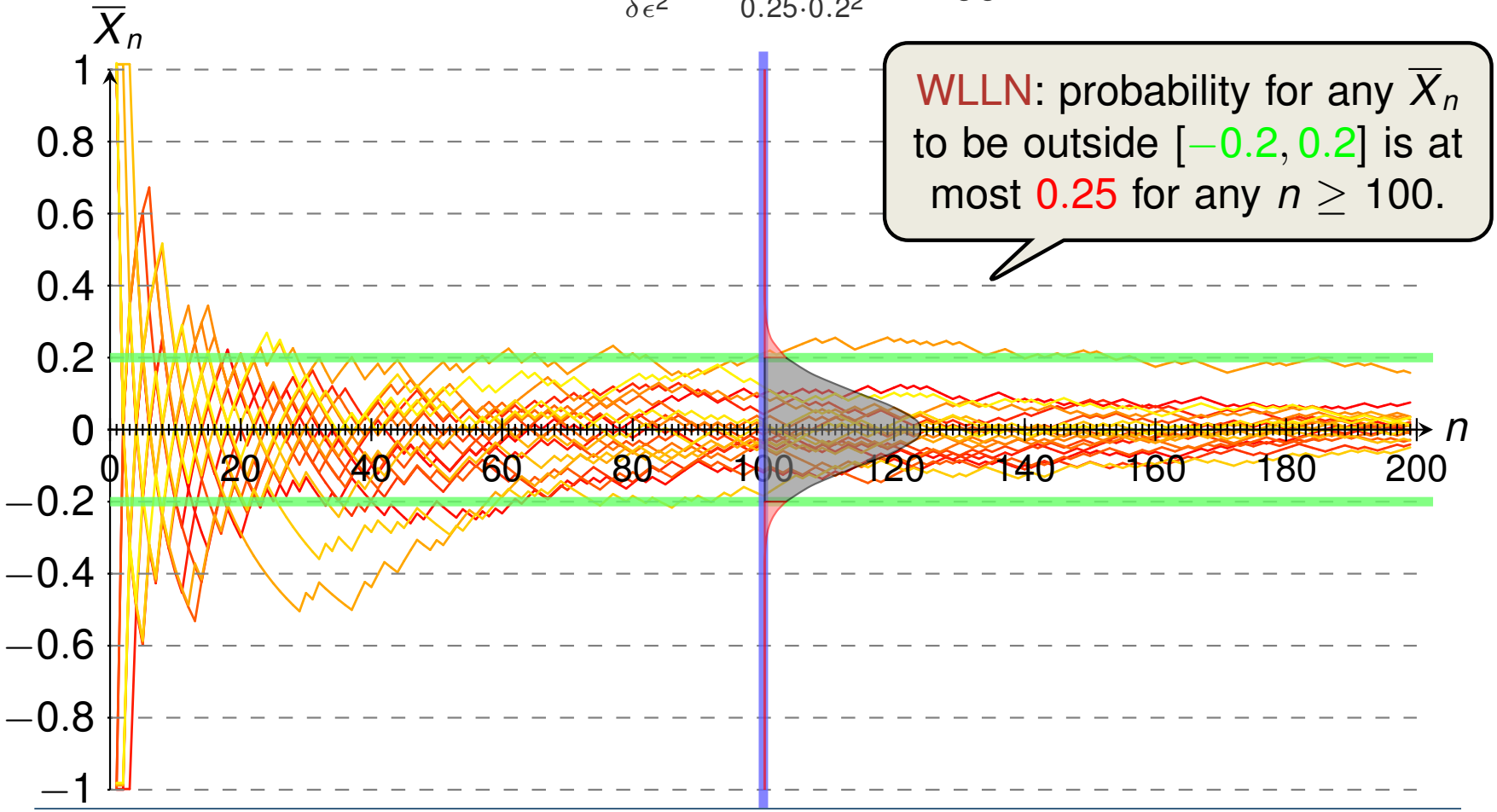

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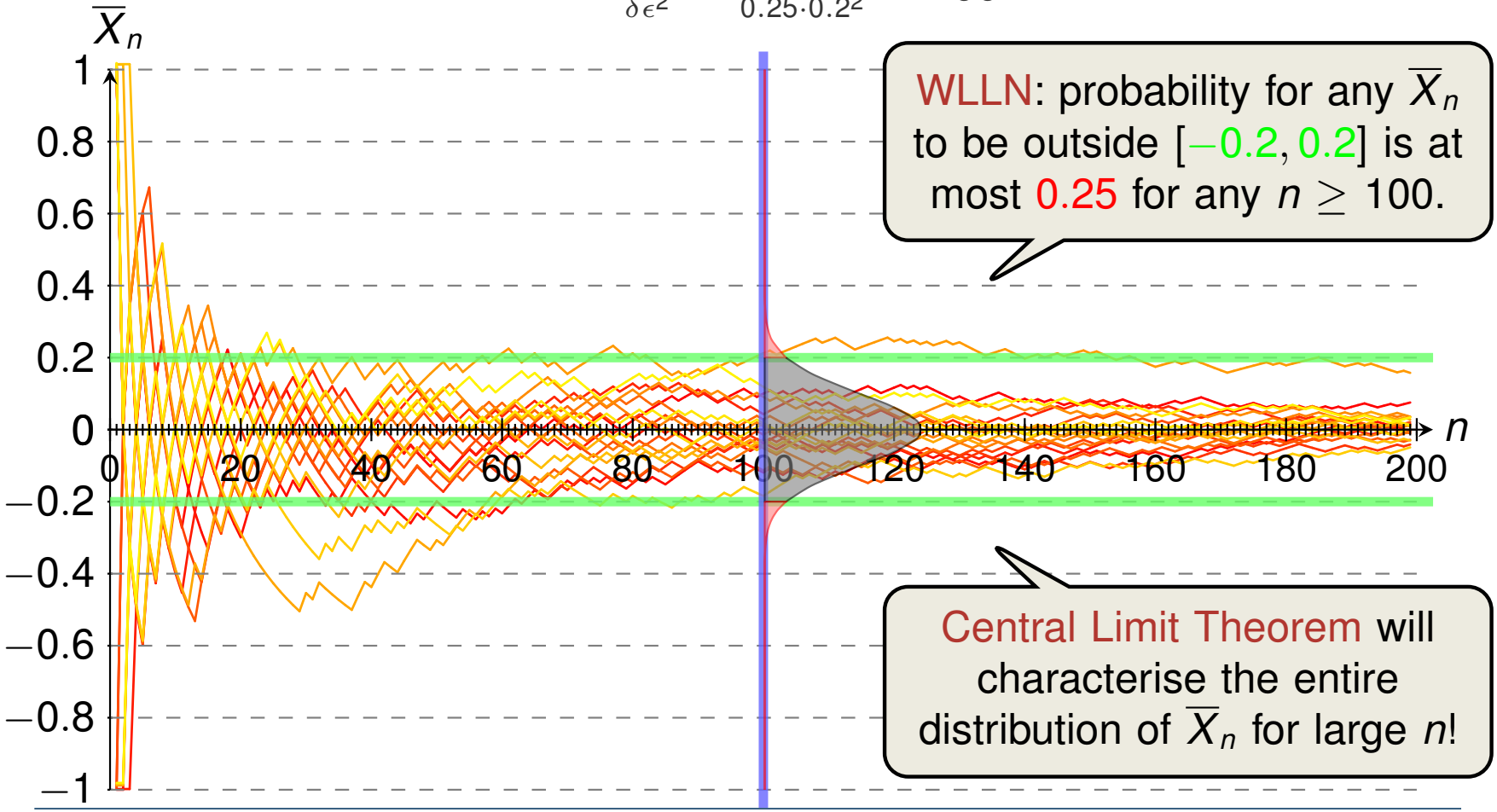

Outline

Towards the CLT: Finding the Right Scaling

- Let $X_1, X_2, \ldots$ i.i.d. with $\mu = 0$ and finite σ^2

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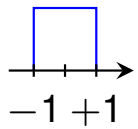

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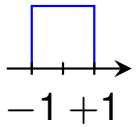

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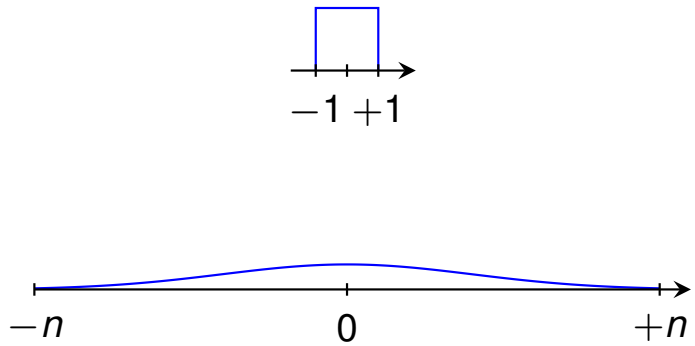

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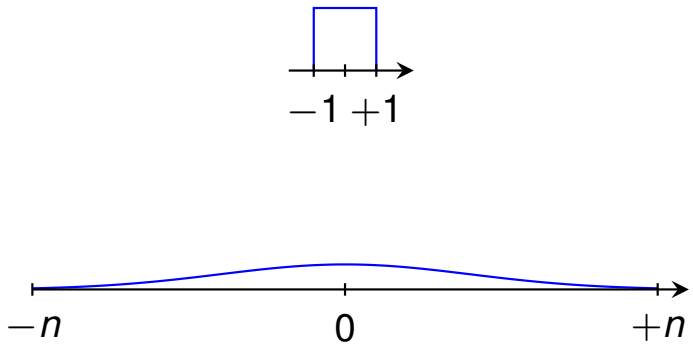

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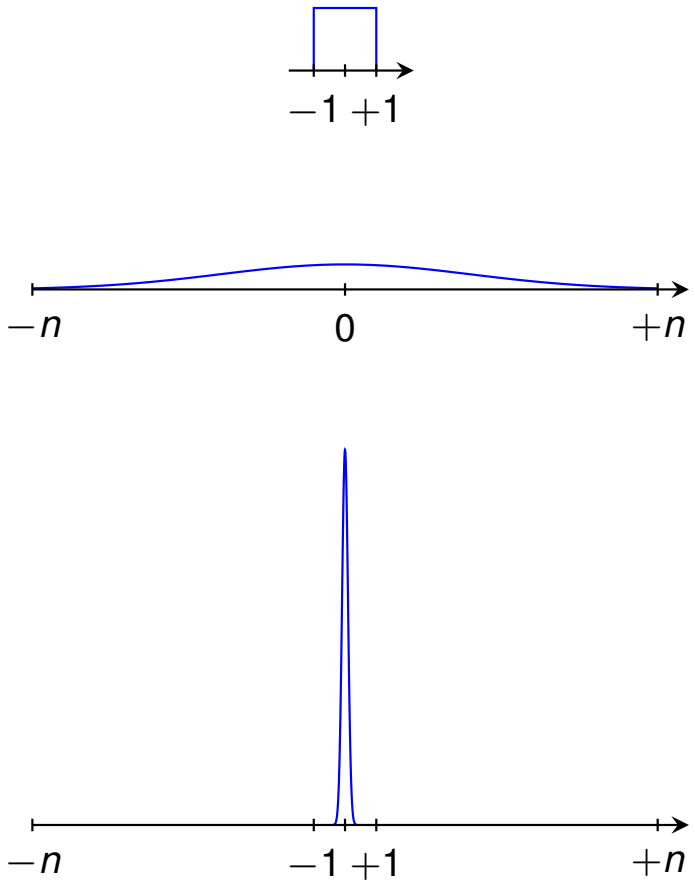

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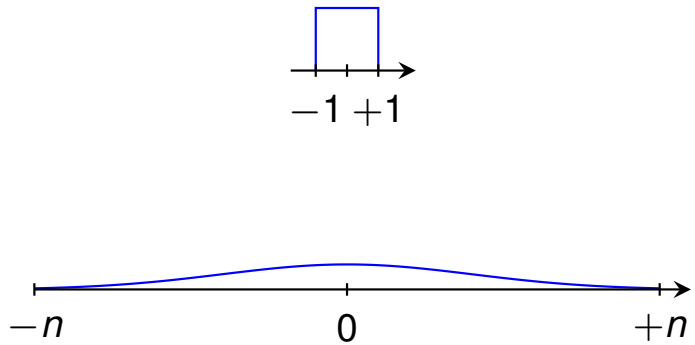

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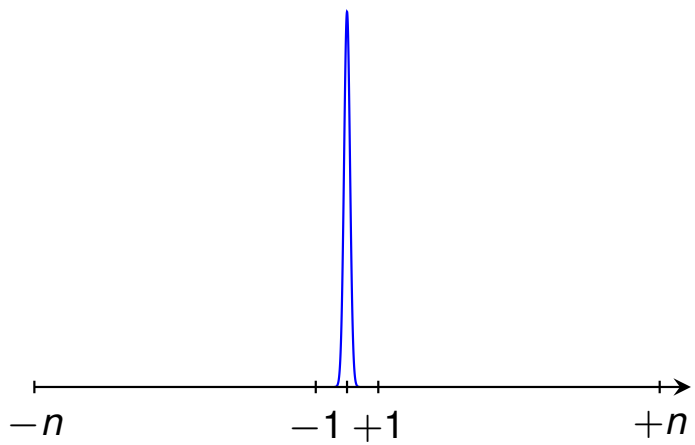

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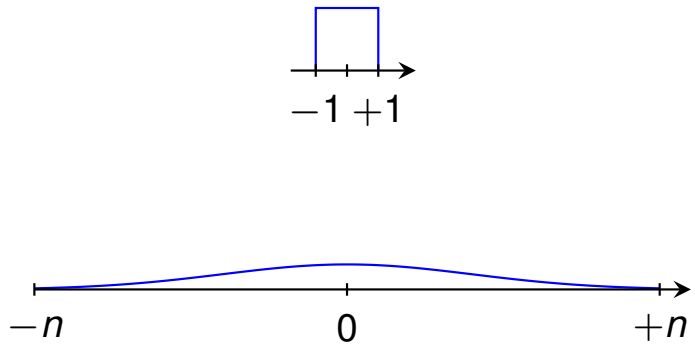

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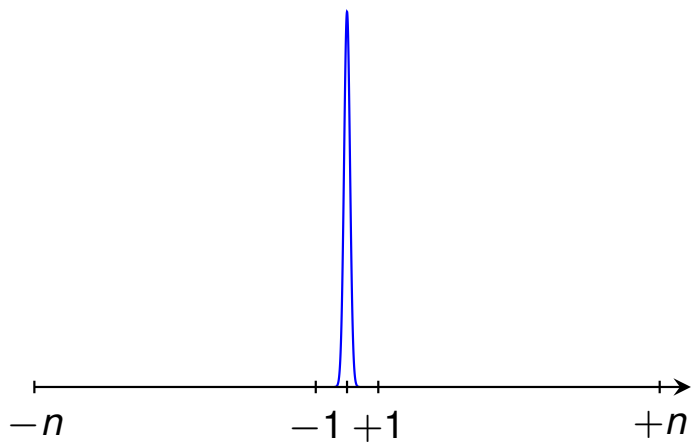

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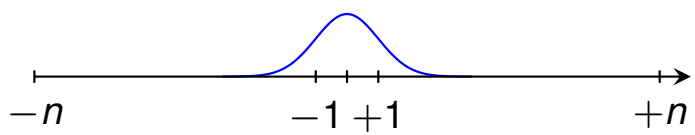
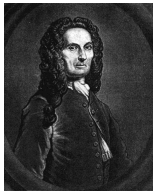


Central Limit Theorem



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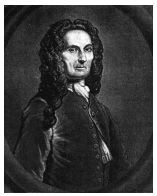
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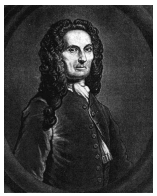
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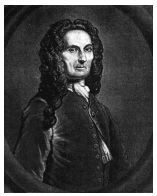
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Then for any number $a \in \mathbb{R}$, it holds that

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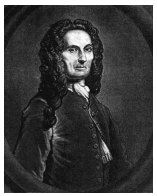
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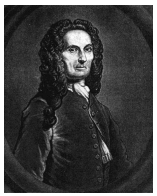
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In words: the distribution of Z_n **always** converges to the distribution function Φ of the standard normal distribution.

Comments on the CLT

- one of the most remarkable results in probability/statistics
- extremely useful tool in data analysis or physical measurements
  - we may not know the actual distribution in real-world, and CLT says we don't have to(!)
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When is the approximation good?

- usually $n \geq 10$ or $n \geq 15$ is sufficient in practice
- approximation tends to be worse when threshold a is far from 0, distribution of X_i's asymmetric, bimodal or discrete
- (for a result quantifying the approximation error: Berry-Esseen-Theorem)

Outline

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{1} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
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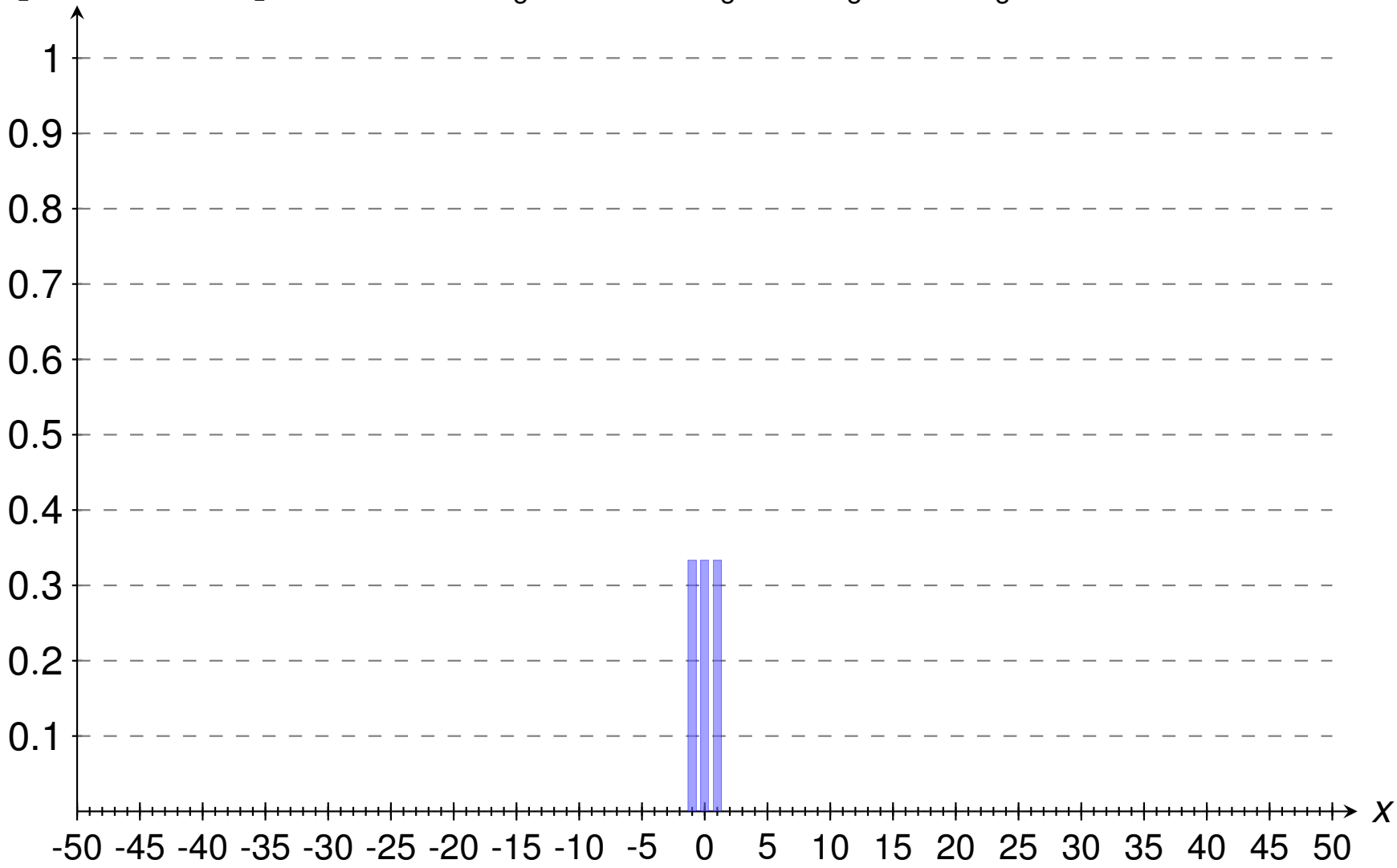

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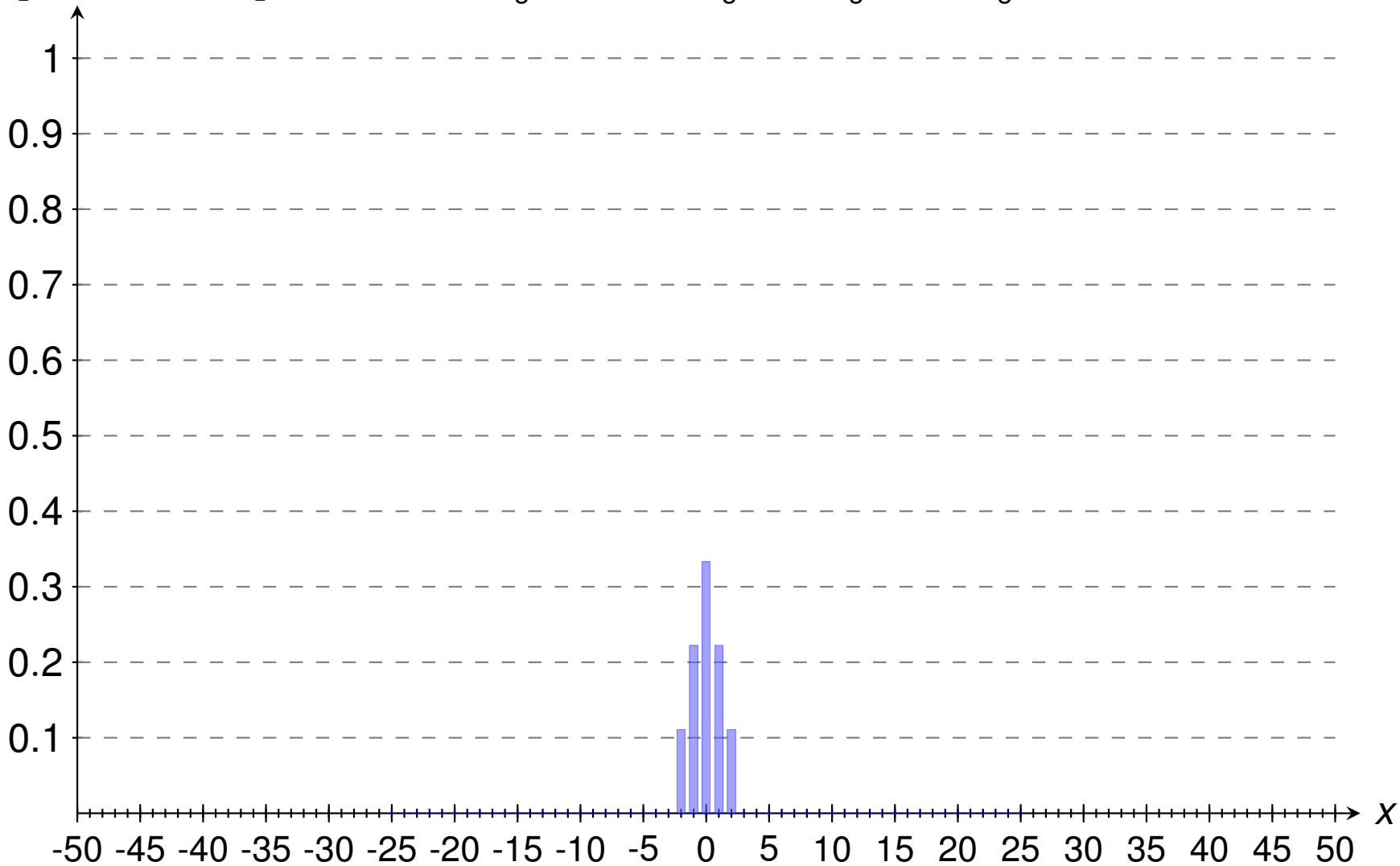

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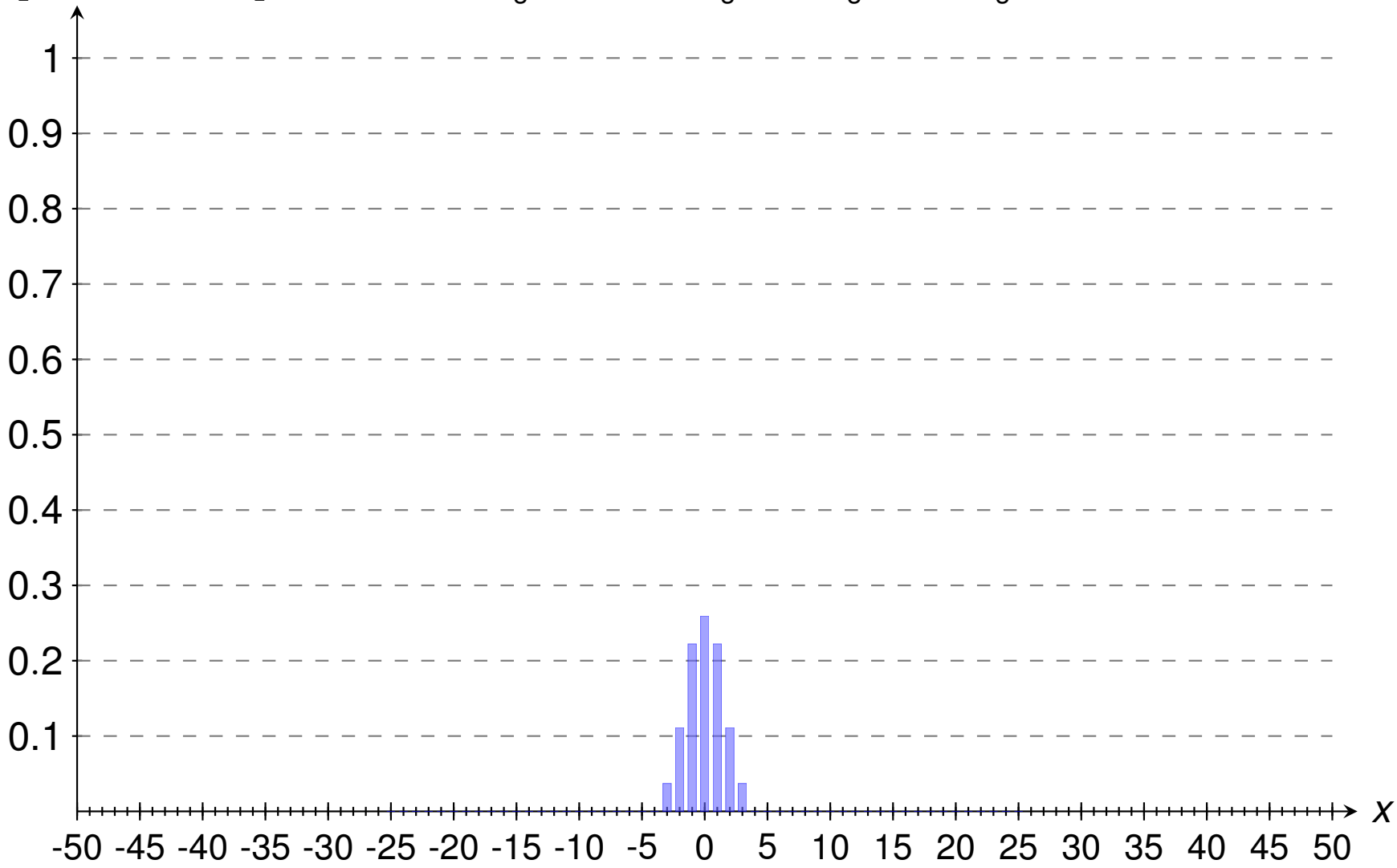

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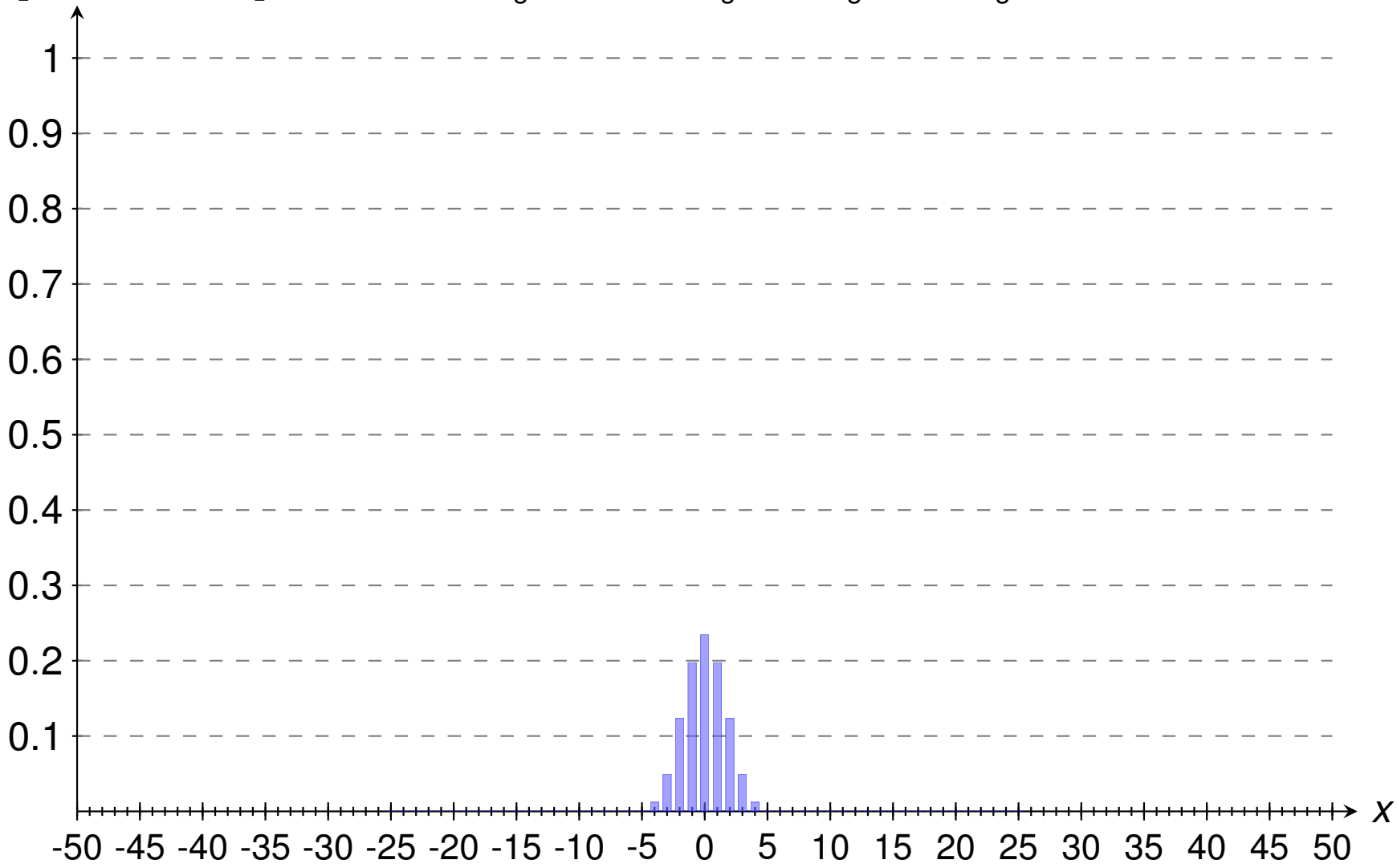

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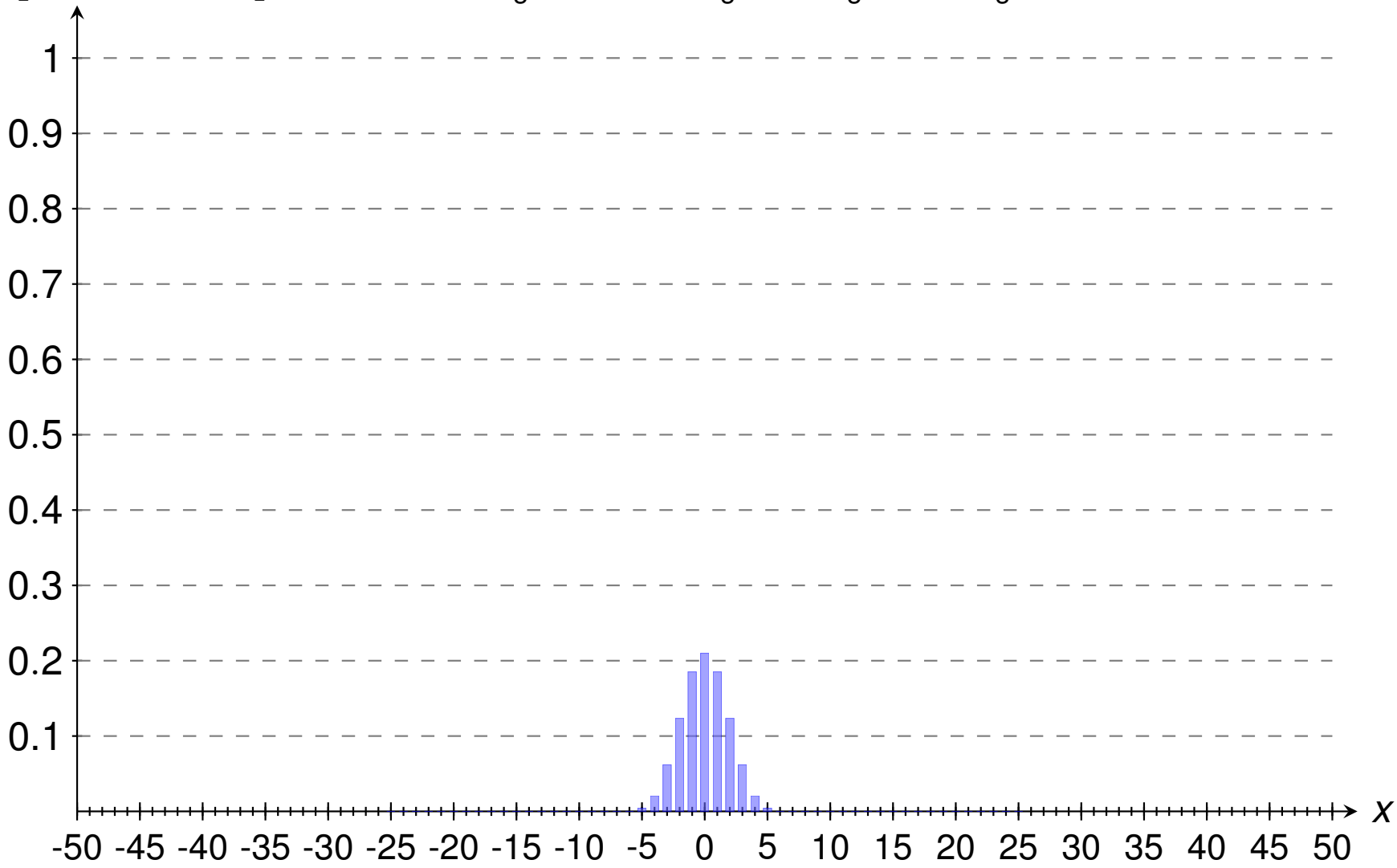

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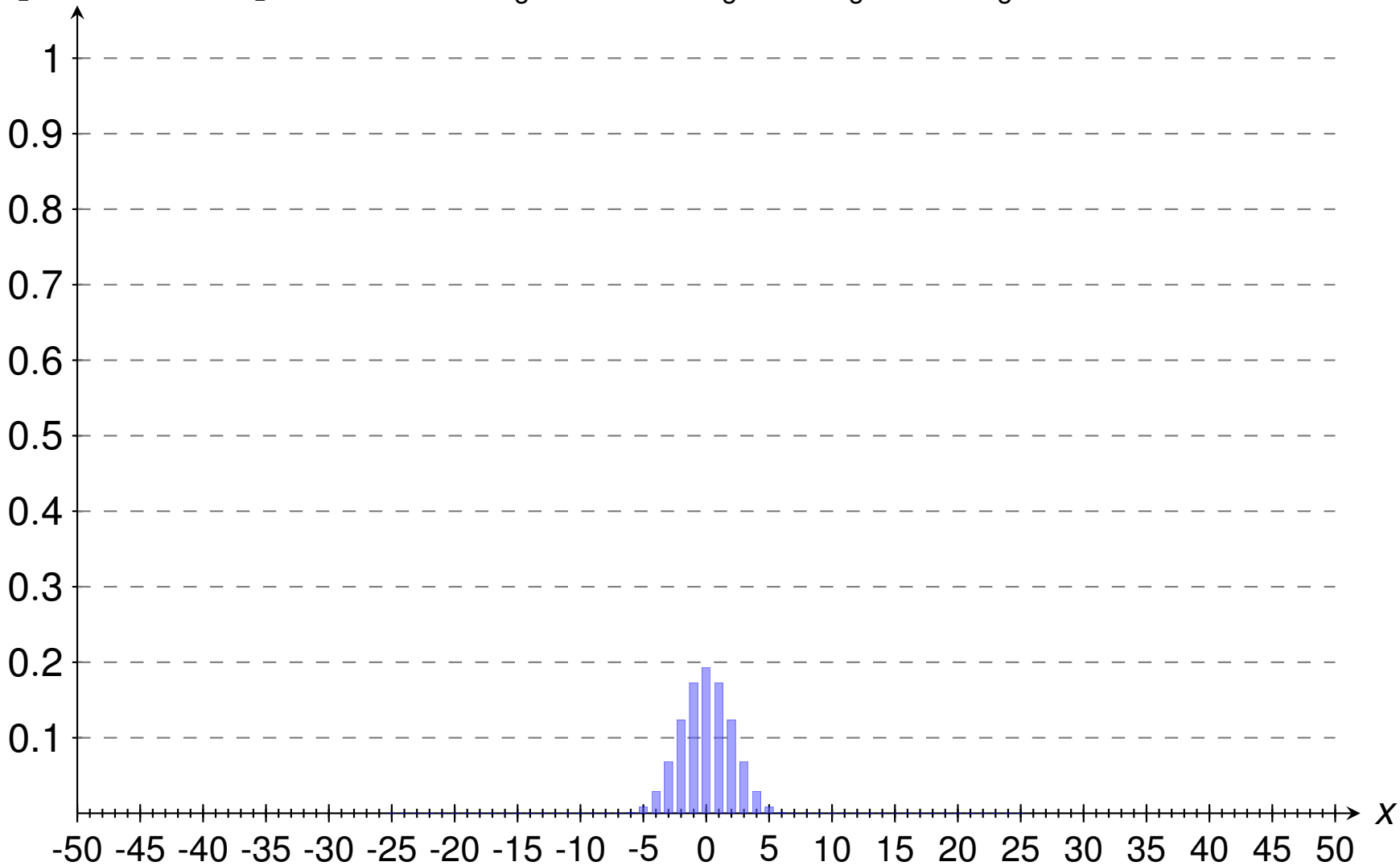

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- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

$\mathbf{P}\left[\sum_{j=1}^{7} X_j = x\right]$

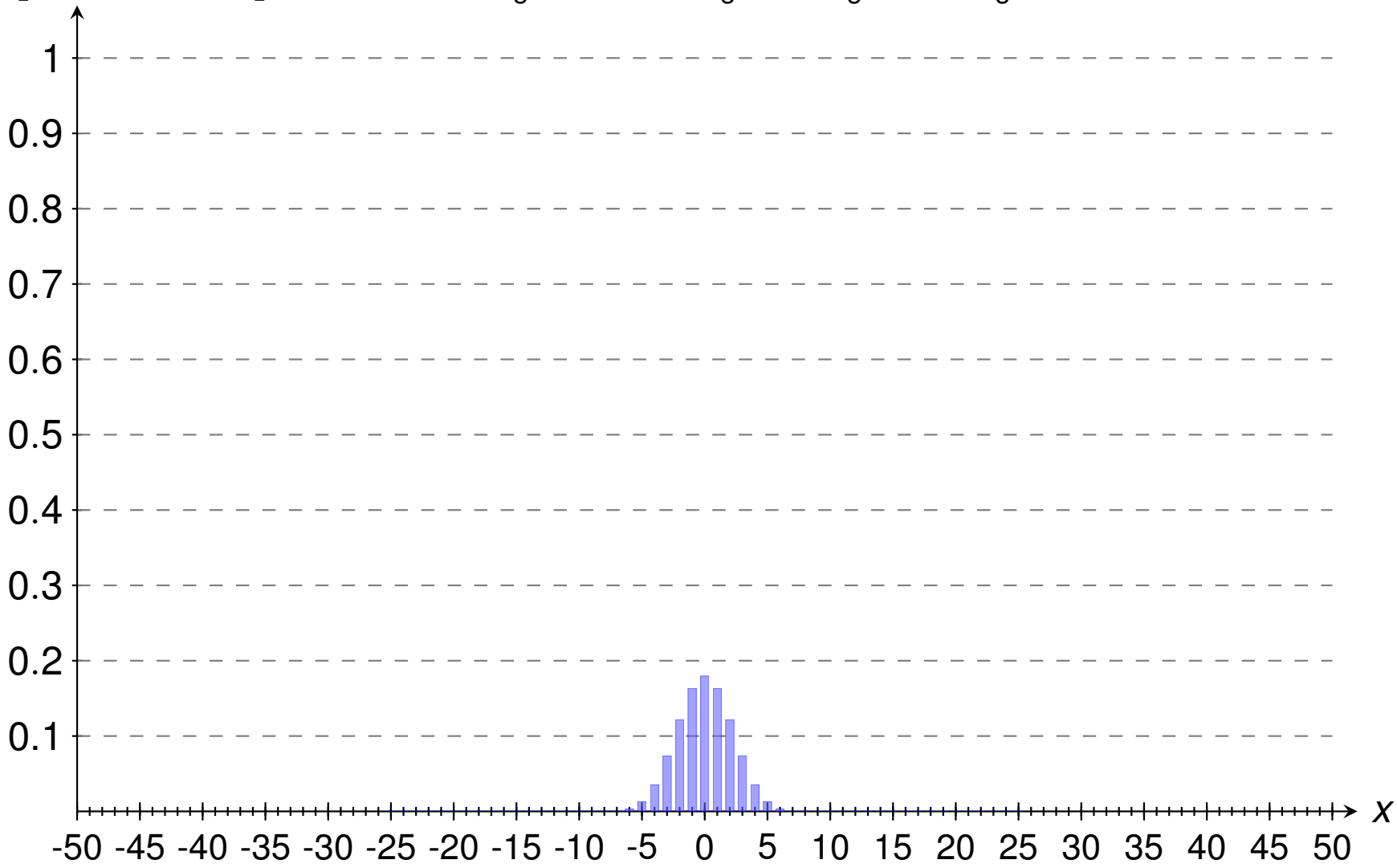

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{8} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

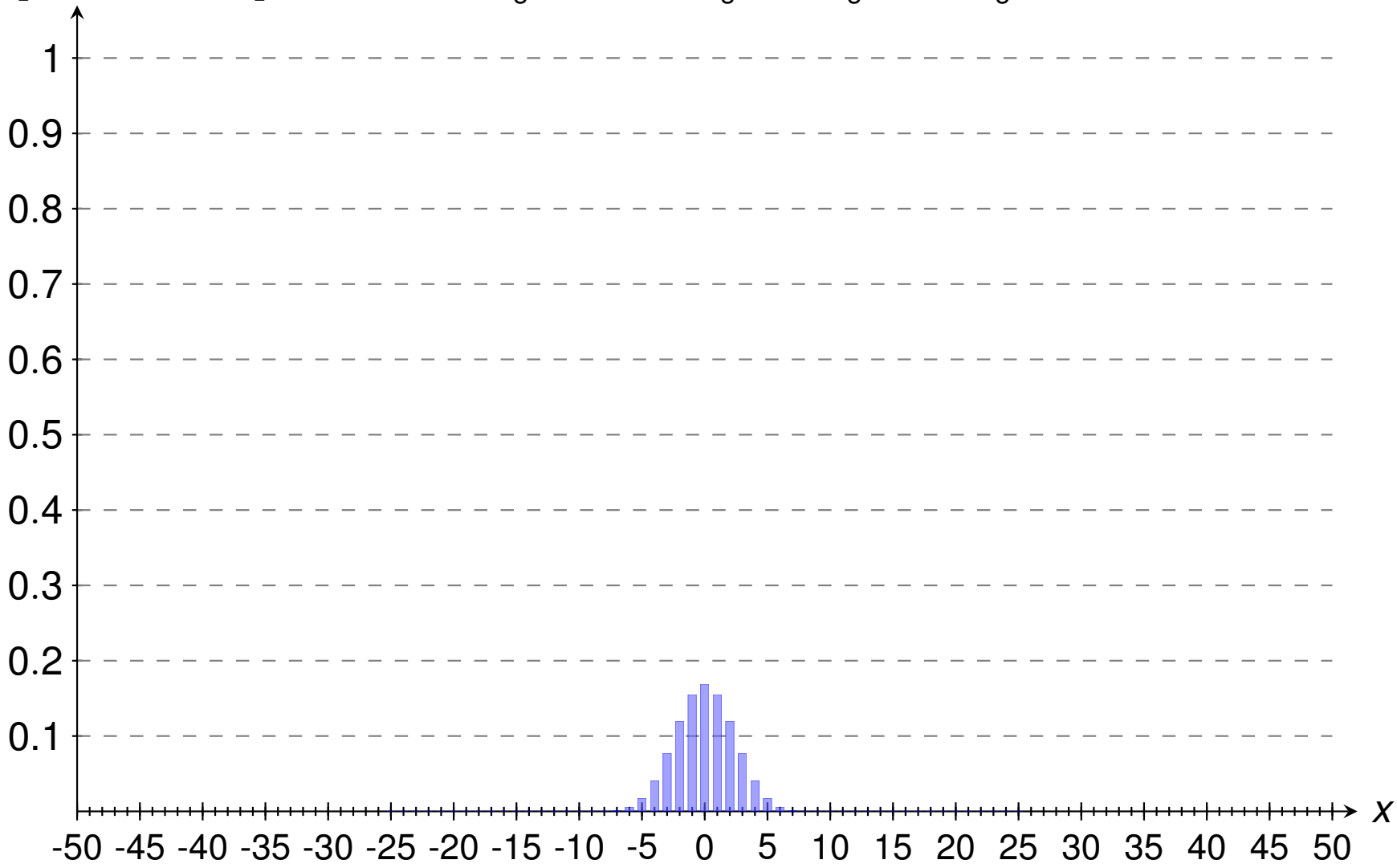

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{9} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

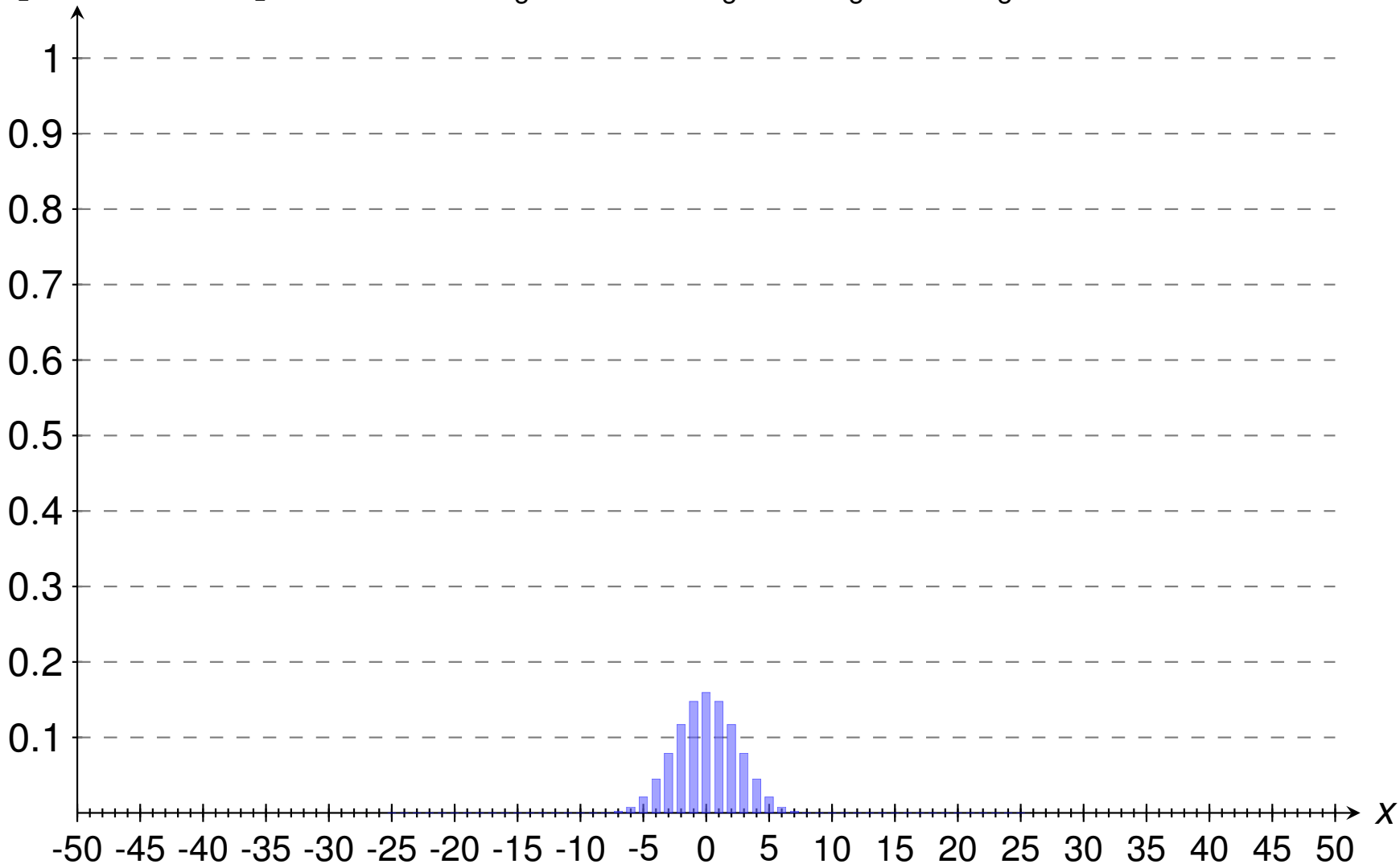

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{10} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

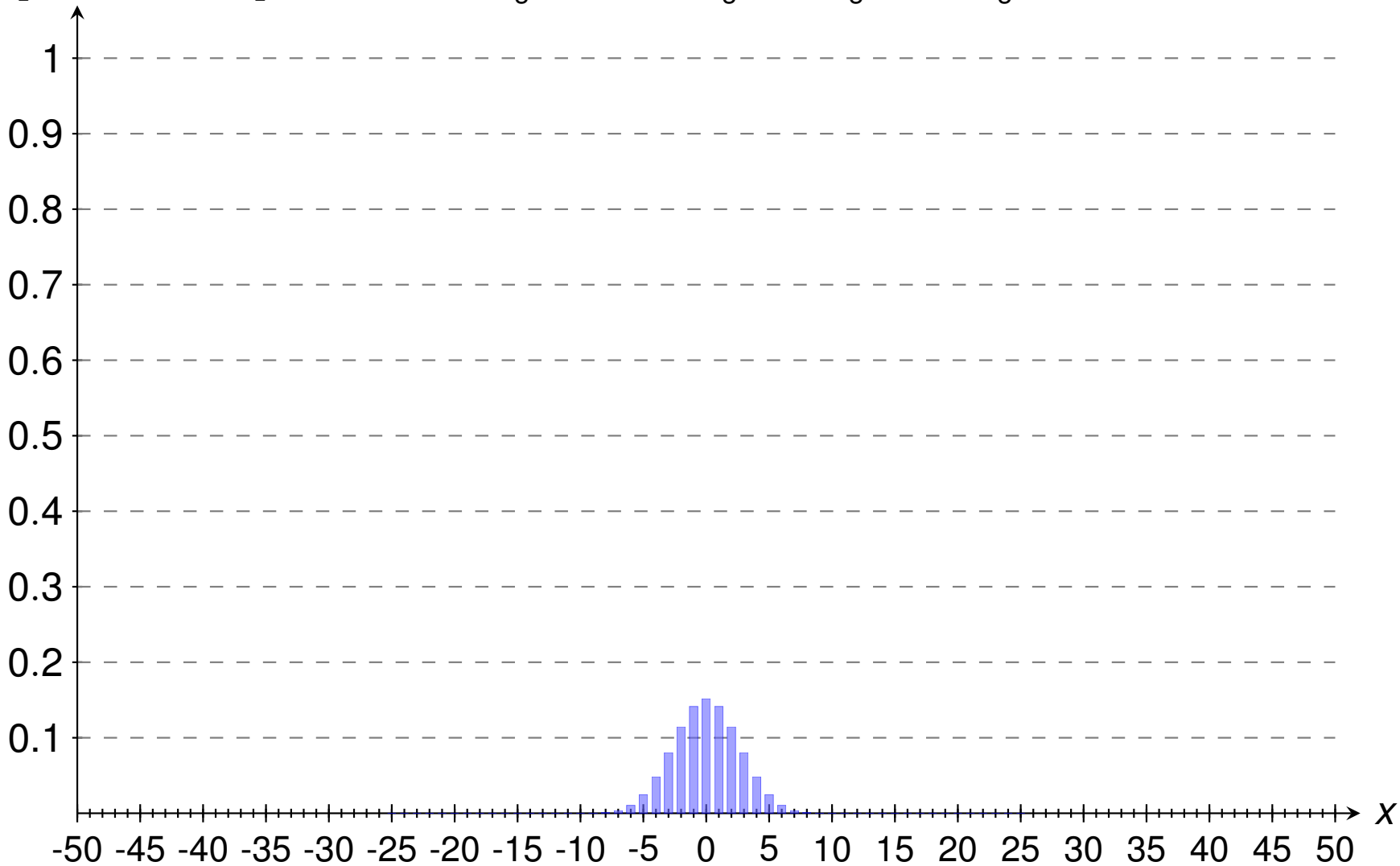

Illustration of CLT (1/4)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

$\mathbf{P}\left[\sum_{j=1}^{11} X_j = x\right]$

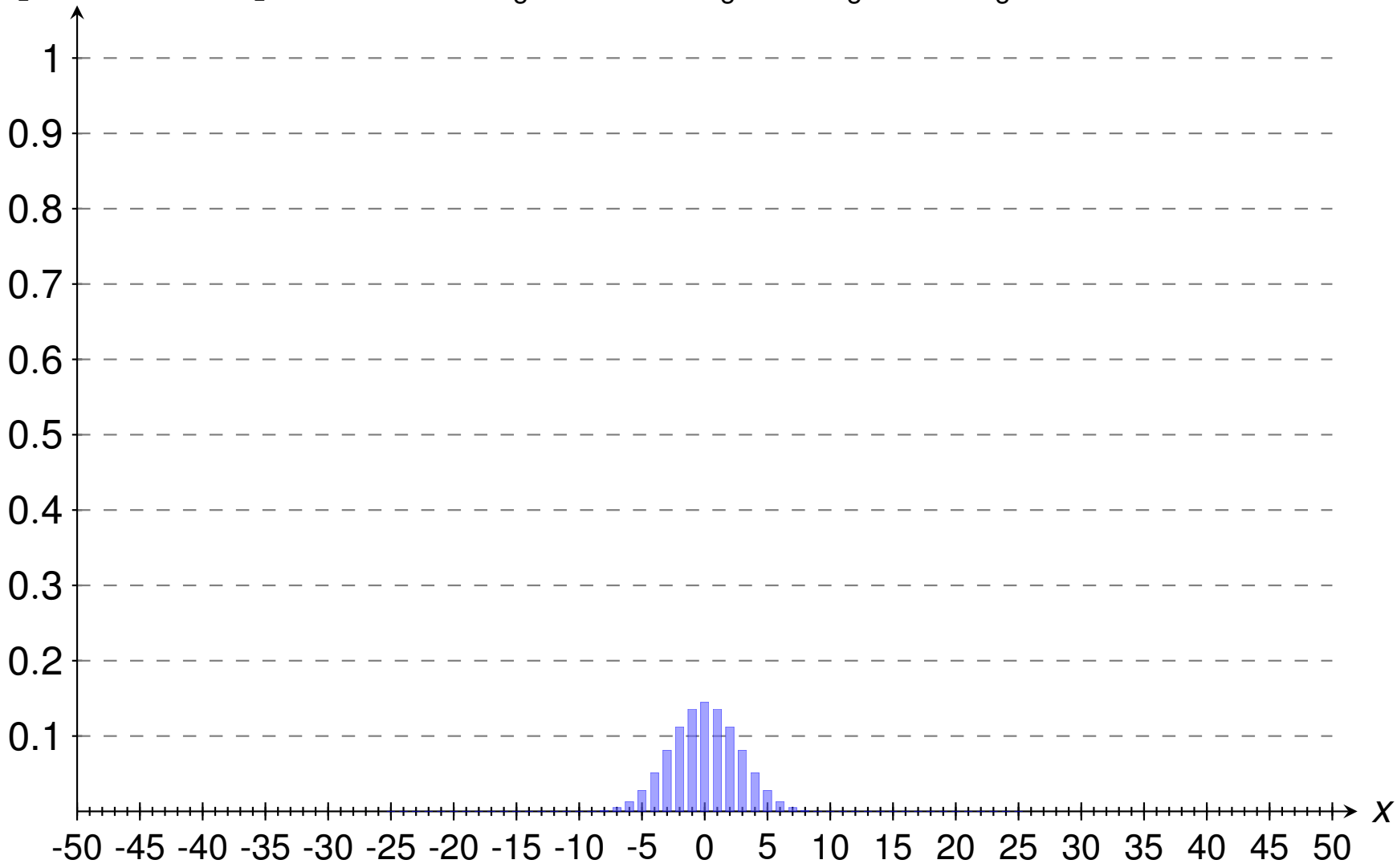

Illustration of CLT (1/4)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

$\mathbf{P}\left[\sum_{j=1}^{12} X_j = x\right]$

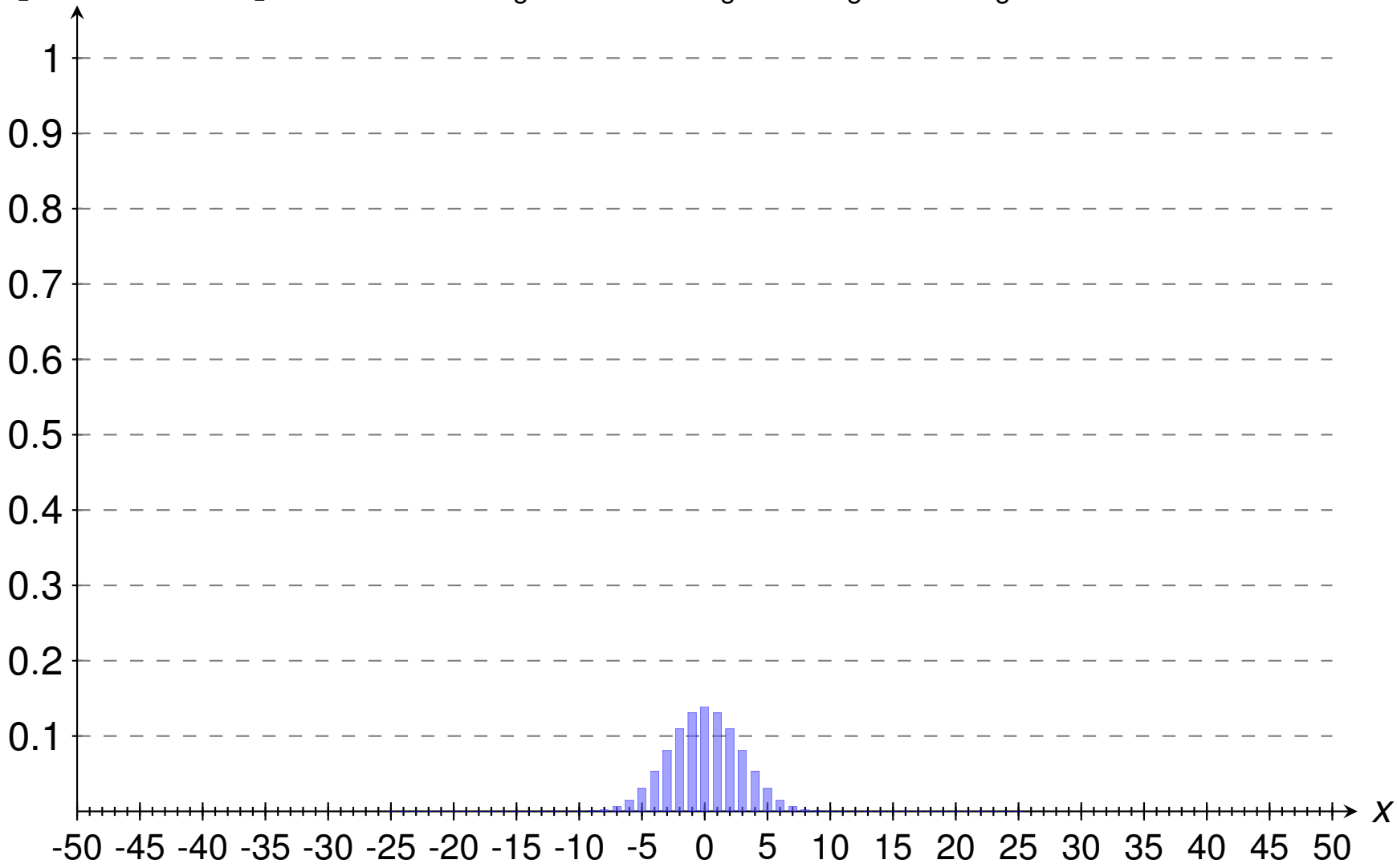

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{13} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

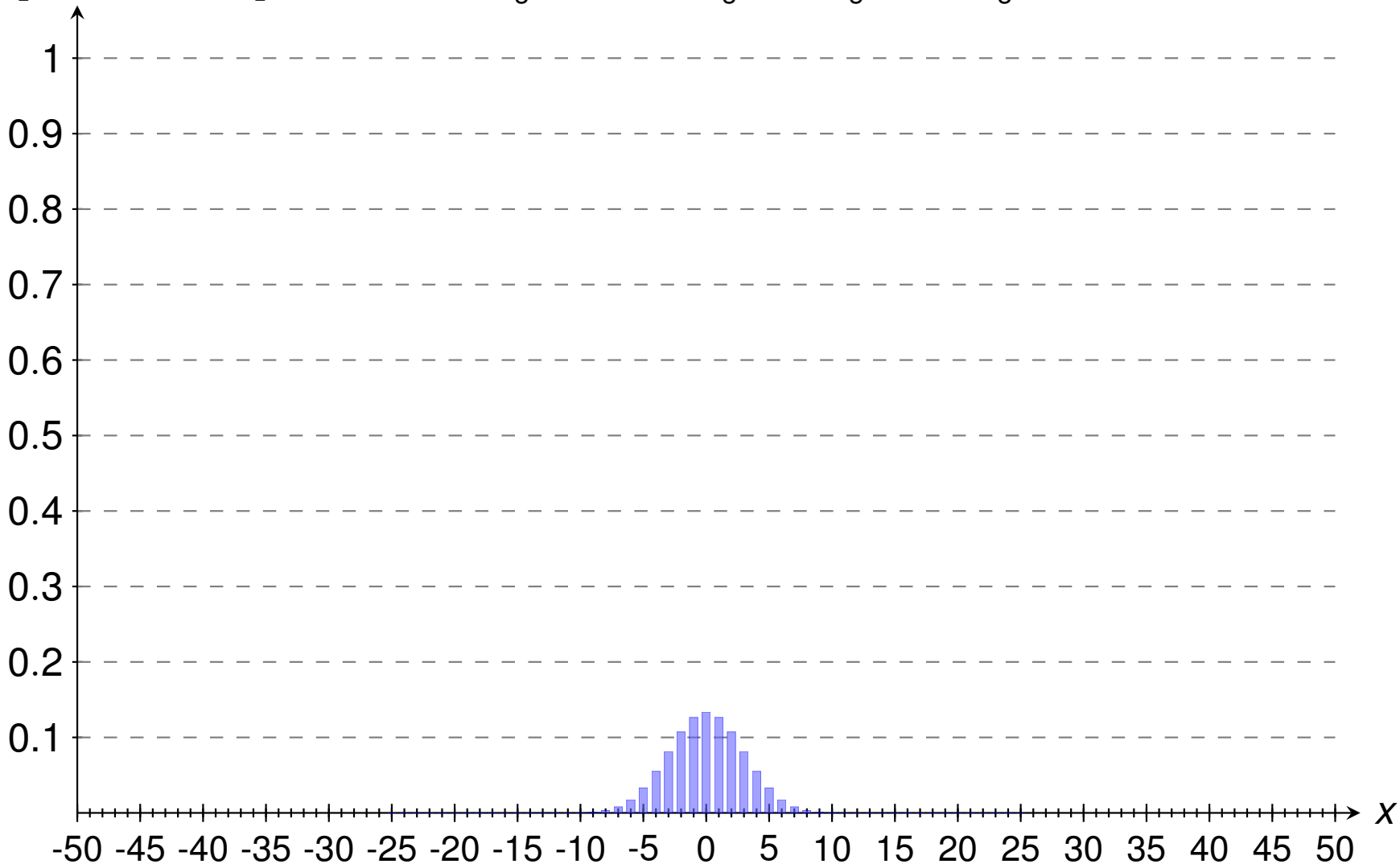

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{14} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

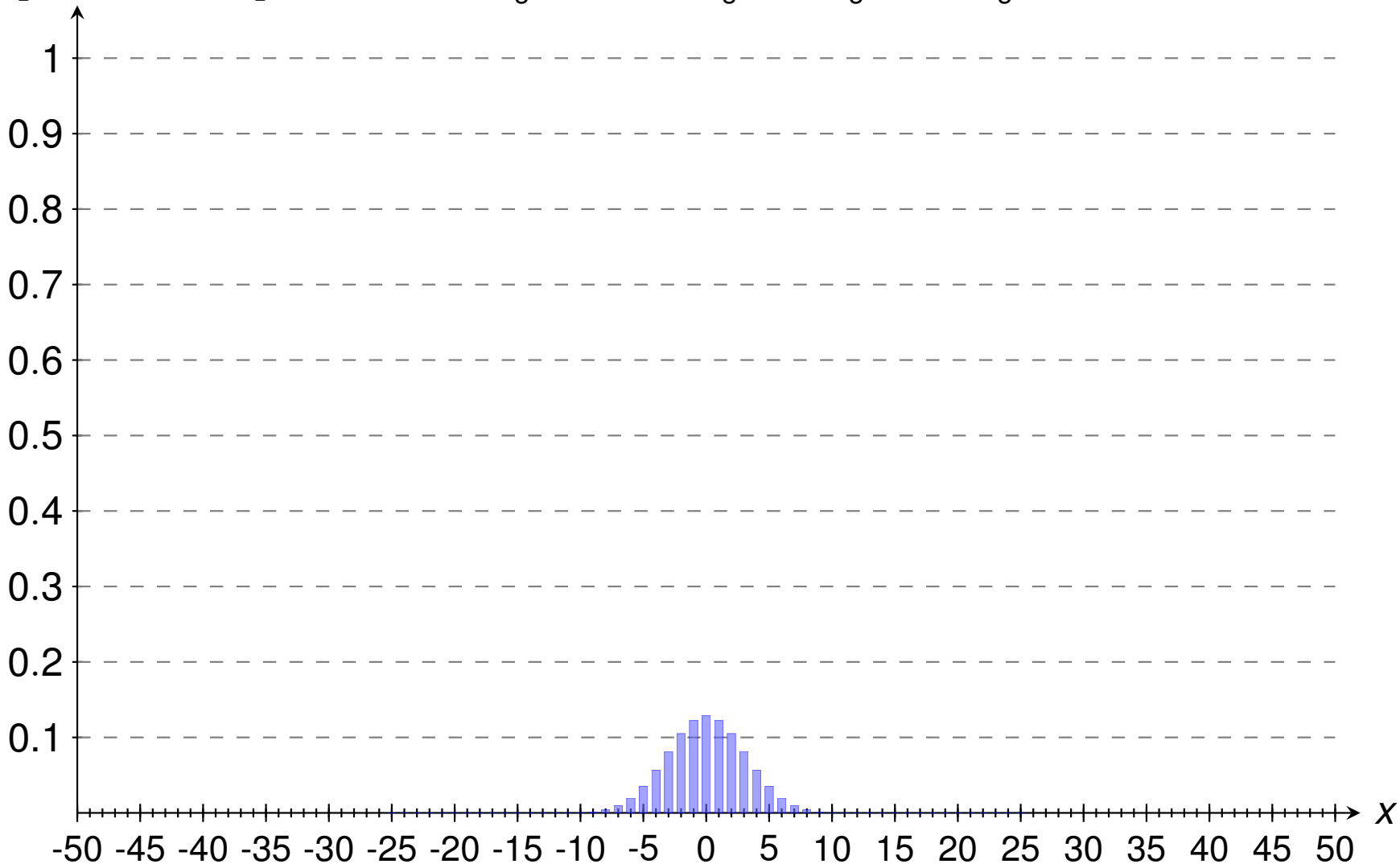

Illustration of CLT (1/4)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

$\mathbf{P}\left[\sum_{j=1}^{15} X_j = x\right]$

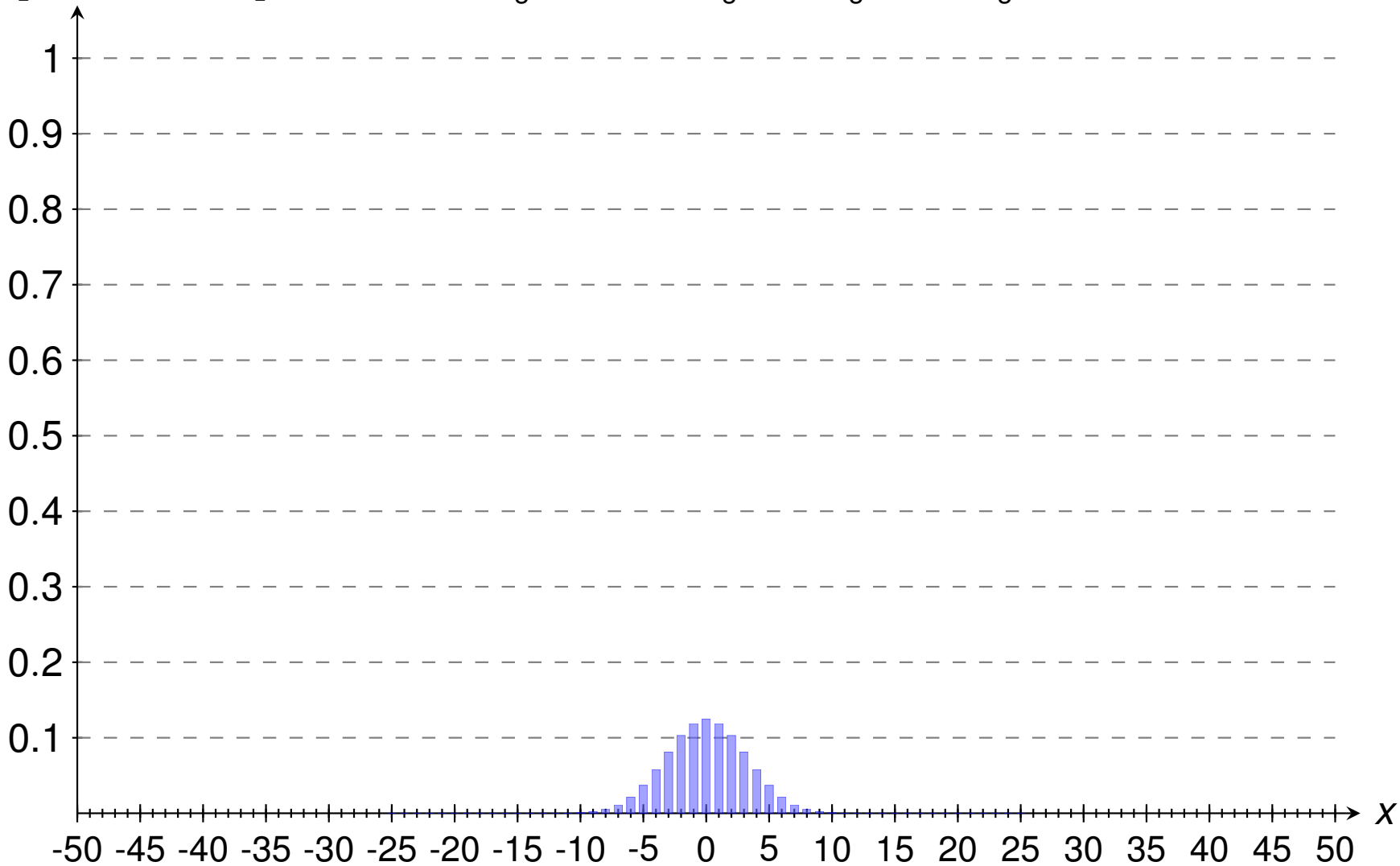

Illustration of CLT (1/4)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

$\mathbf{P}\left[\sum_{j=1}^{16} X_j = x\right]$

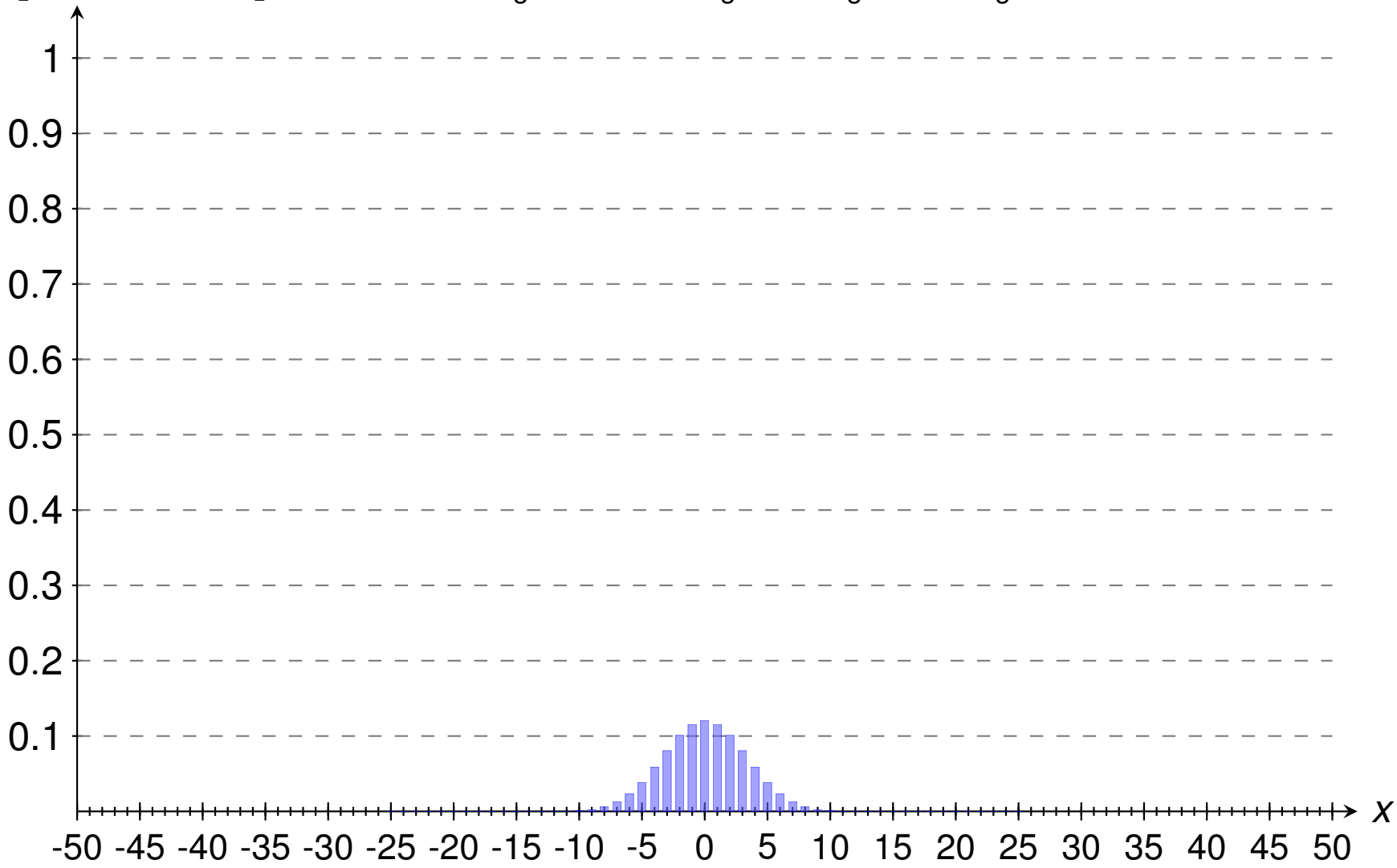

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{17} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

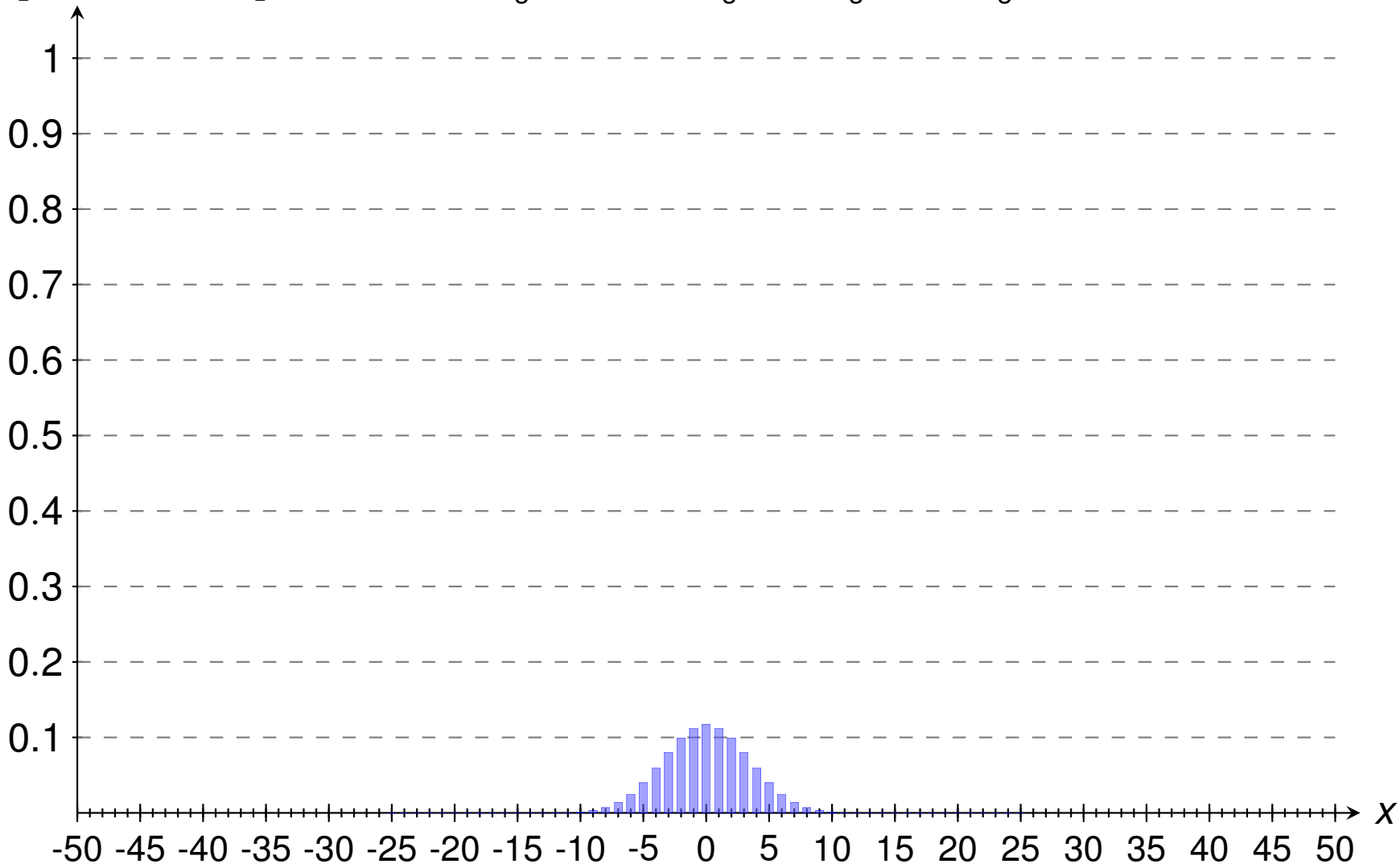

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{18} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

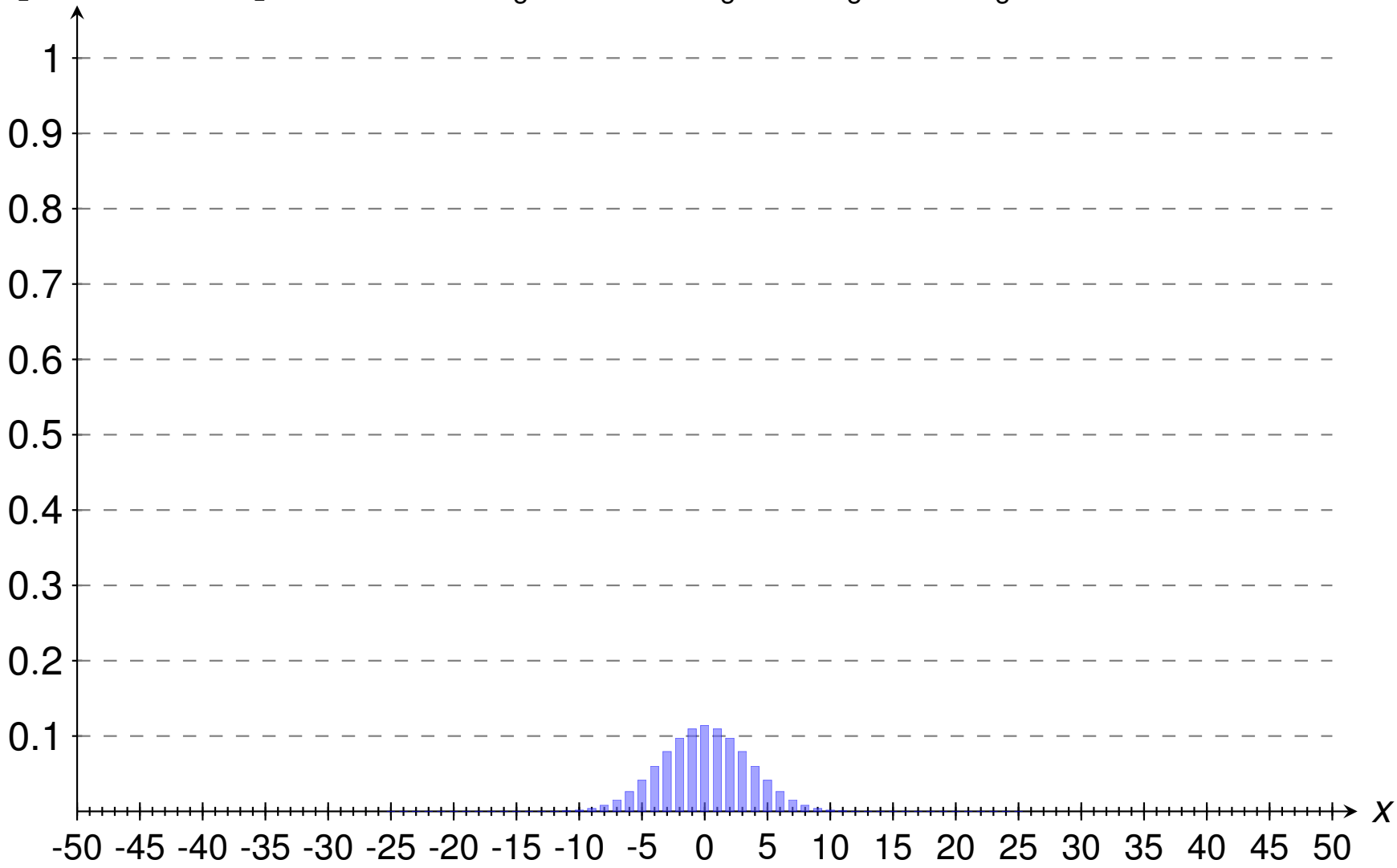

Illustration of CLT (1/4)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

$\mathbf{P}\left[\sum_{j=1}^{19} X_j = x\right]$

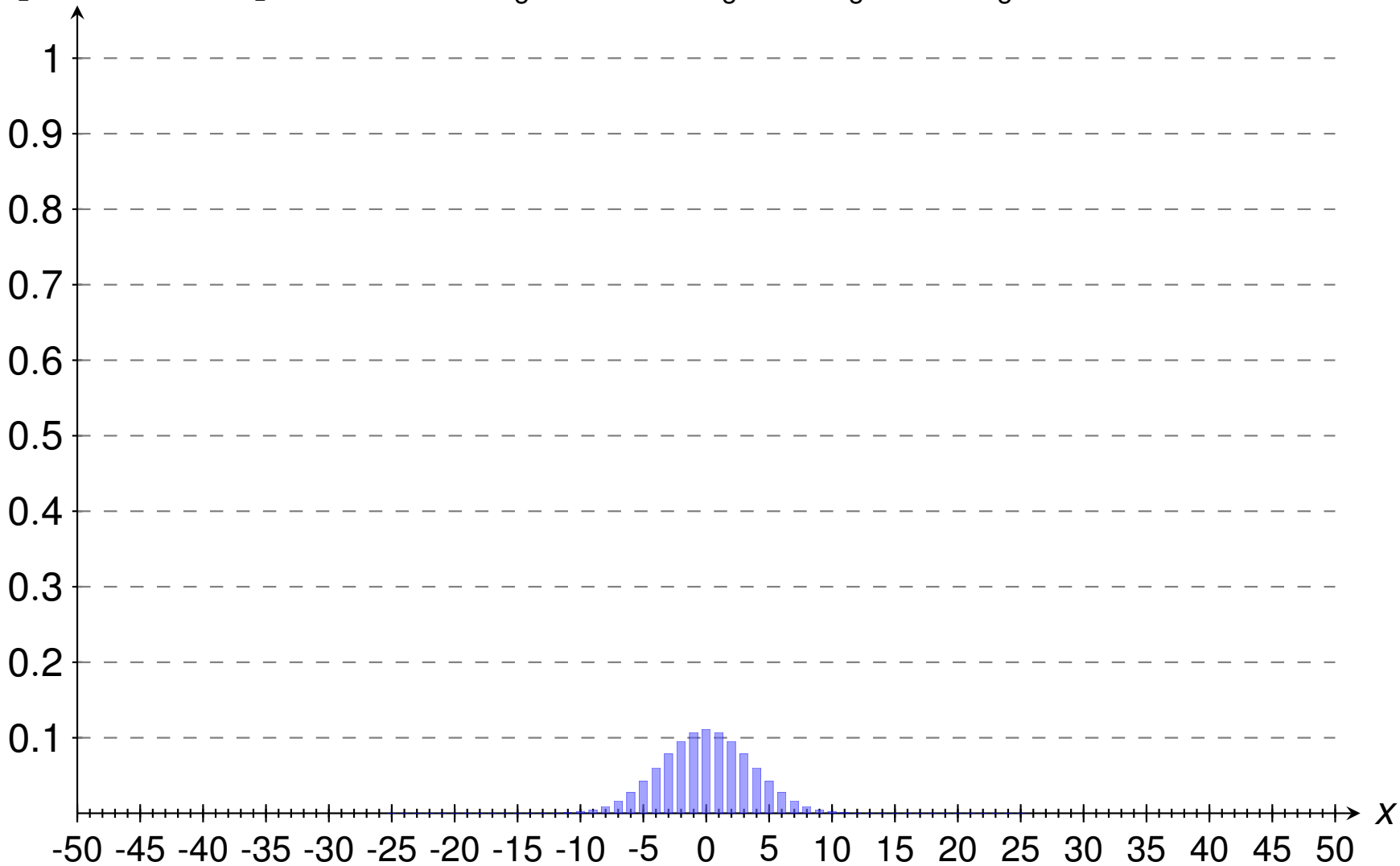

Illustration of CLT (1/4)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

$\mathbf{P}\left[\sum_{j=1}^{20} X_j = x\right]$

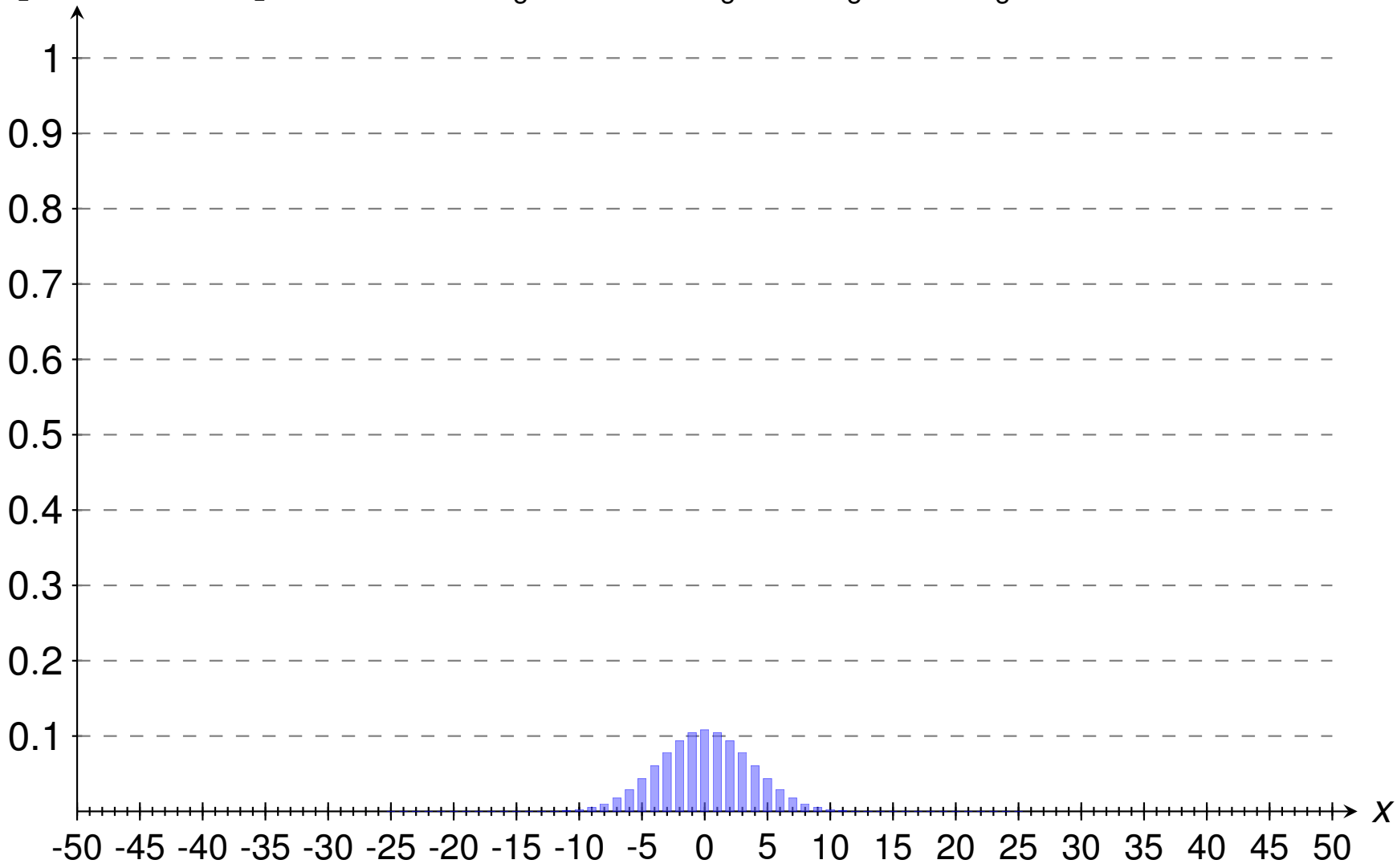

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{21} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

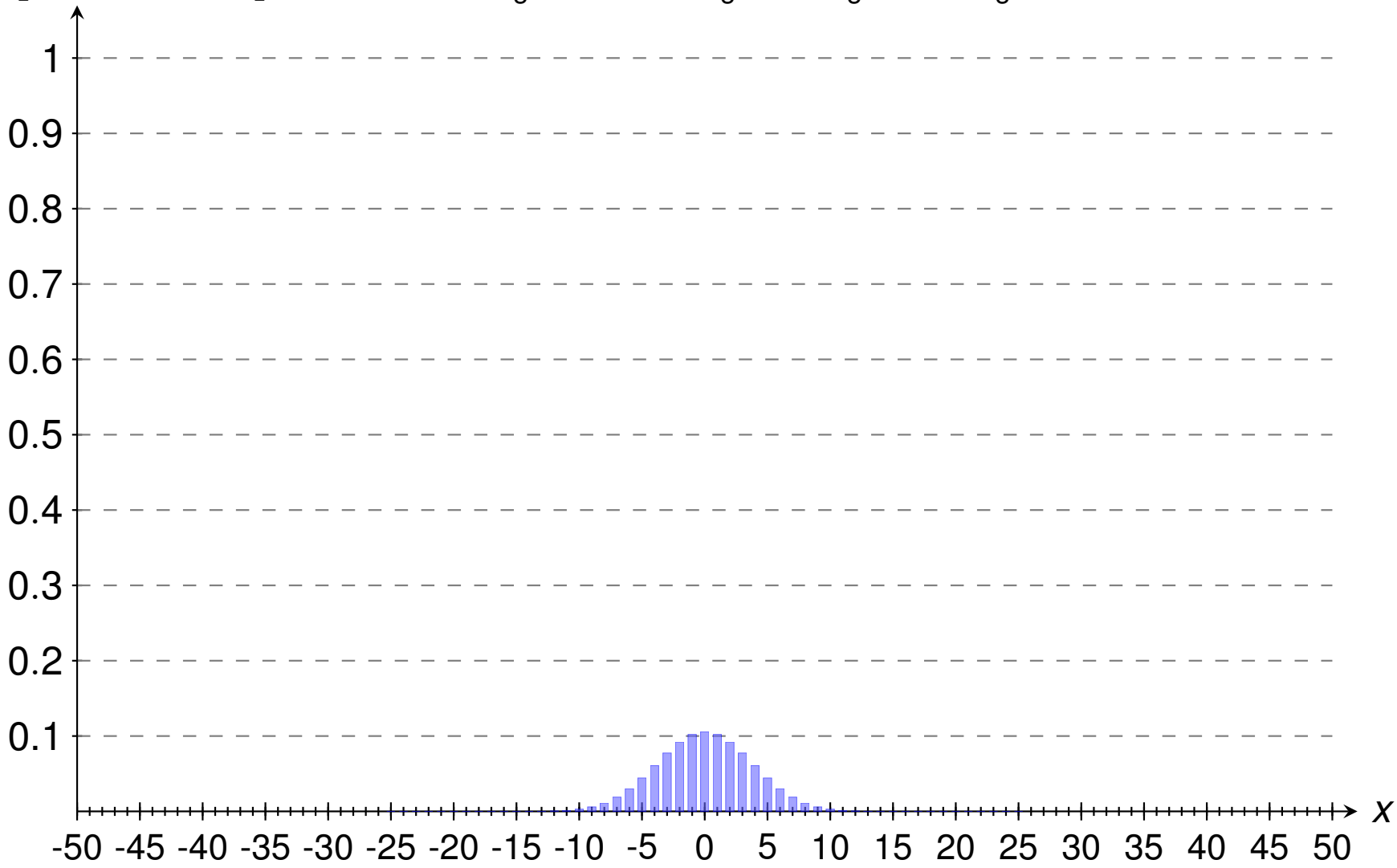

Illustration of CLT (1/4)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

$\mathbf{P}\left[\sum_{j=1}^{22} X_j = x\right]$

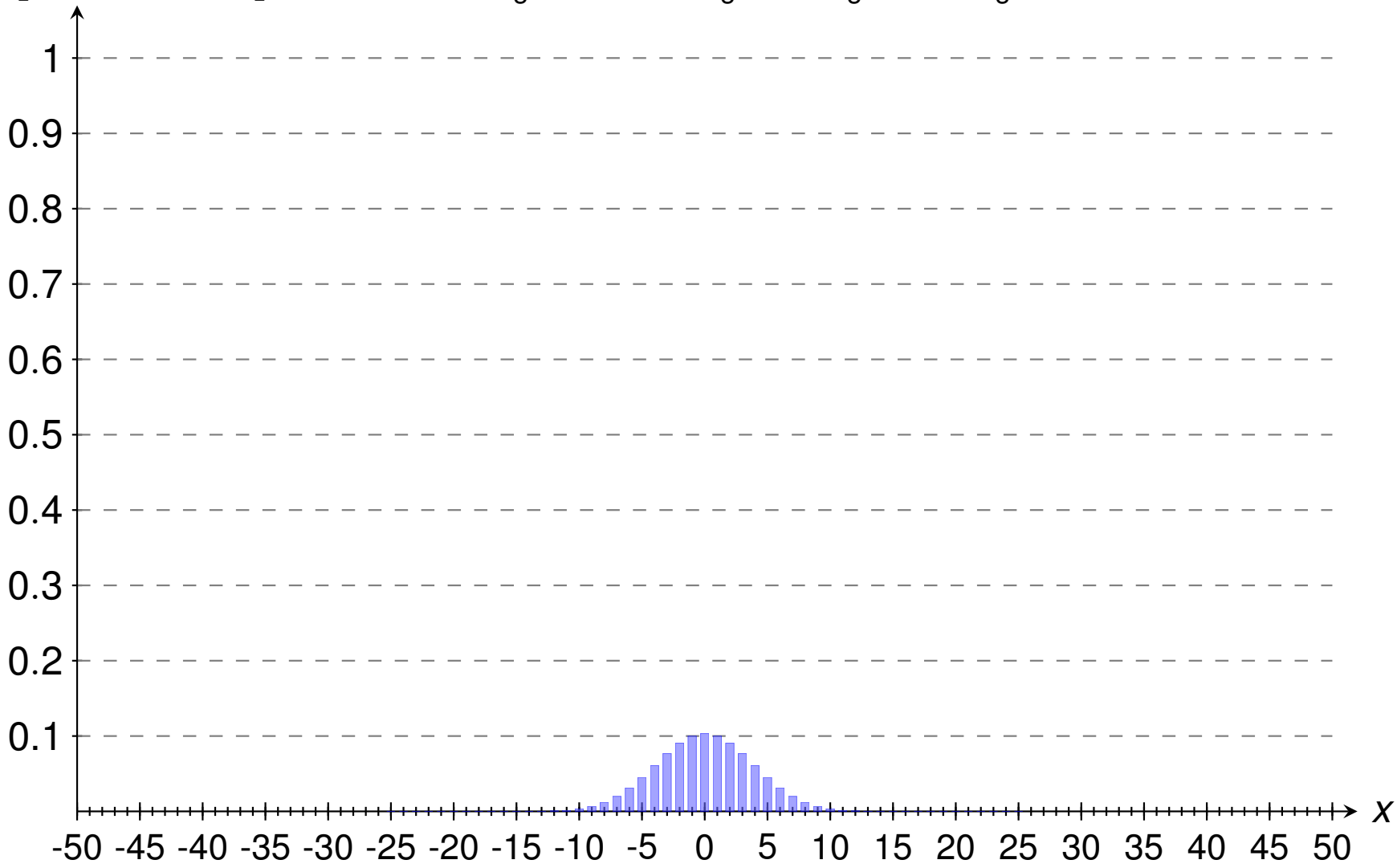

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{23} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

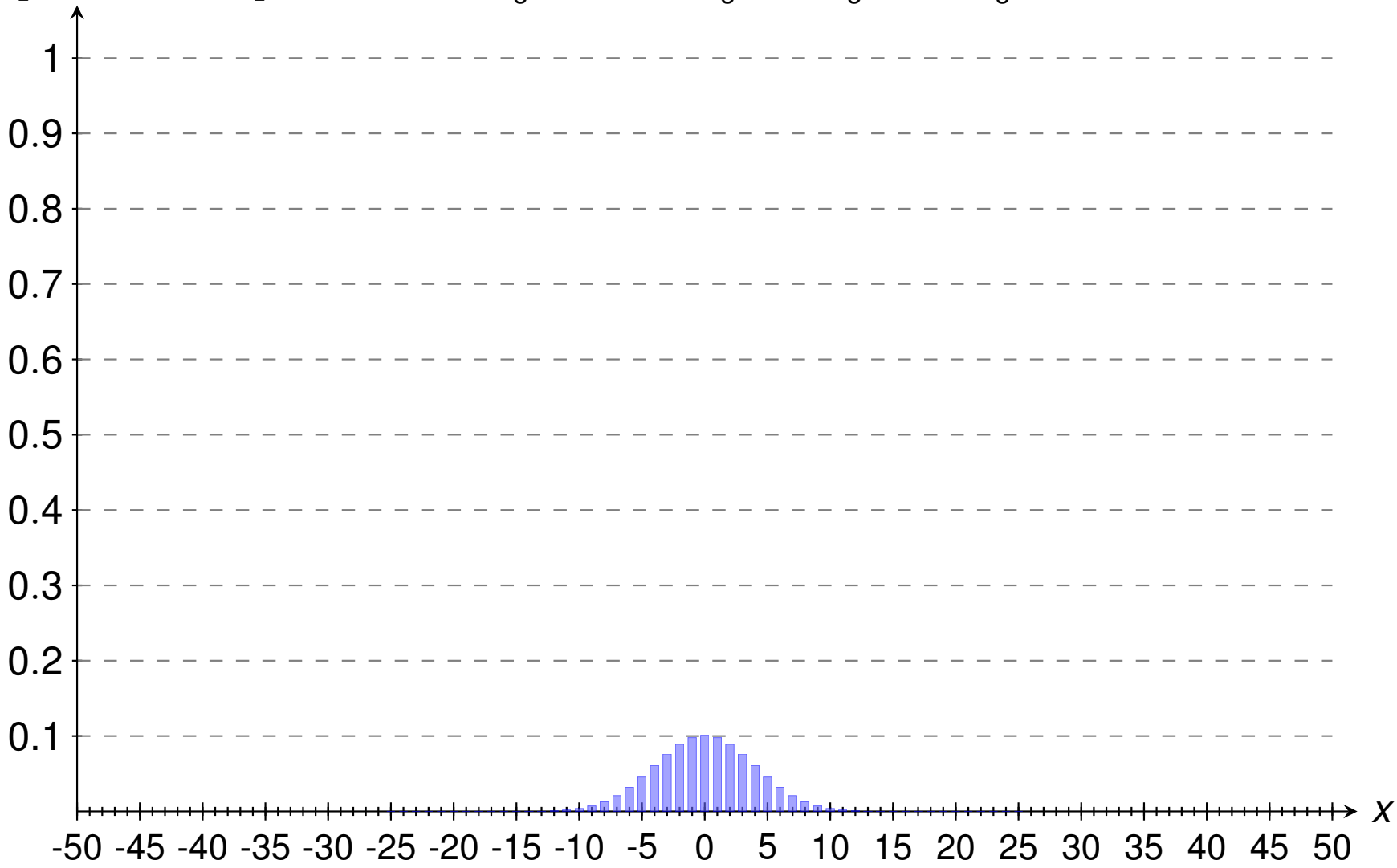

Illustration of CLT (1/4)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

$\mathbf{P}\left[\sum_{j=1}^{24} X_j = x\right]$

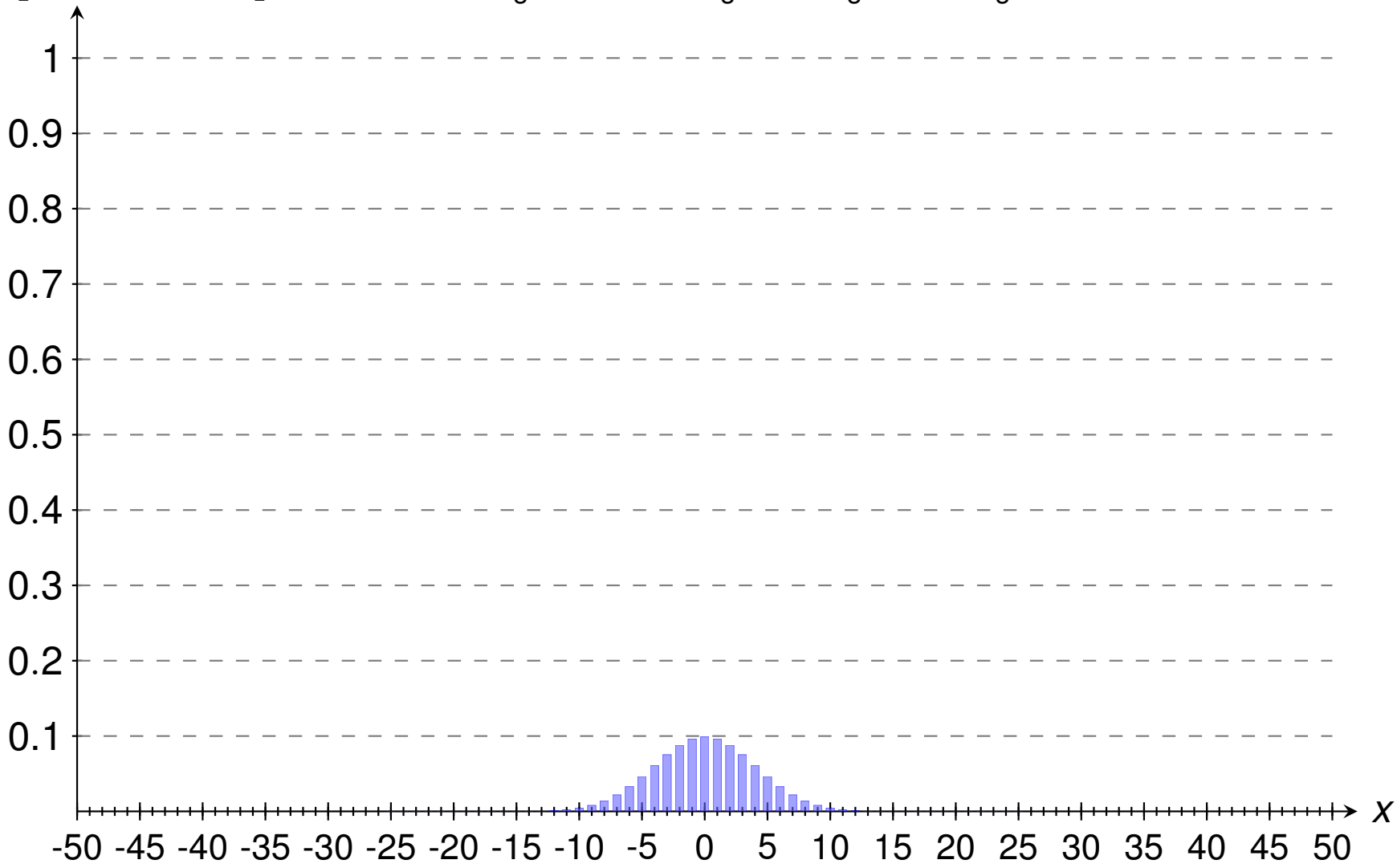

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{25} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

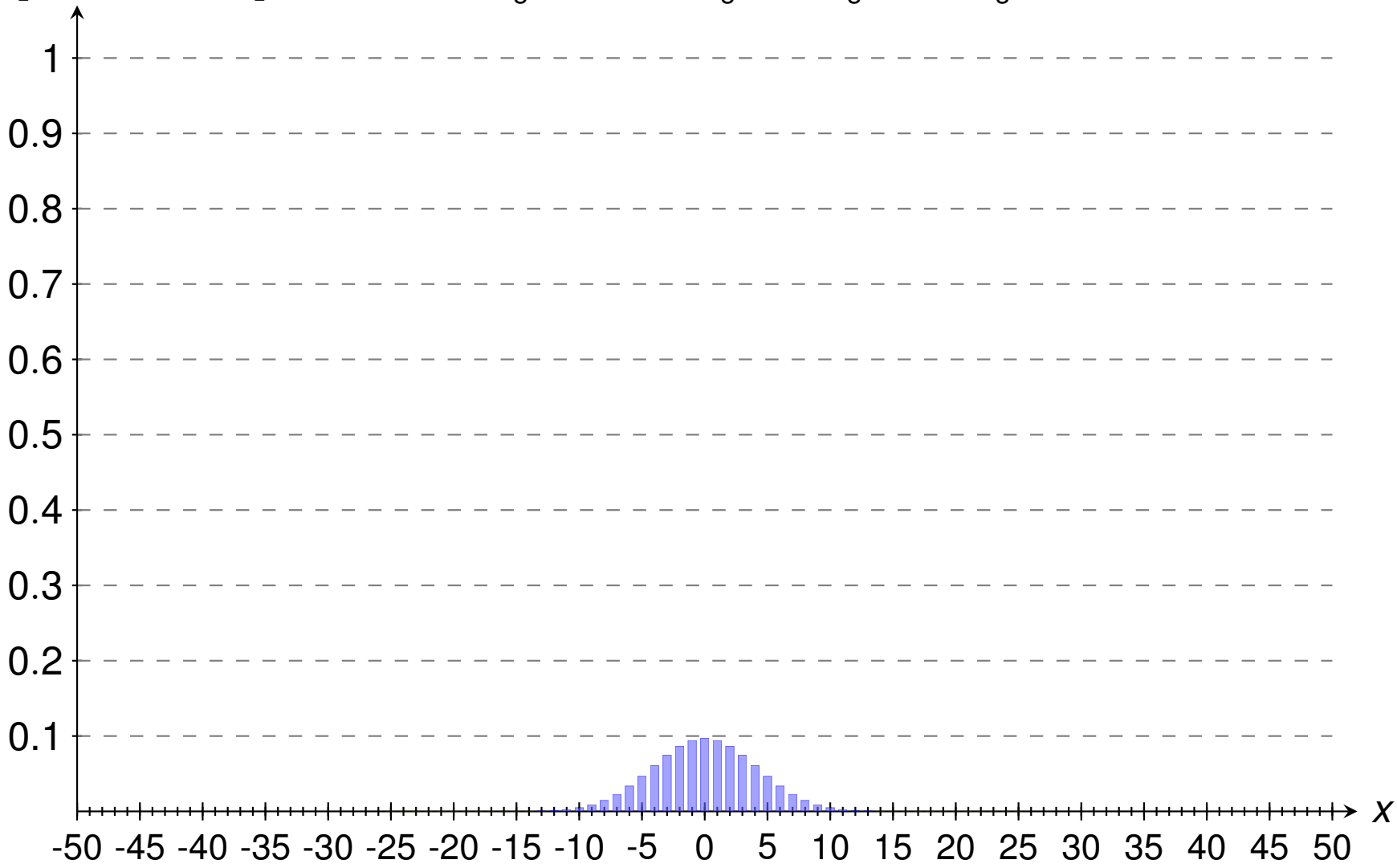

Illustration of CLT (1/4)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

$\mathbf{P}\left[\sum_{j=1}^{26} X_j = x\right]$

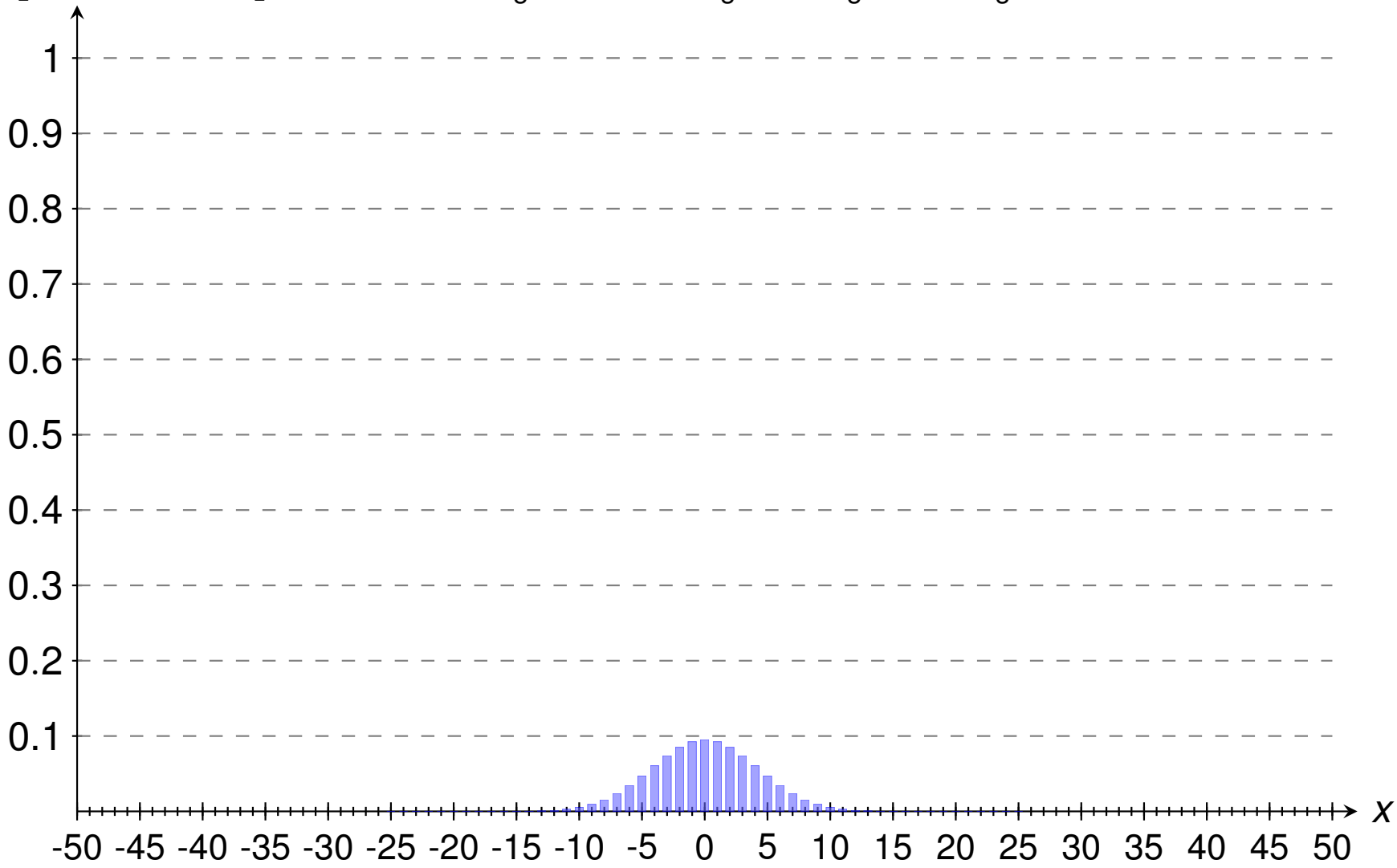

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{27} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

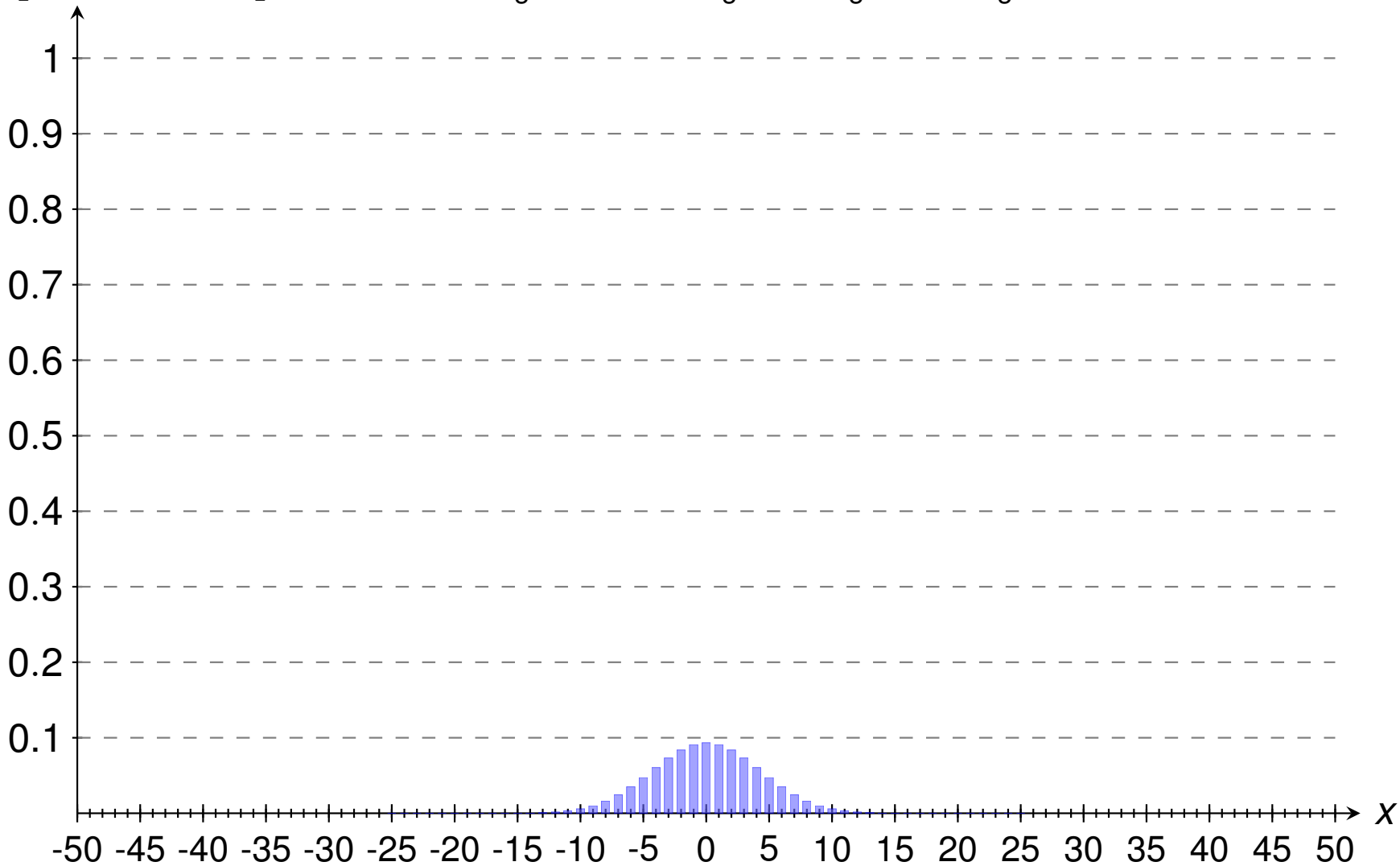

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{28} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

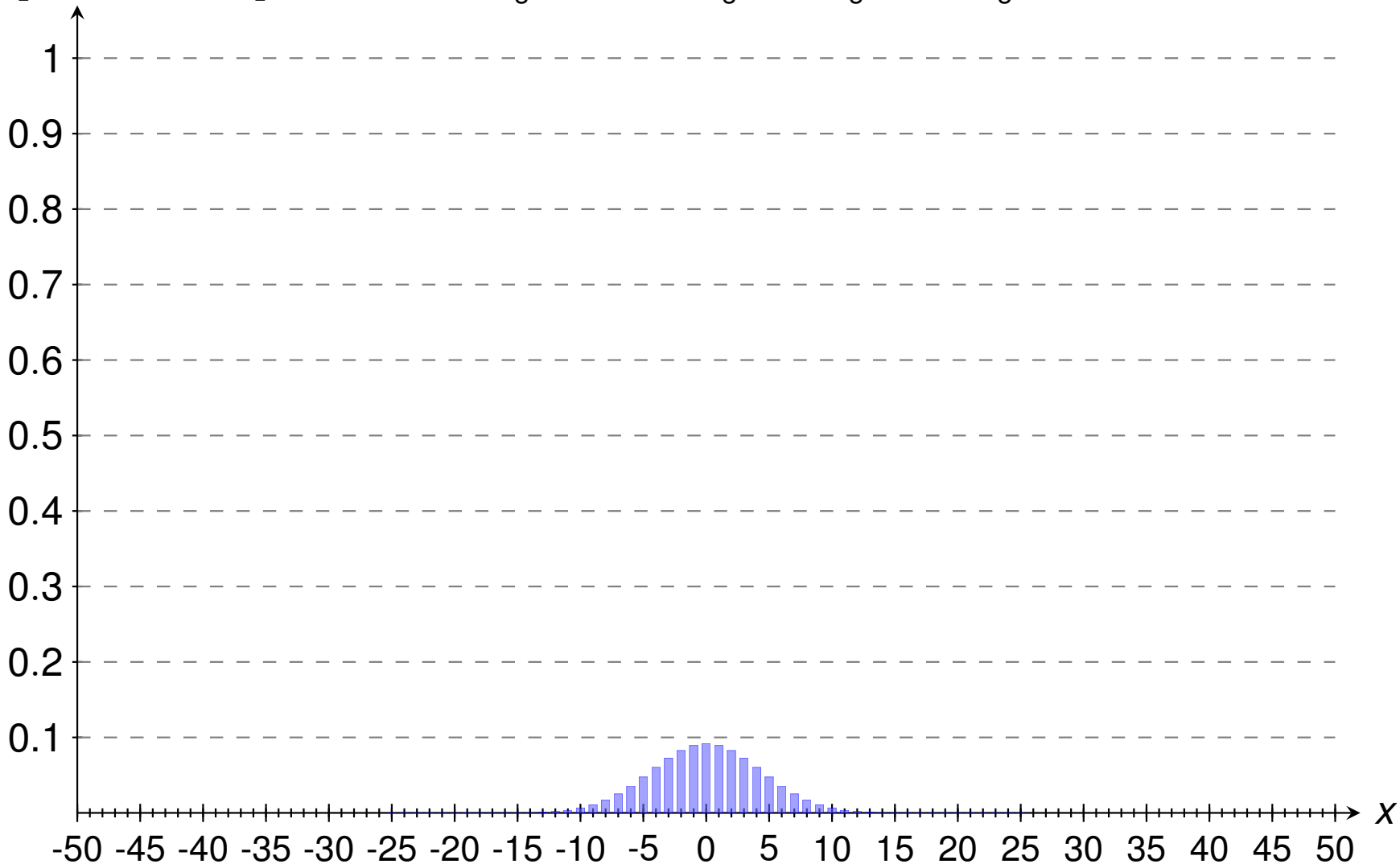

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{29} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

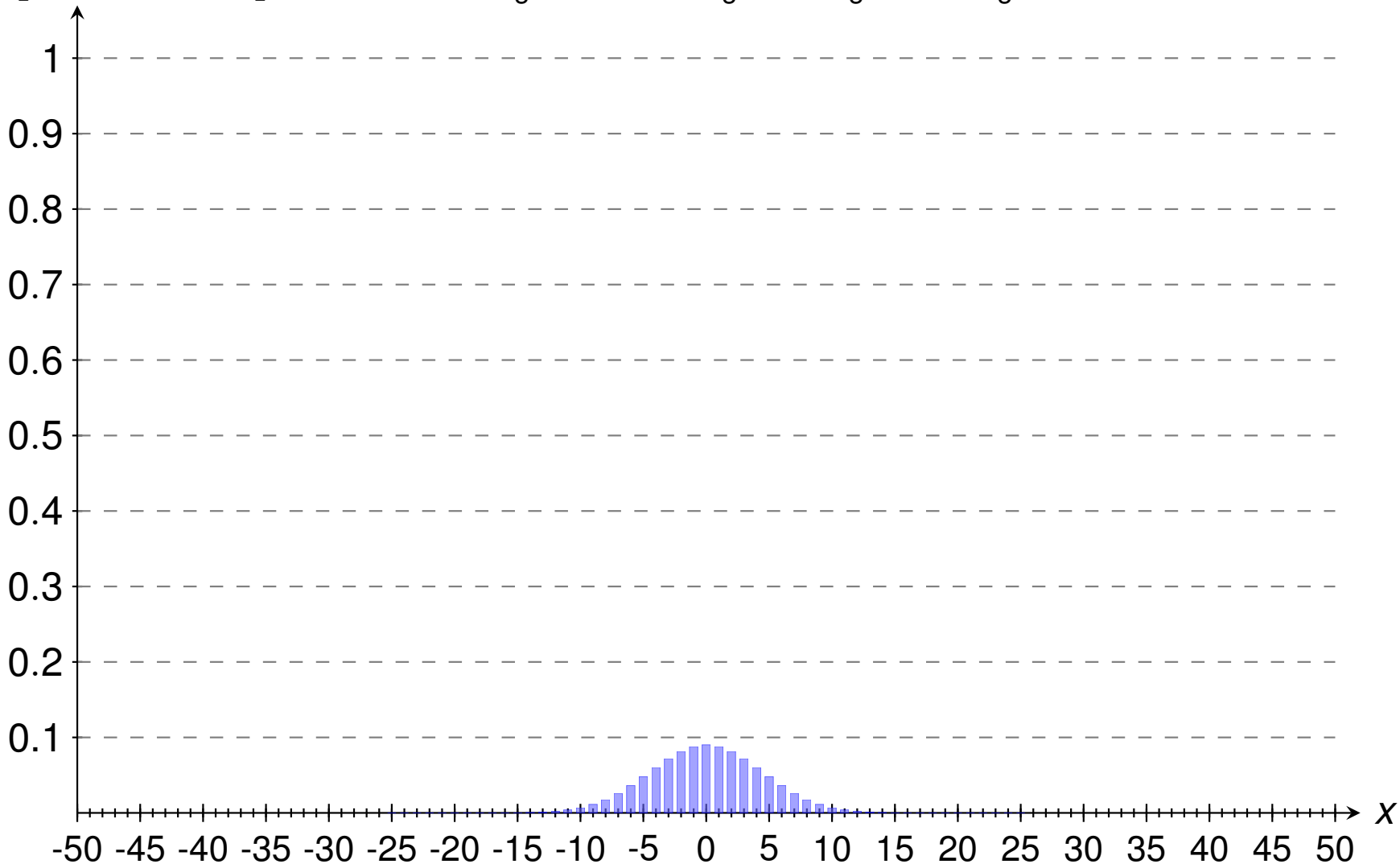

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

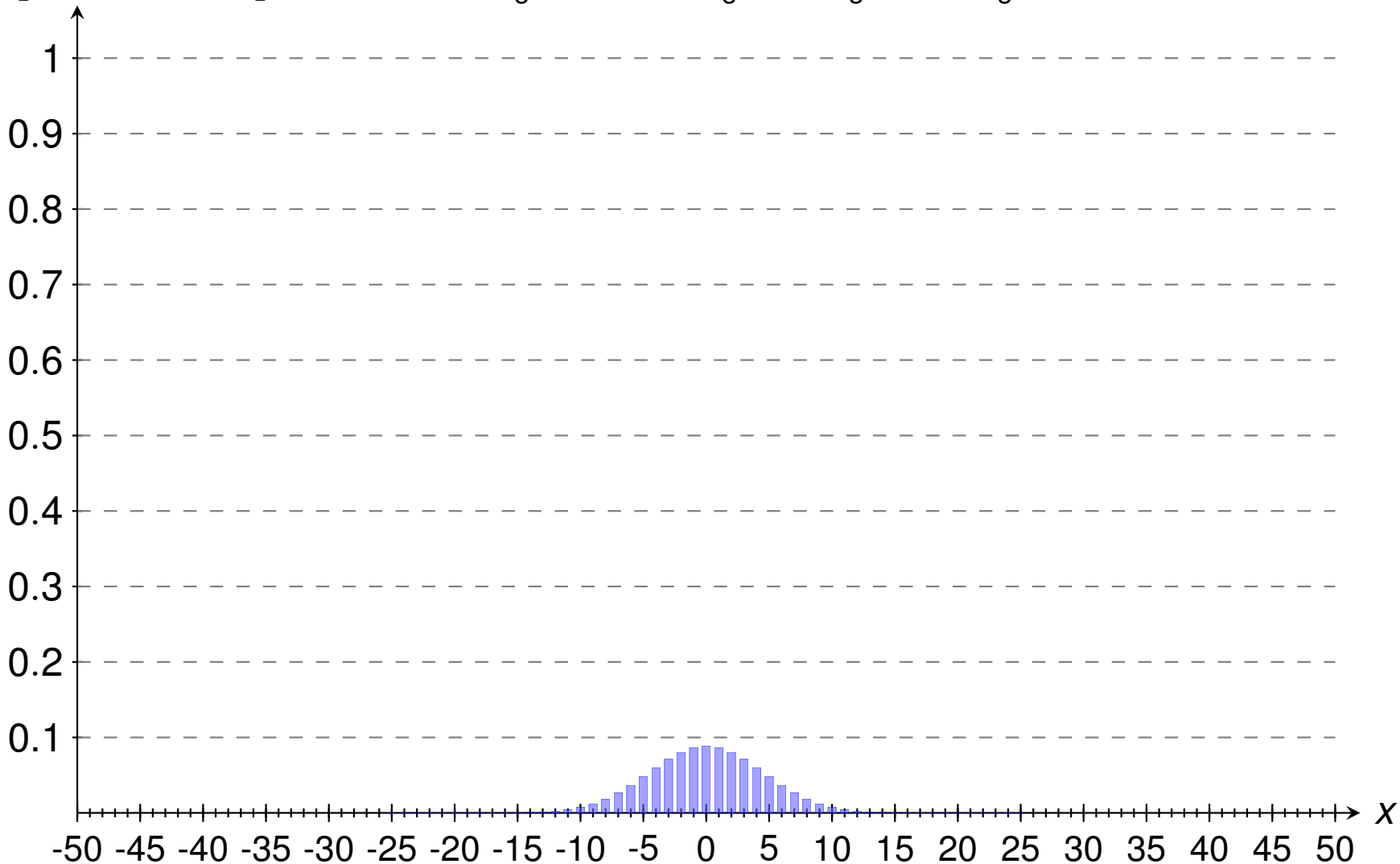

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

By the CLT:

$$Z_n = \frac{1}{\sqrt{n} \cdot \sigma} \cdot \left(\sum_{i=1}^{n} X_i - n \cdot \mu \right) \overset{n \to \infty}{\longrightarrow} Z \sim \mathcal{N}(0, 1)$$

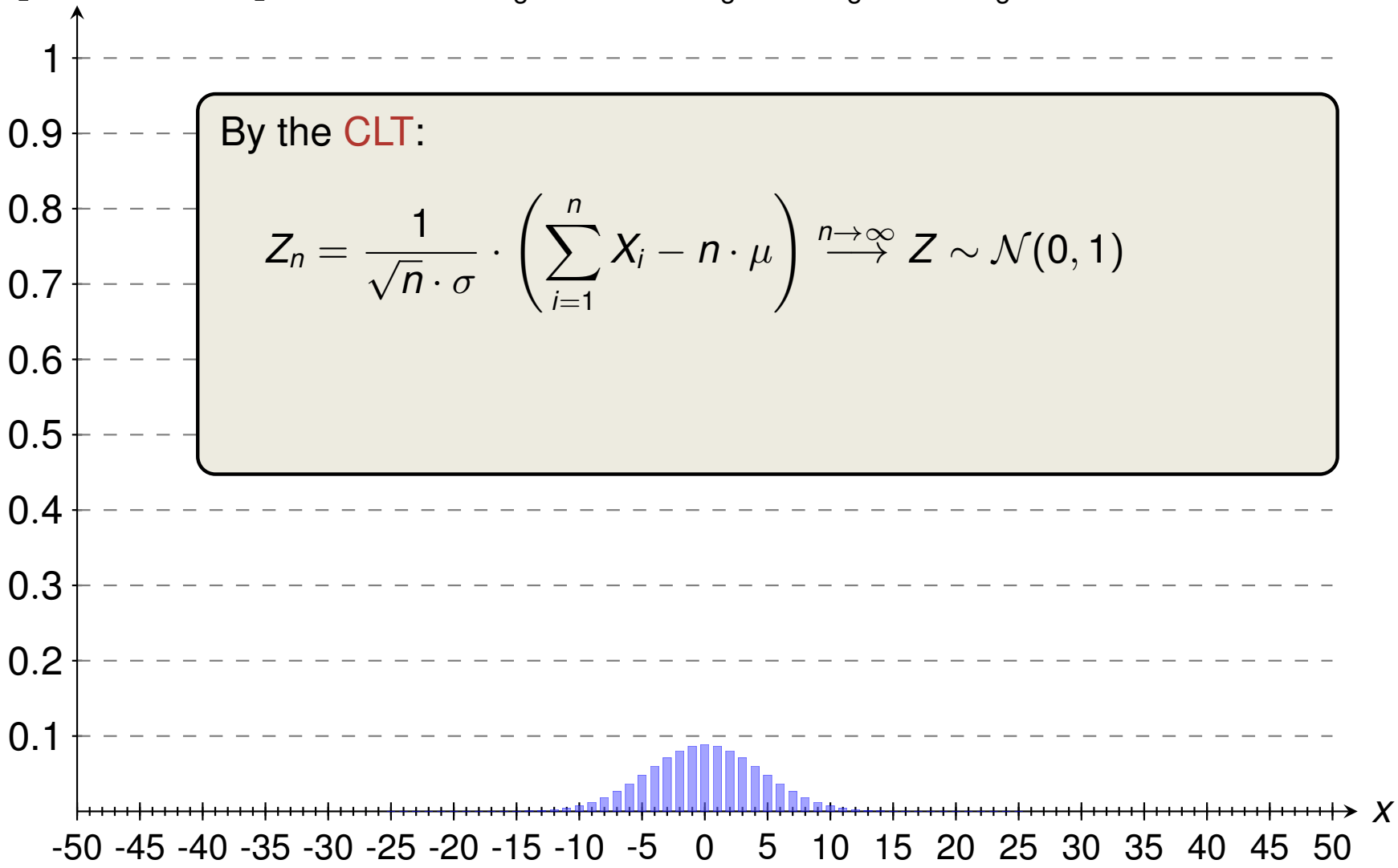

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

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$$\Rightarrow \quad \sum_{i=1}^{n} X_i$$

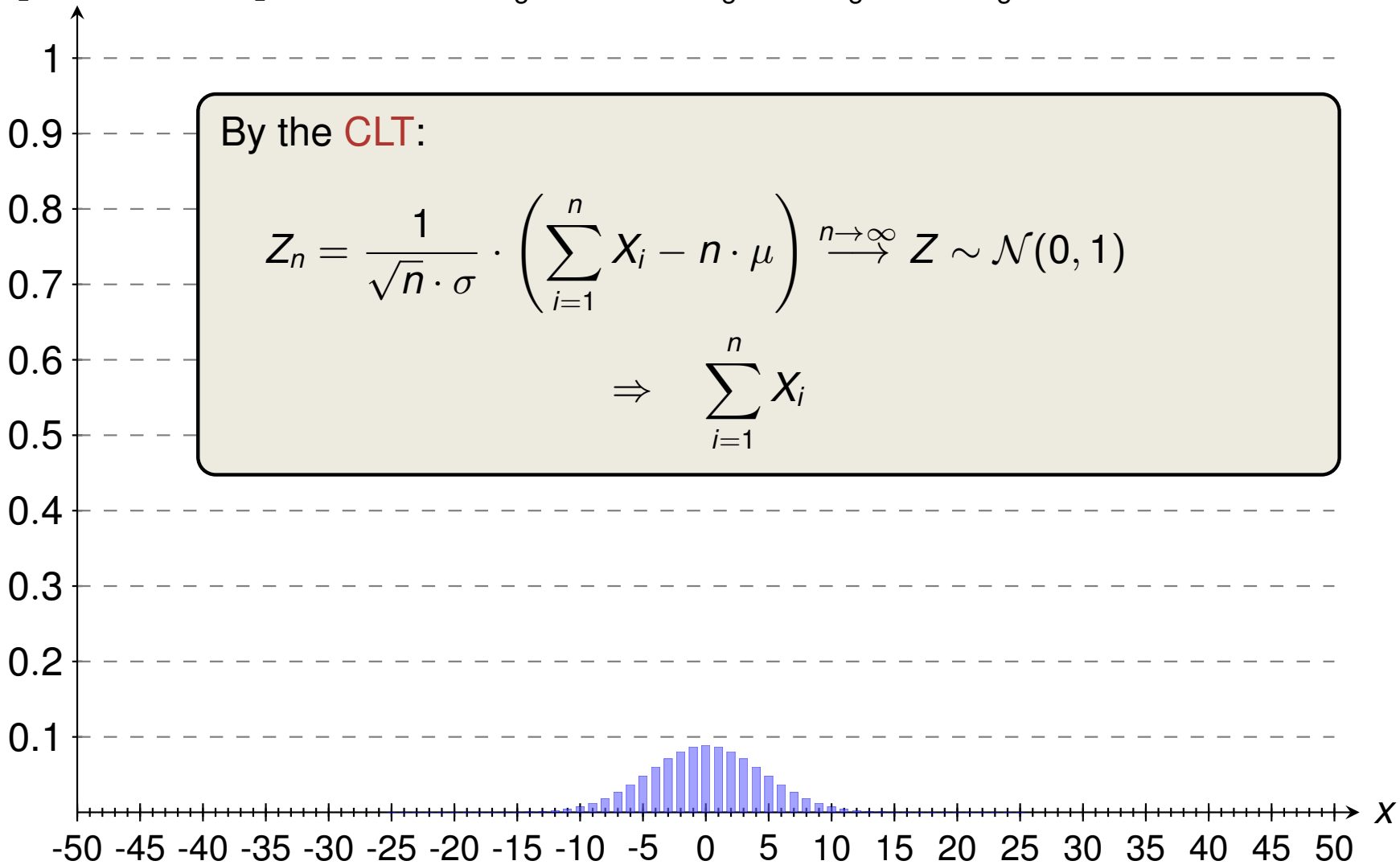

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

By the CLT:

$$Z_n = \frac{1}{\sqrt{n} \cdot \sigma} \cdot \left(\sum_{i=1}^{n} X_i - n \cdot \mu \right) \overset{n \to \infty}{\longrightarrow} Z \sim \mathcal{N}(0,1)$$

$$\Rightarrow \quad \sum_{i=1}^{n} X_i \approx \sqrt{n} \cdot \sigma Z$$

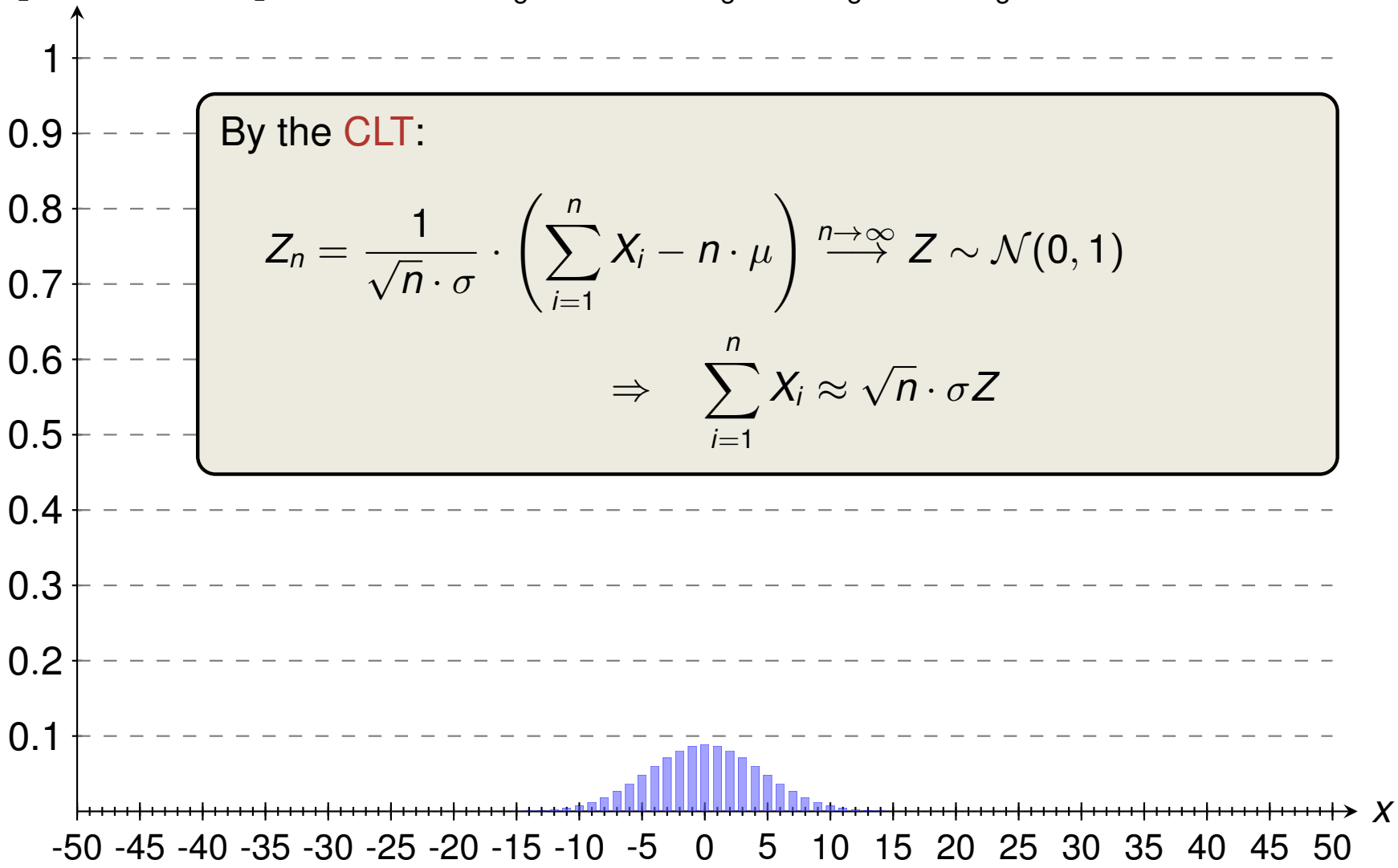

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

By the CLT:

$$Z_n = \frac{1}{\sqrt{n} \cdot \sigma} \cdot \left(\sum_{i=1}^{n} X_i - n \cdot \mu \right) \overset{n \to \infty}{\longrightarrow} Z \sim \mathcal{N}(0, 1)$$

$$\Rightarrow \quad \sum_{i=1}^{n} X_i \approx \sqrt{n} \cdot \sigma Z \sim \mathcal{N}(0, n \cdot \sigma^2)$$

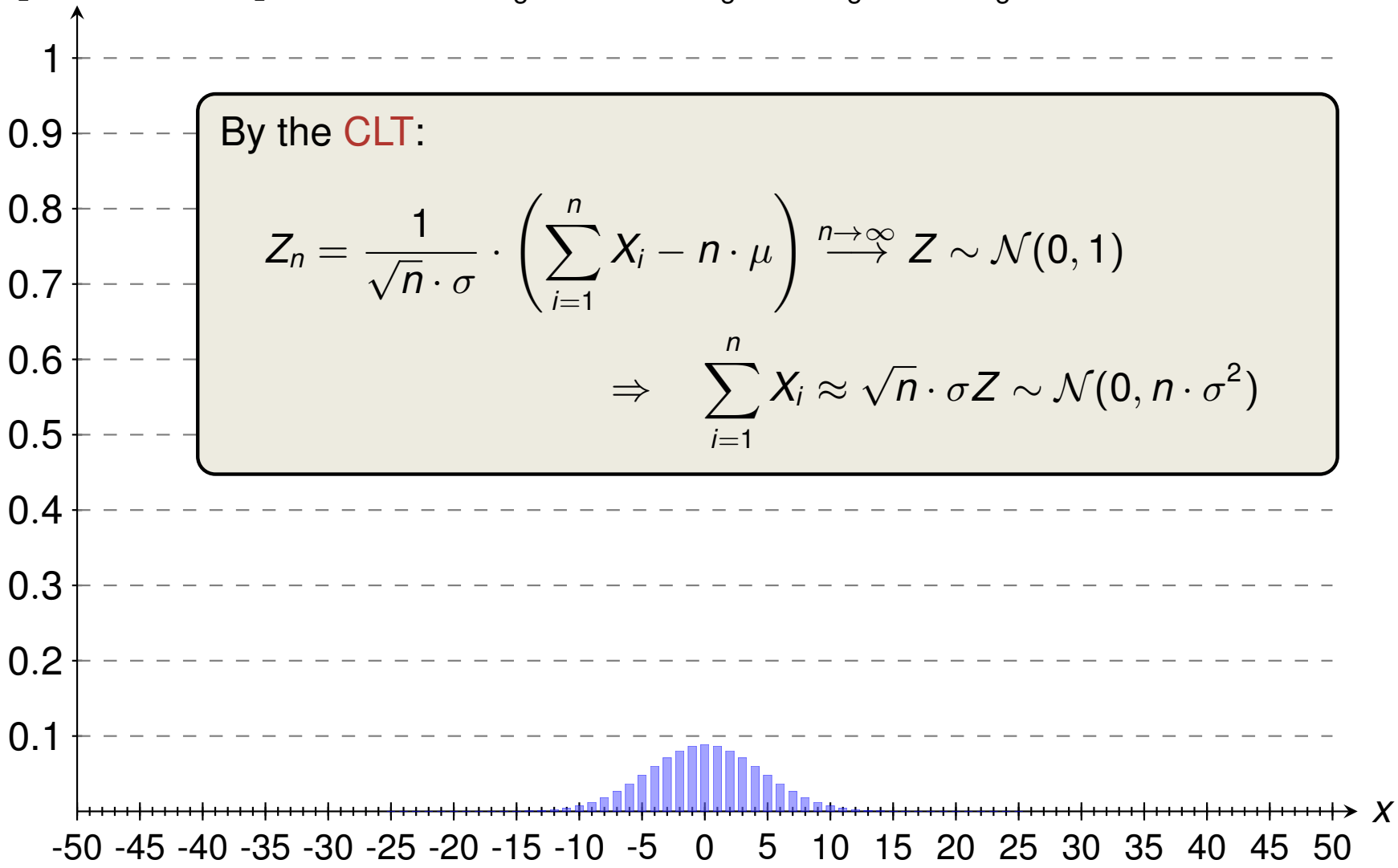

Illustration of CLT (1/4)

$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$

- $\mu = \frac{1}{3}\cdot(-1) + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 = 0$
- $\sigma^2 = \frac{1}{3}\cdot(-1)^2 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1^2 = \frac{2}{3}$

By the CLT:

$$Z_n = \frac{1}{\sqrt{n}\cdot\sigma}\cdot\left(\sum_{i=1}^{n} X_i - n\cdot\mu\right) \overset{n\to\infty}{\longrightarrow} Z \sim \mathcal{N}(0,1)$$

$$\Rightarrow \quad \sum_{i=1}^{n} X_i \approx \sqrt{n}\cdot\sigma Z \sim \mathcal{N}(0, n\cdot\sigma^2)$$

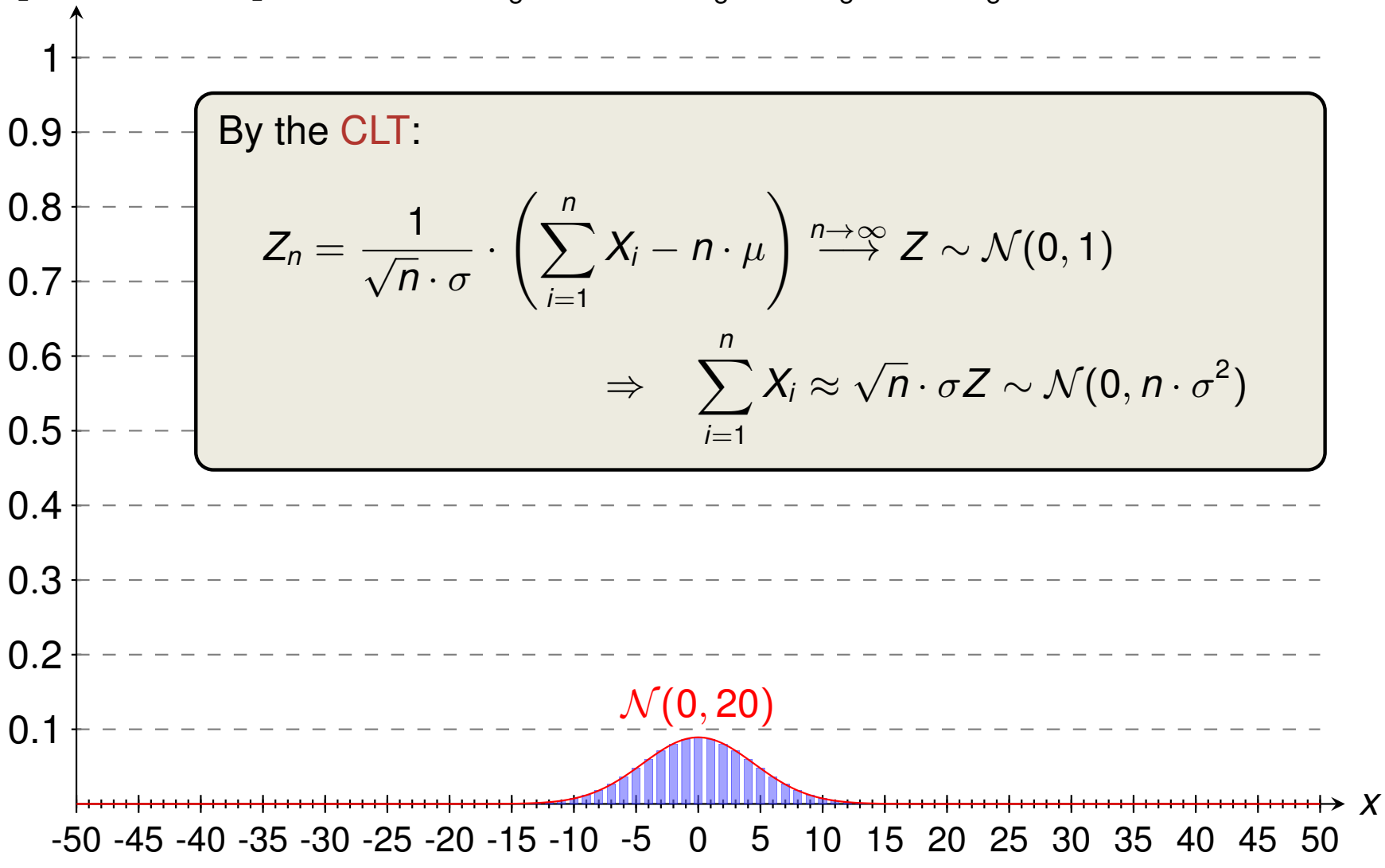

Illustration of CLT (2/4)

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

$\mathbf{P}\left[\sum_{j=1}^{1} X_j = x\right]$

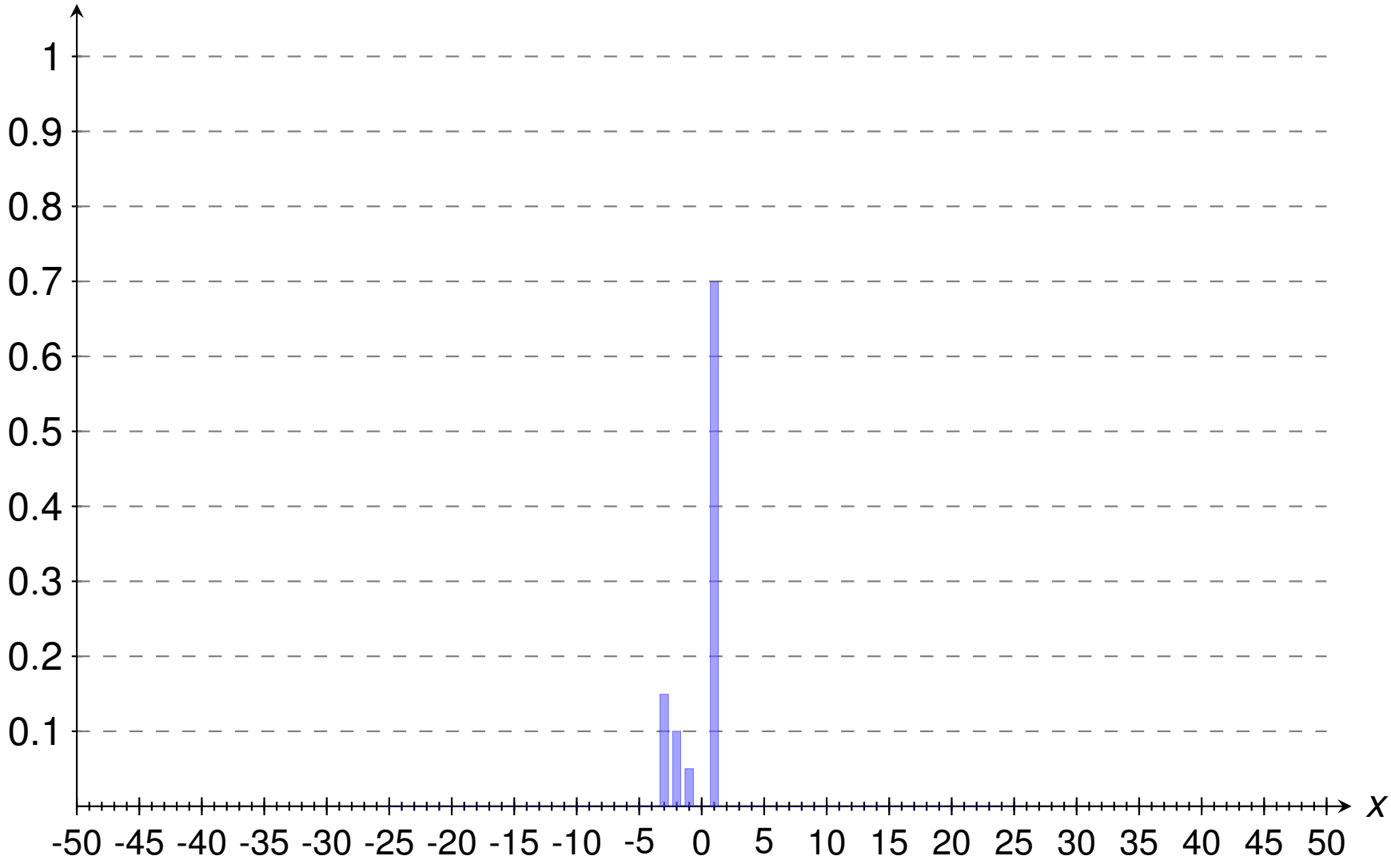

Illustration of CLT (2/4)

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

$\mathbf{P}\left[\sum_{j=1}^{2} X_j = x\right]$

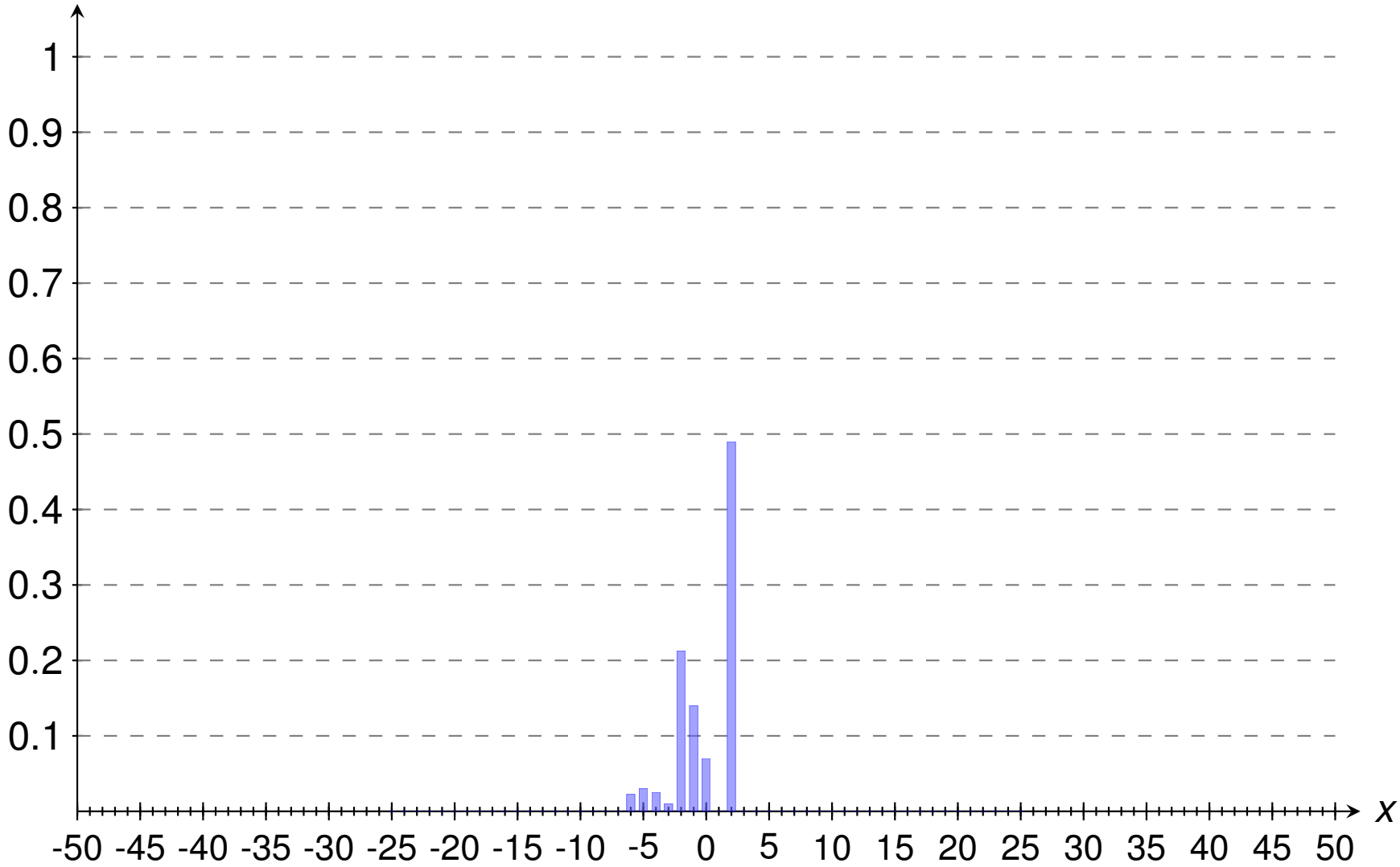

Illustration of CLT (2/4)

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

$\mathbf{P}\left[\sum_{j=1}^{3} X_j = x\right]$

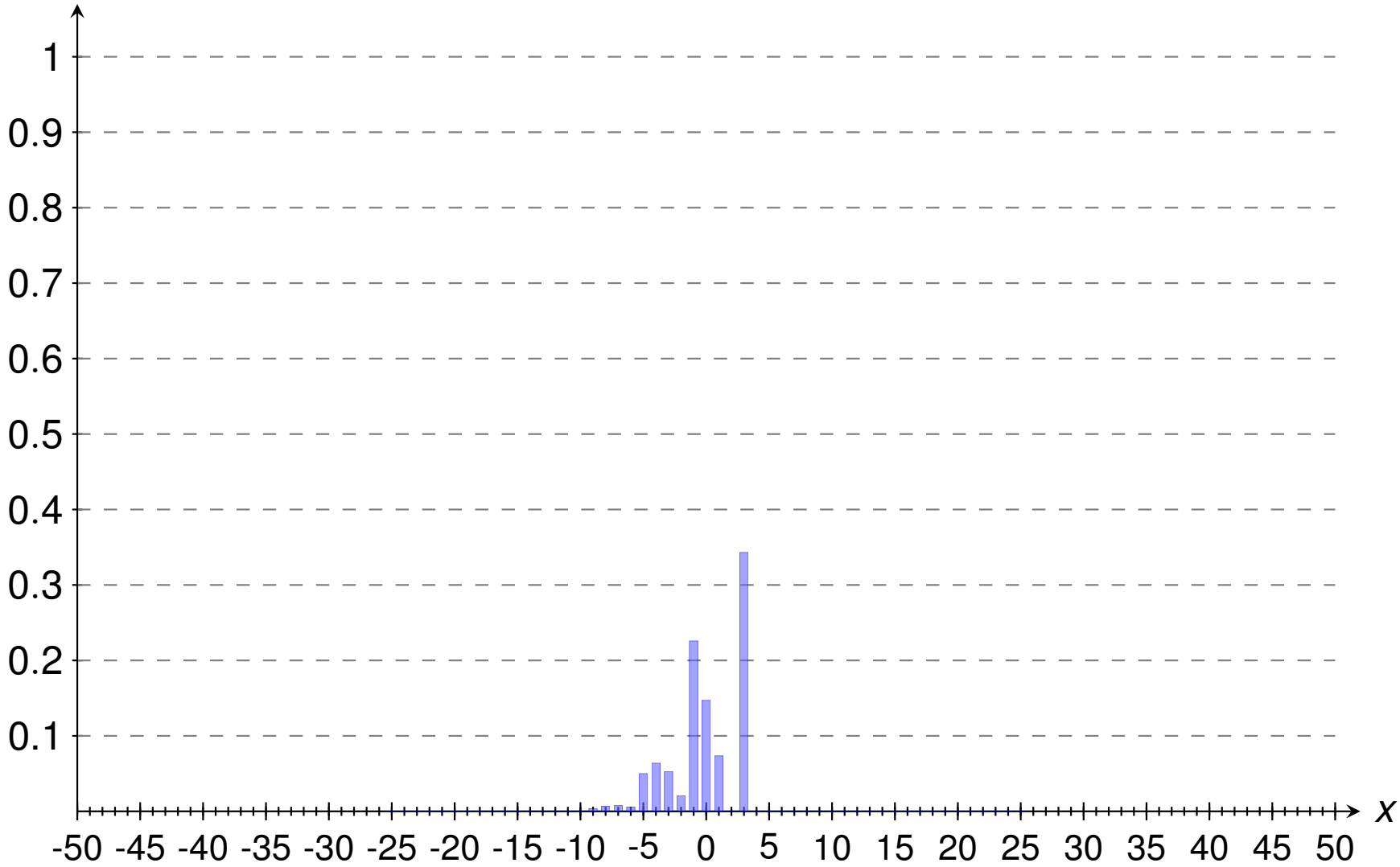

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{4} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

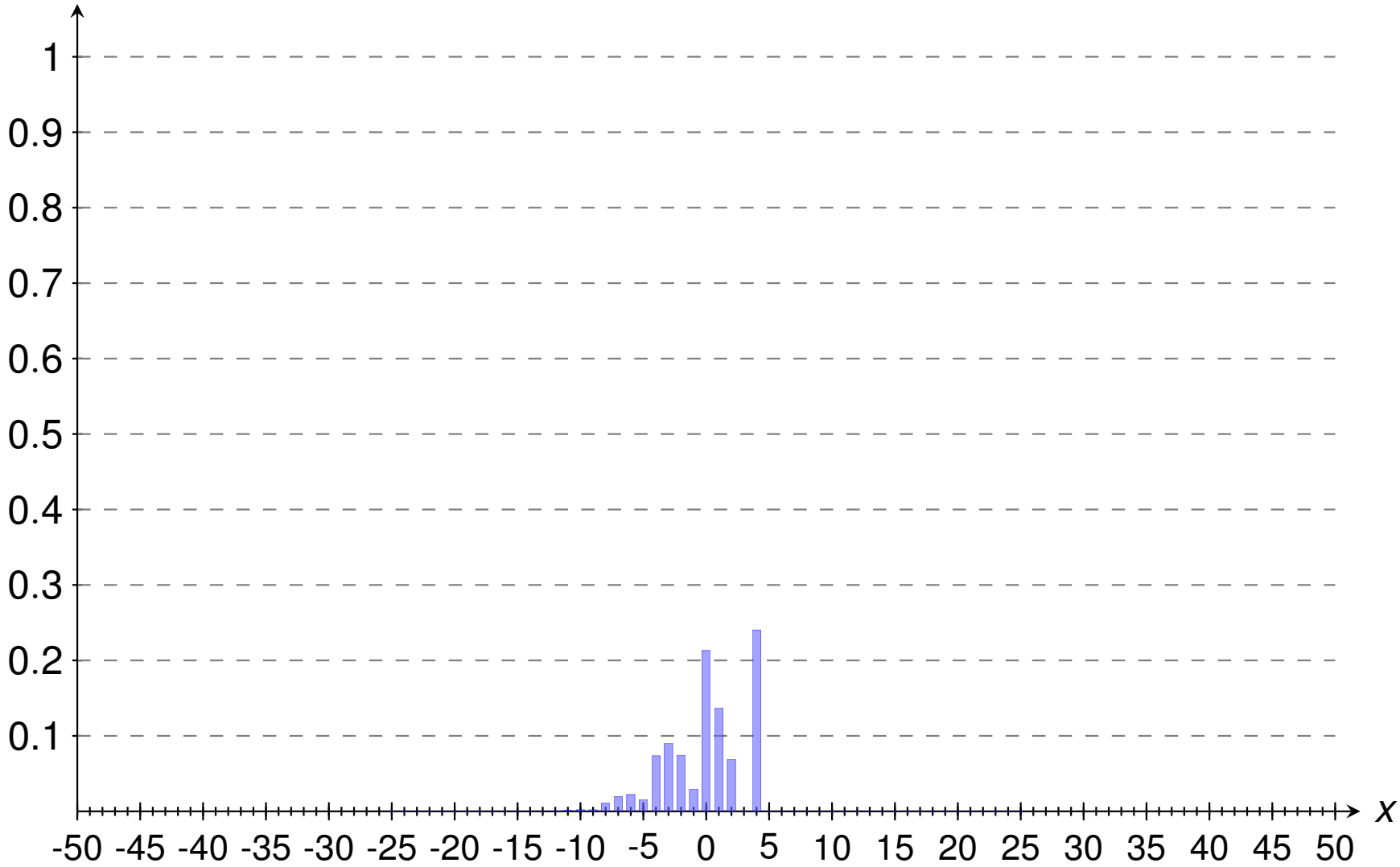

Illustration of CLT (2/4)

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

$\mathbf{P}\left[\sum_{j=1}^{5} X_j = x\right]$

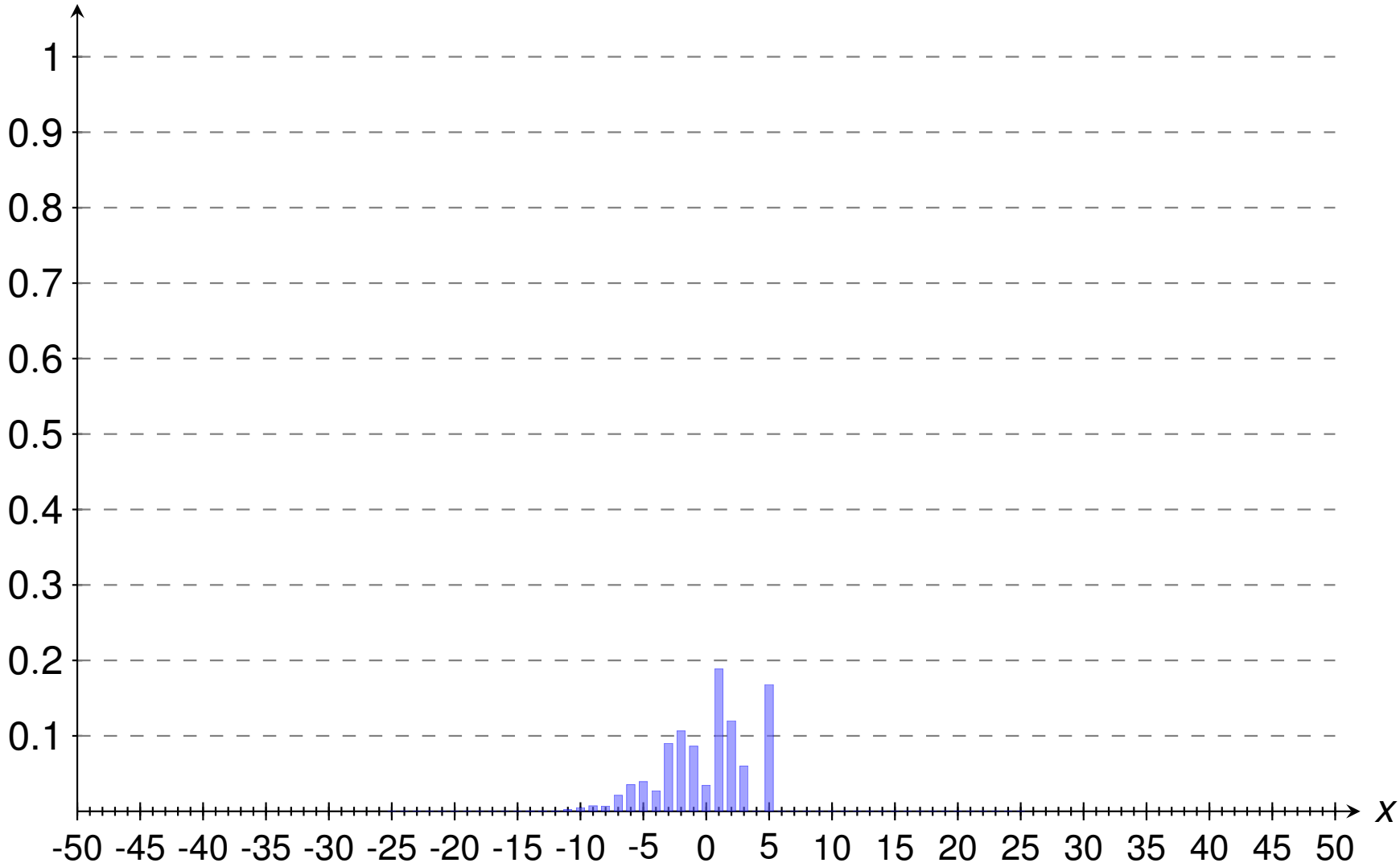

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{6} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

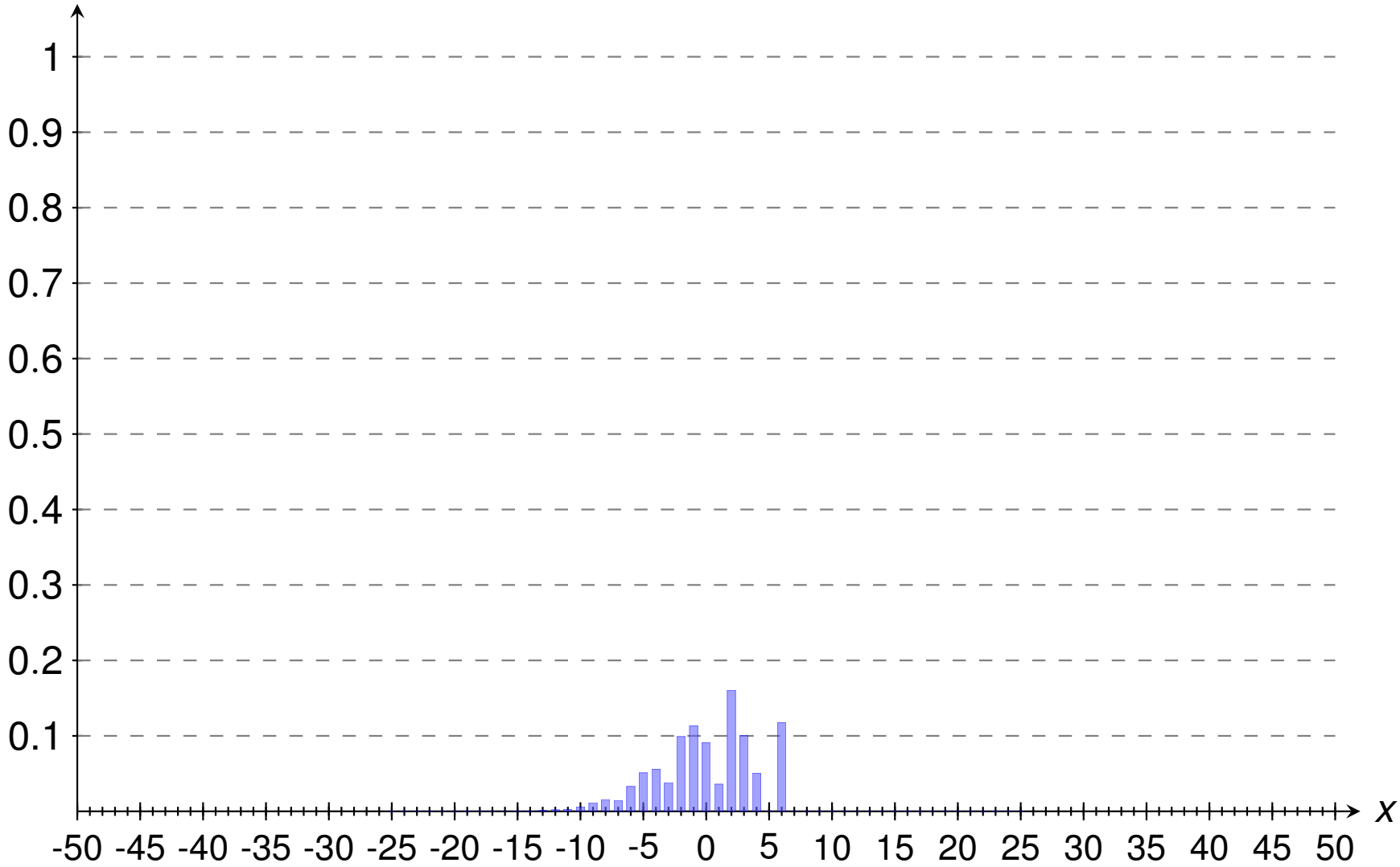

Illustration of CLT (2/4)

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

$\mathbf{P}\left[\sum_{j=1}^{7} X_j = x\right]$

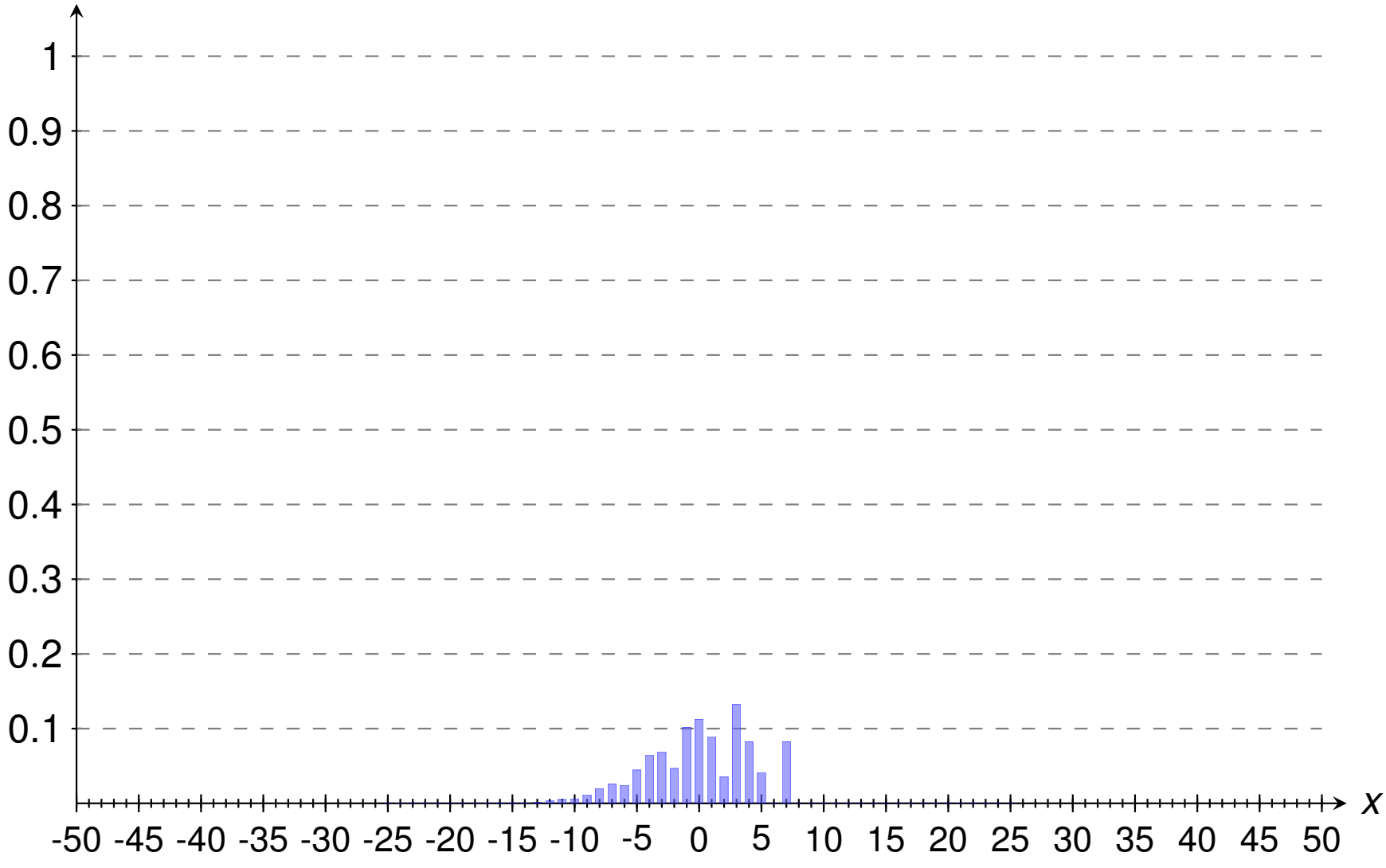

Illustration of CLT (2/4)

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

$\mathbf{P}\left[\sum_{j=1}^{8} X_j = x\right]$

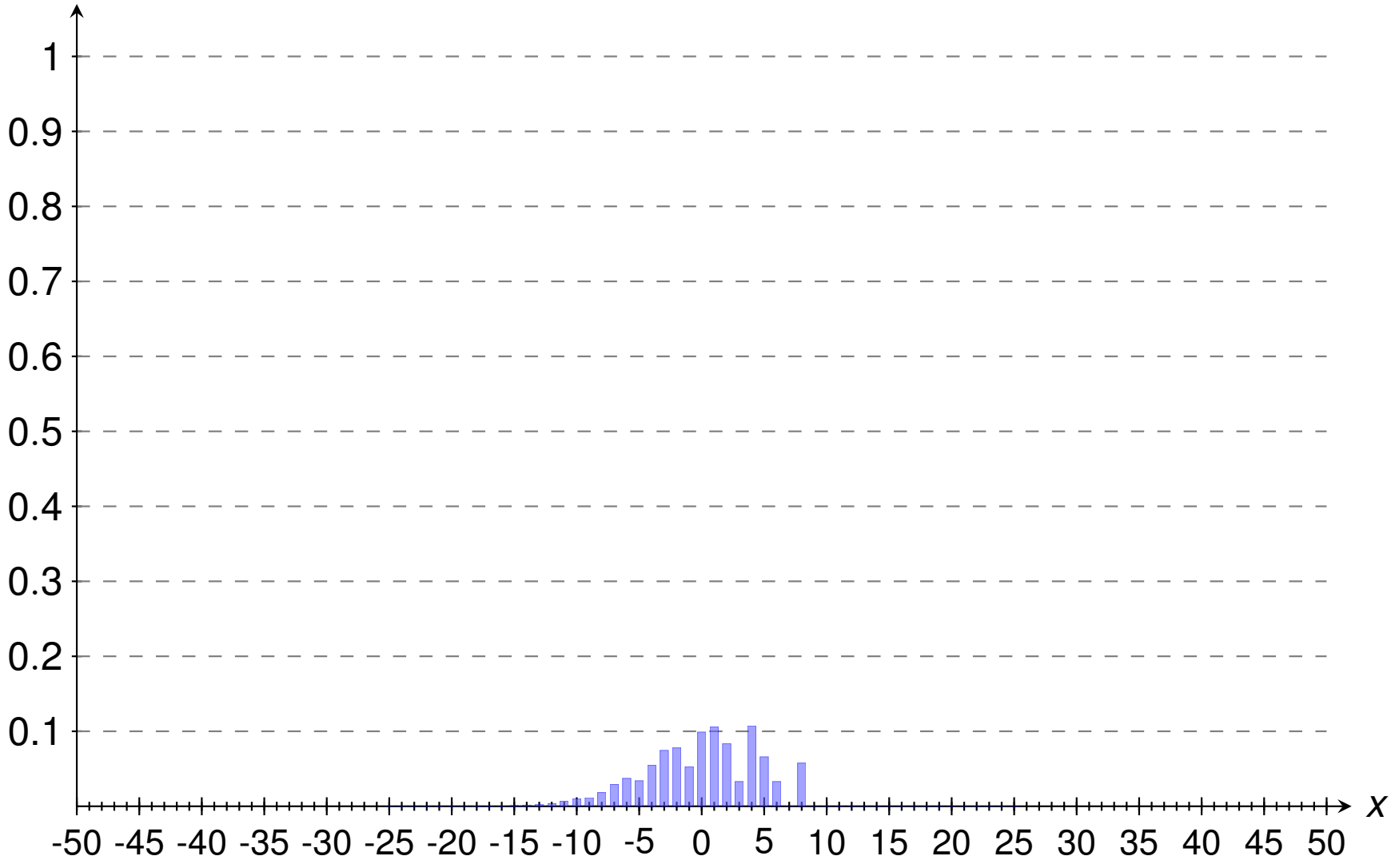

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{9} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

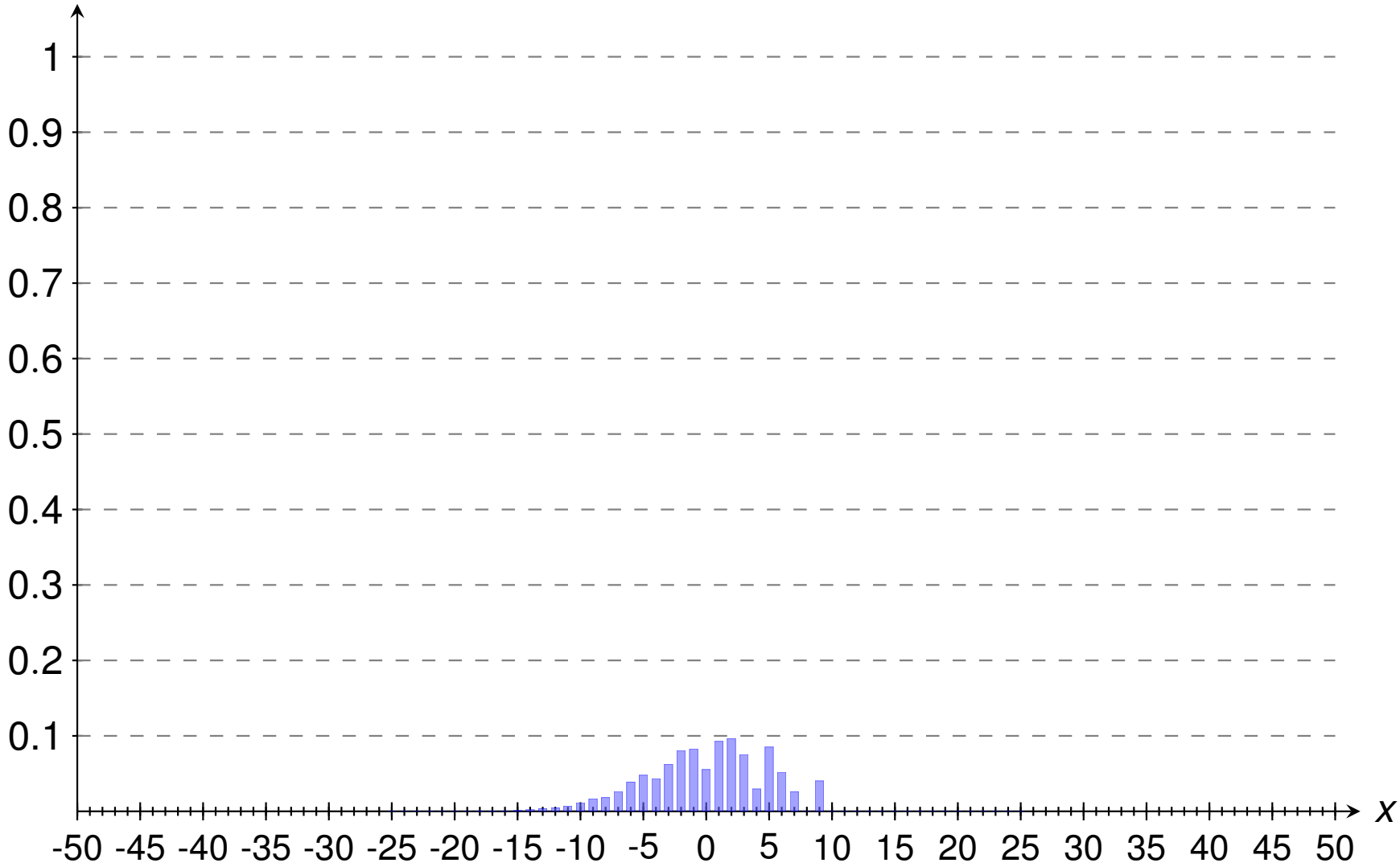

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{10} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

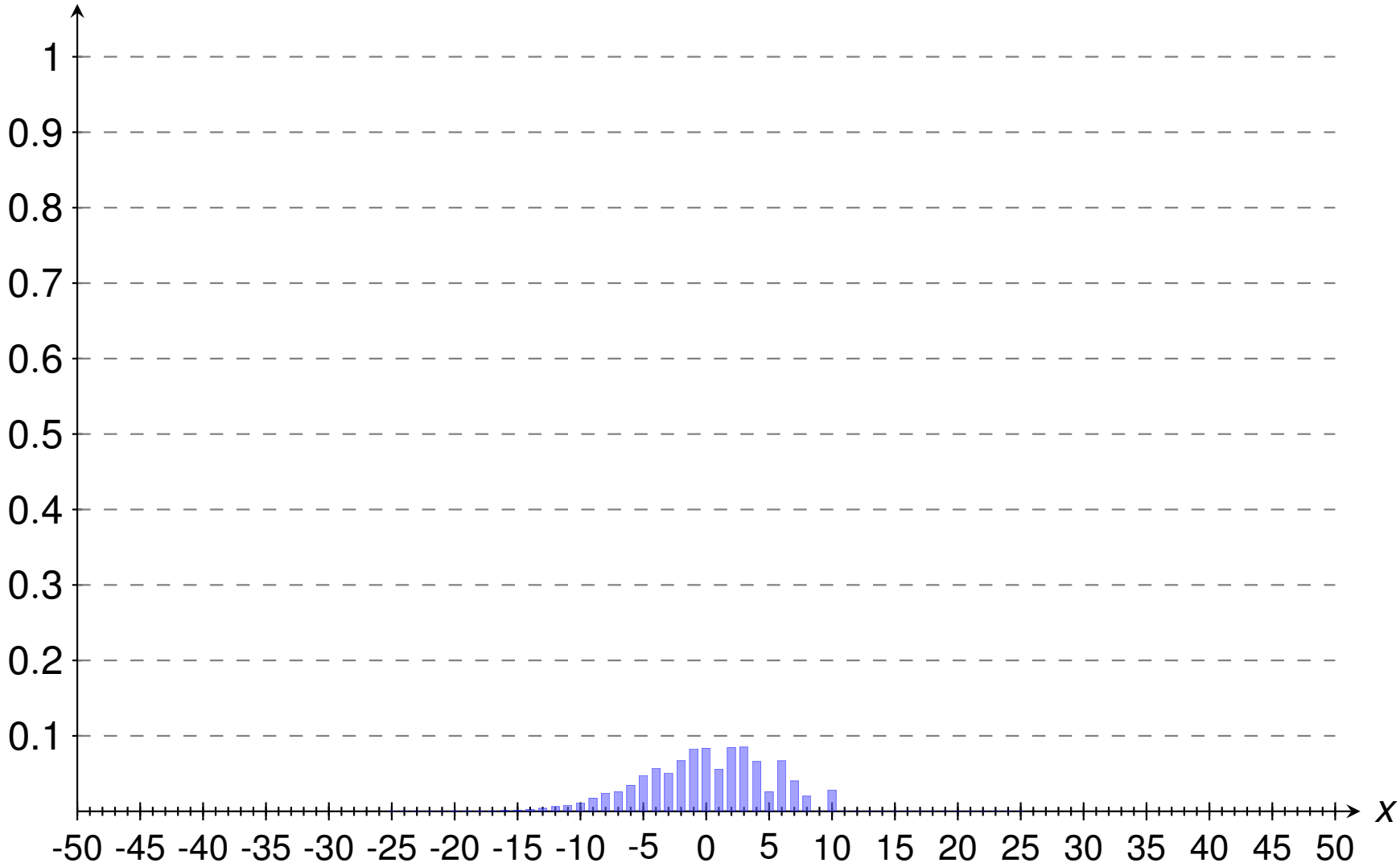

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{11} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

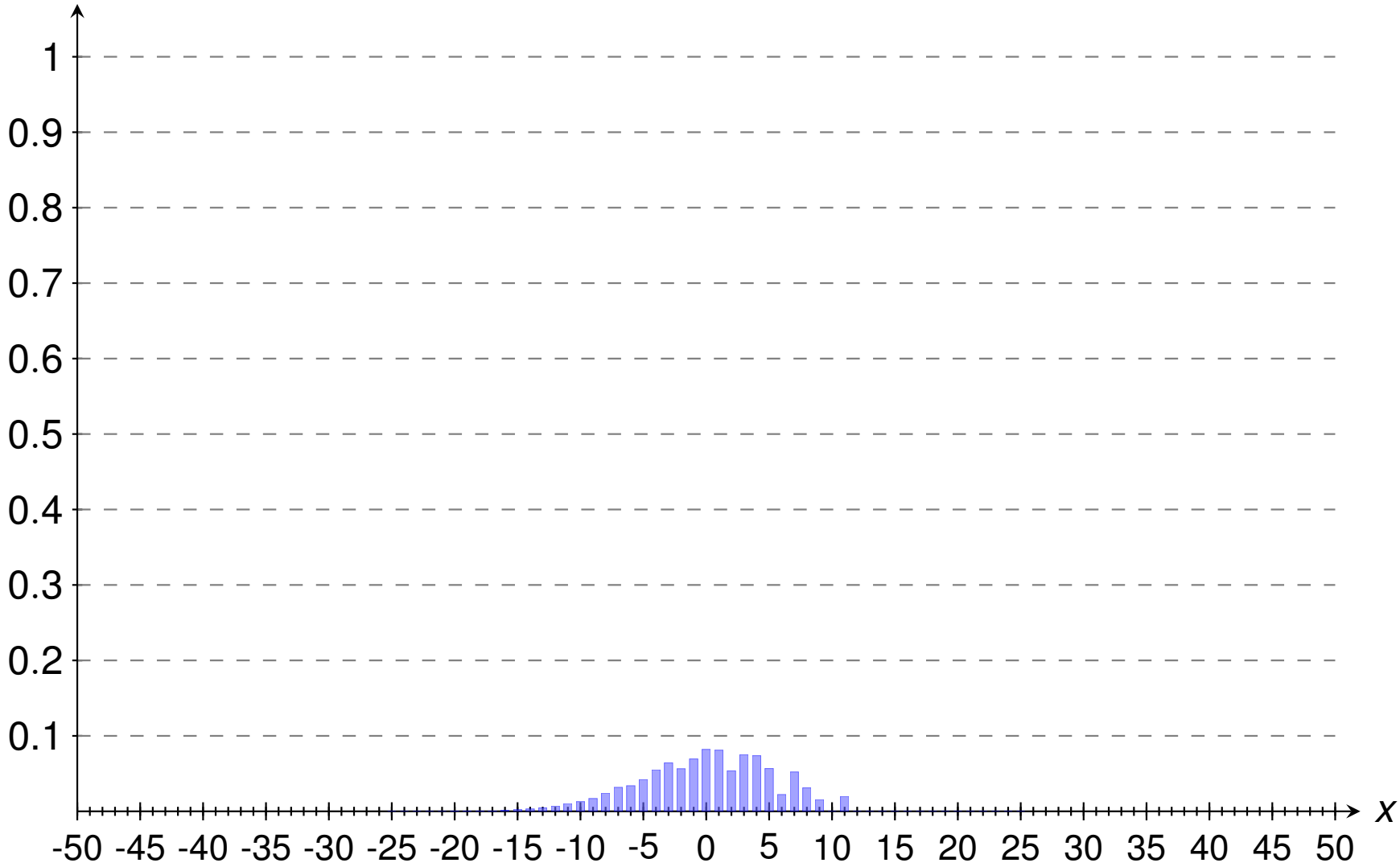

Illustration of CLT (2/4)

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

$\mathbf{P}\left[\sum_{j=1}^{12} X_j = x\right]$

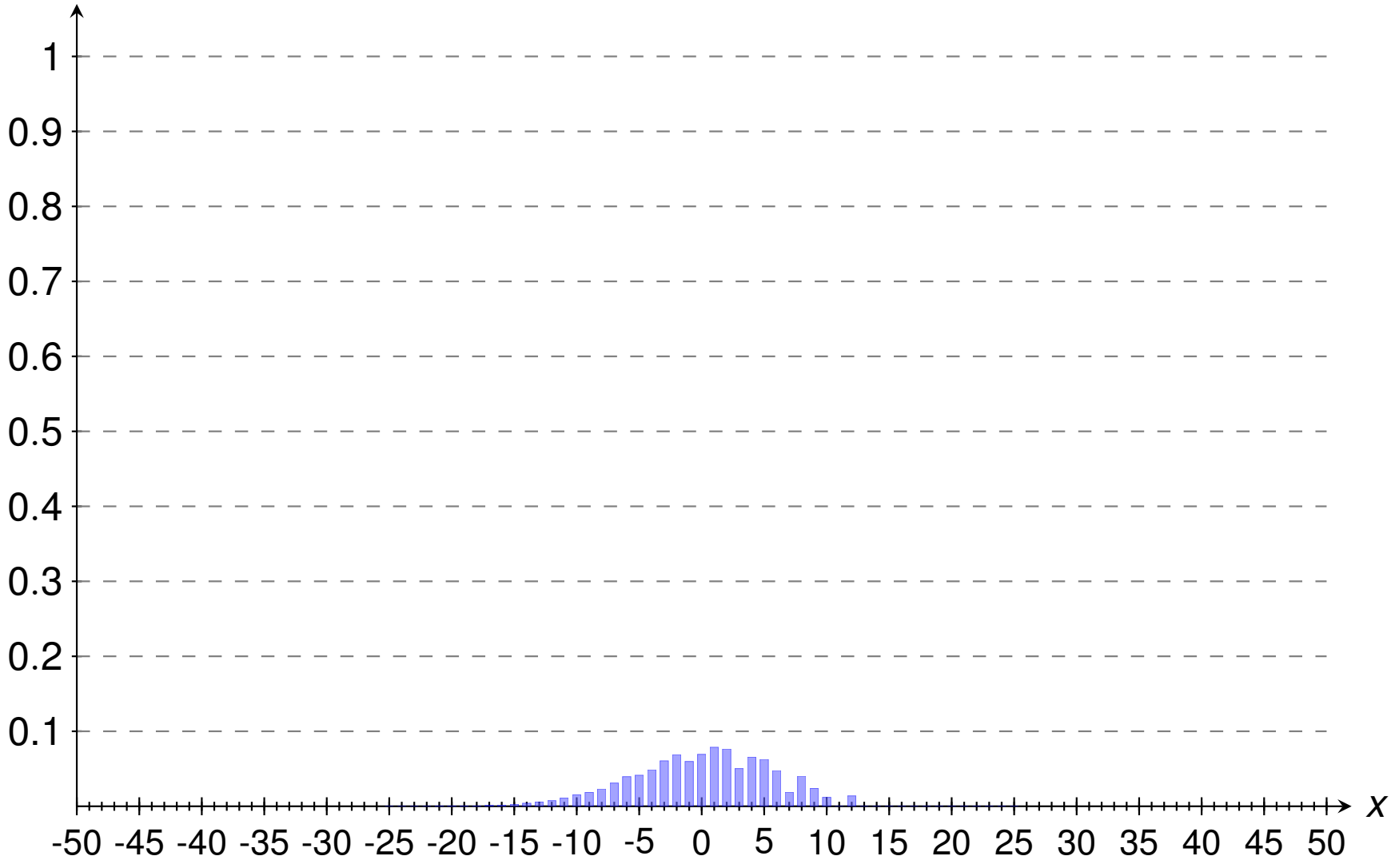

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{13} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

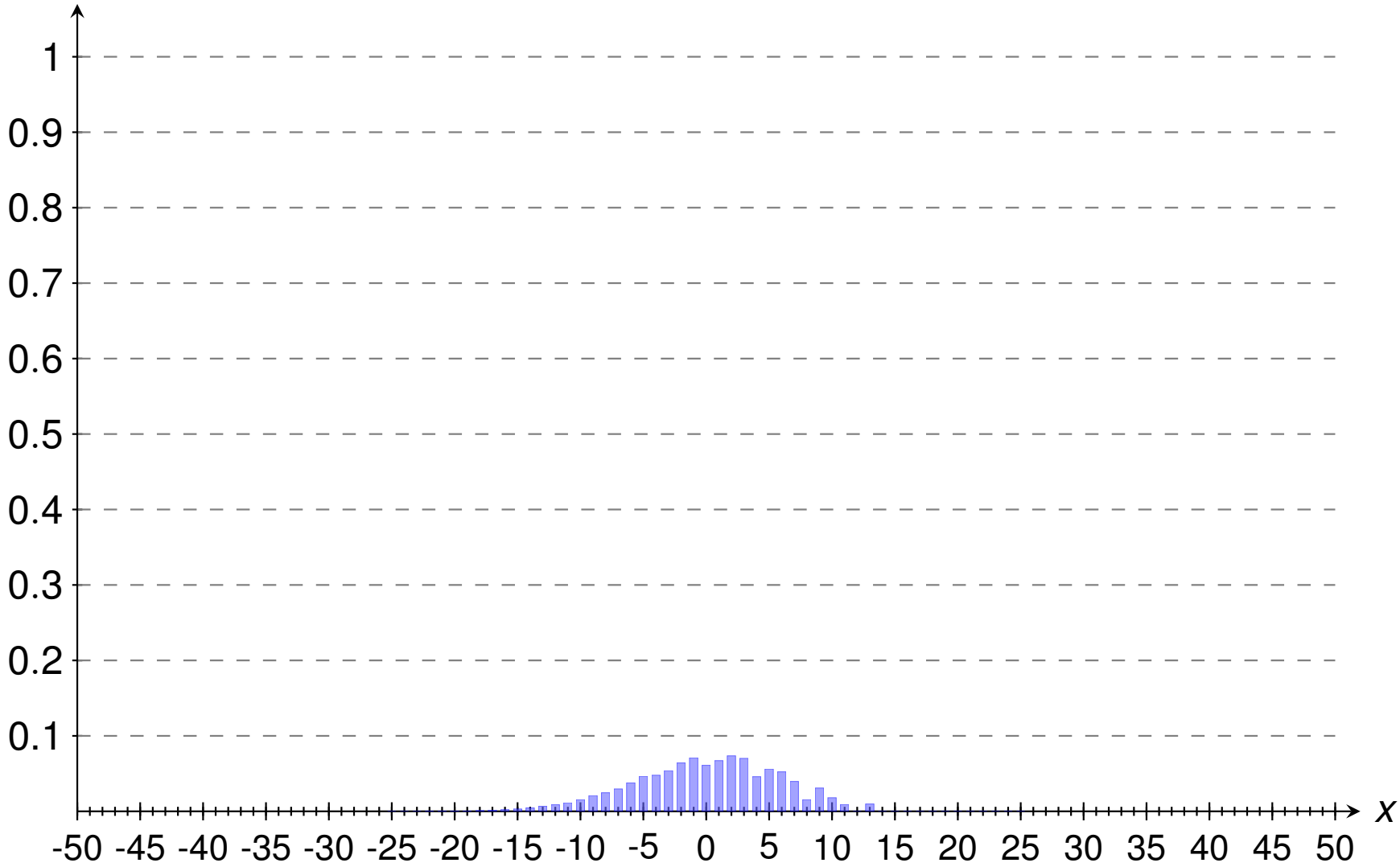

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{14} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

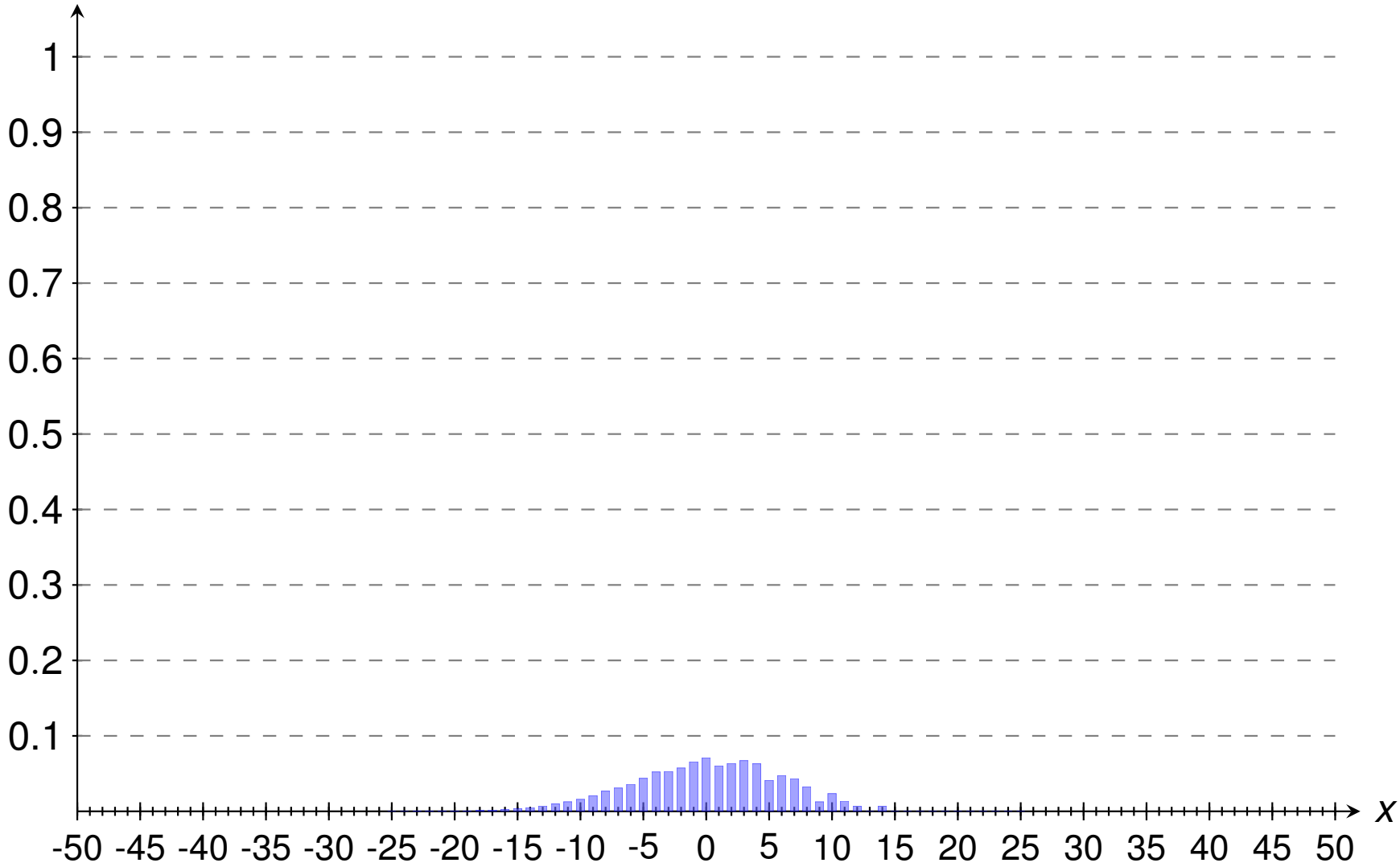

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{15} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

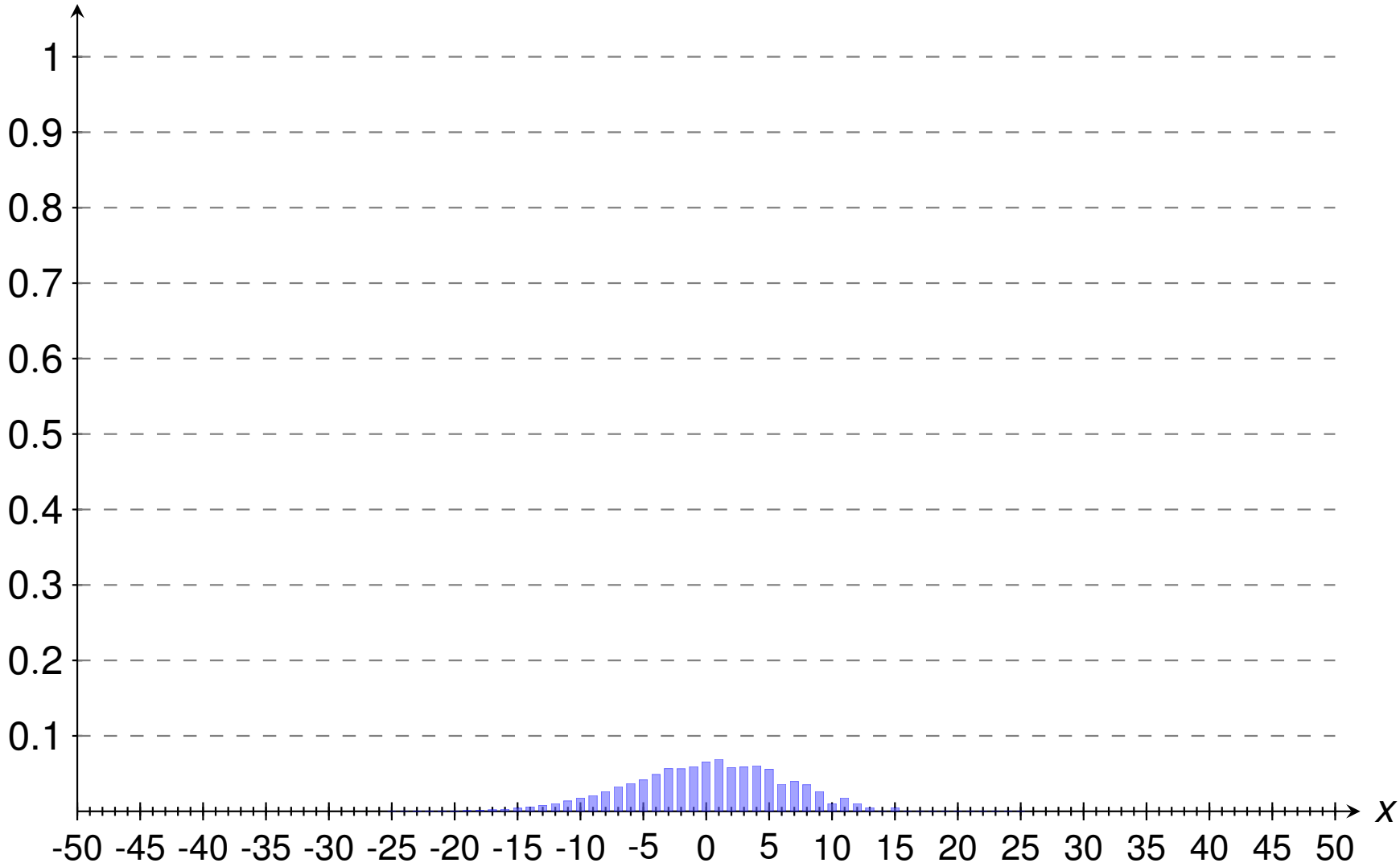

Illustration of CLT (2/4)

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

$\mathbf{P}\left[\sum_{j=1}^{16} X_j = x\right]$

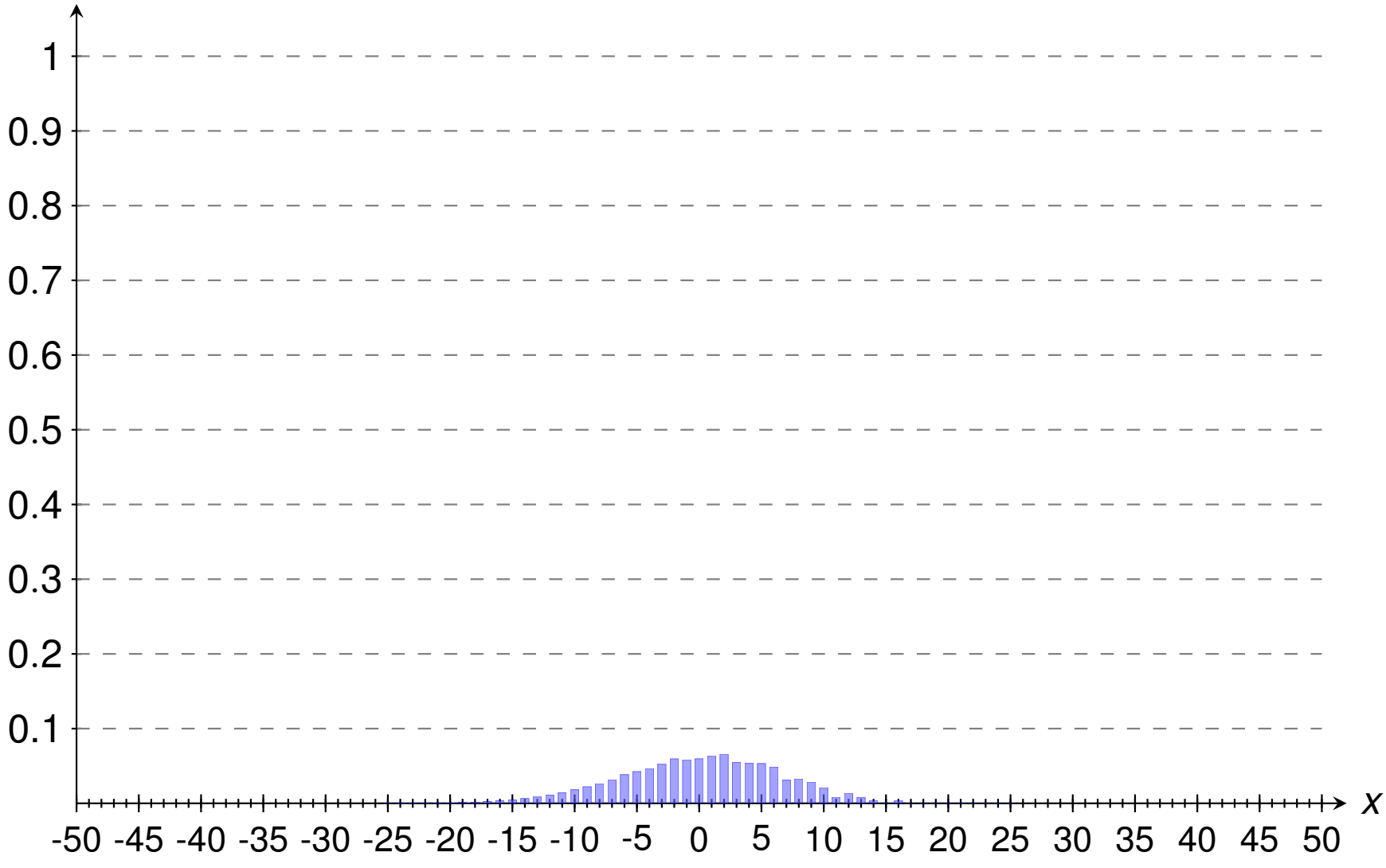

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{17} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

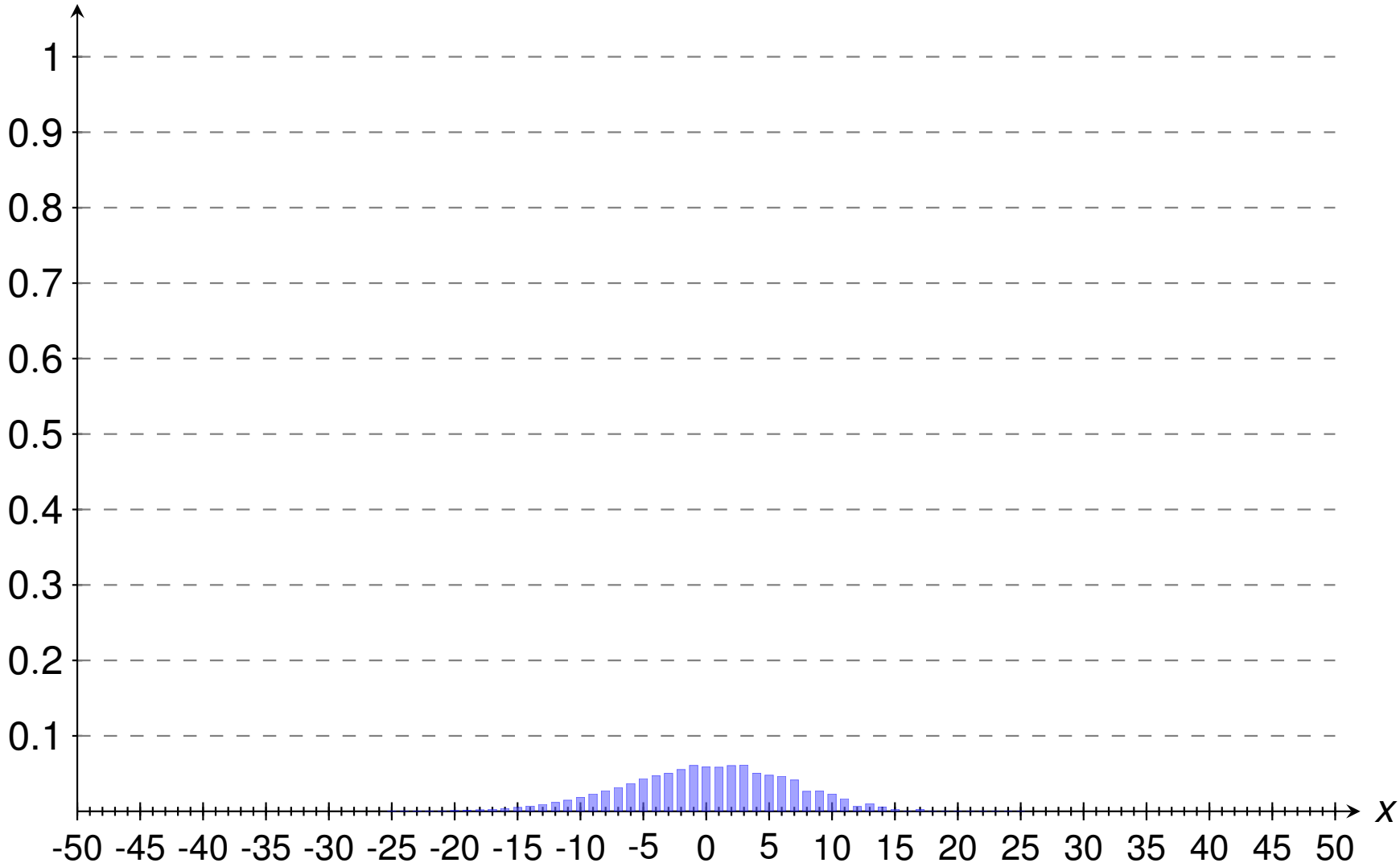

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{18} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

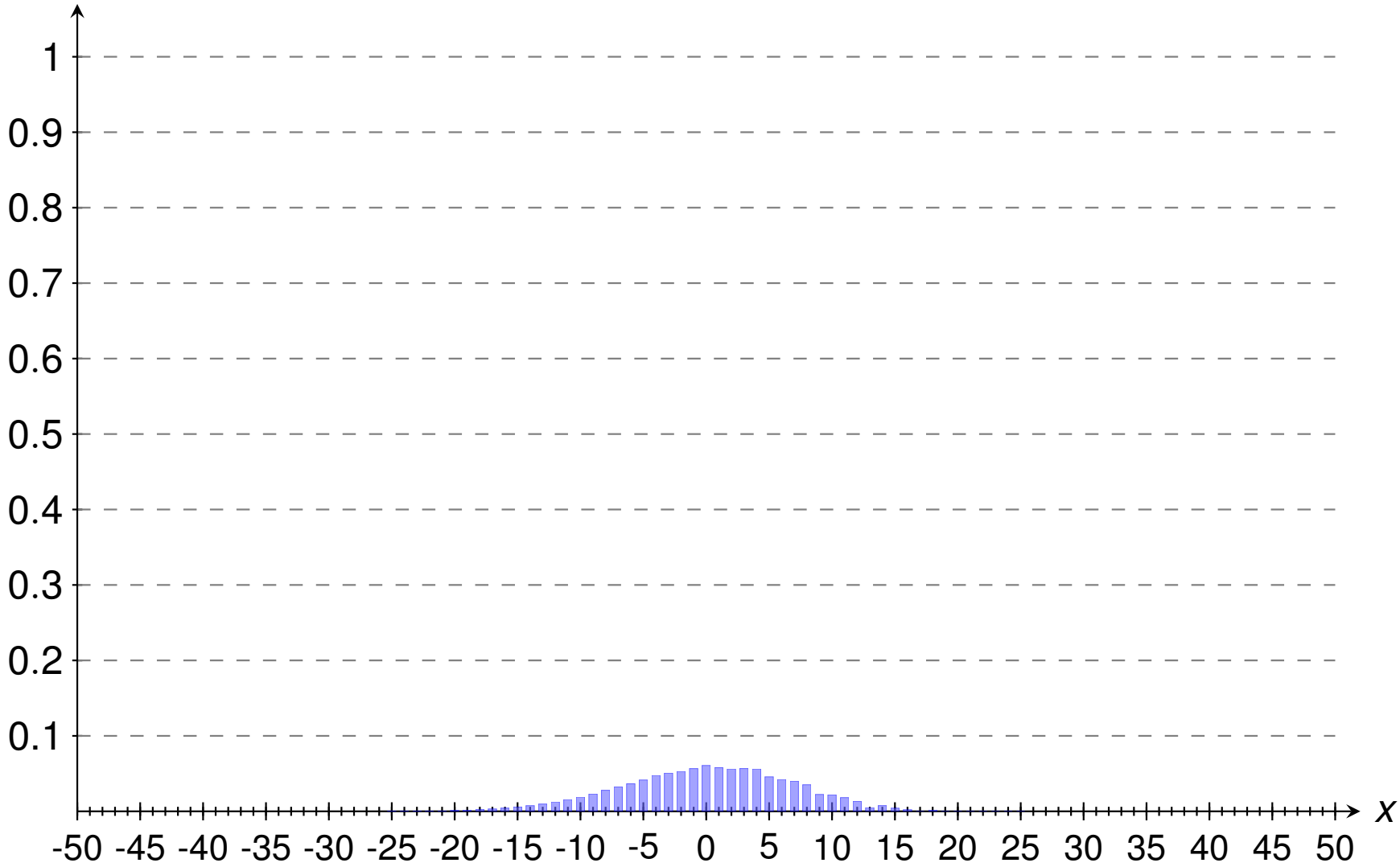

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{19} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

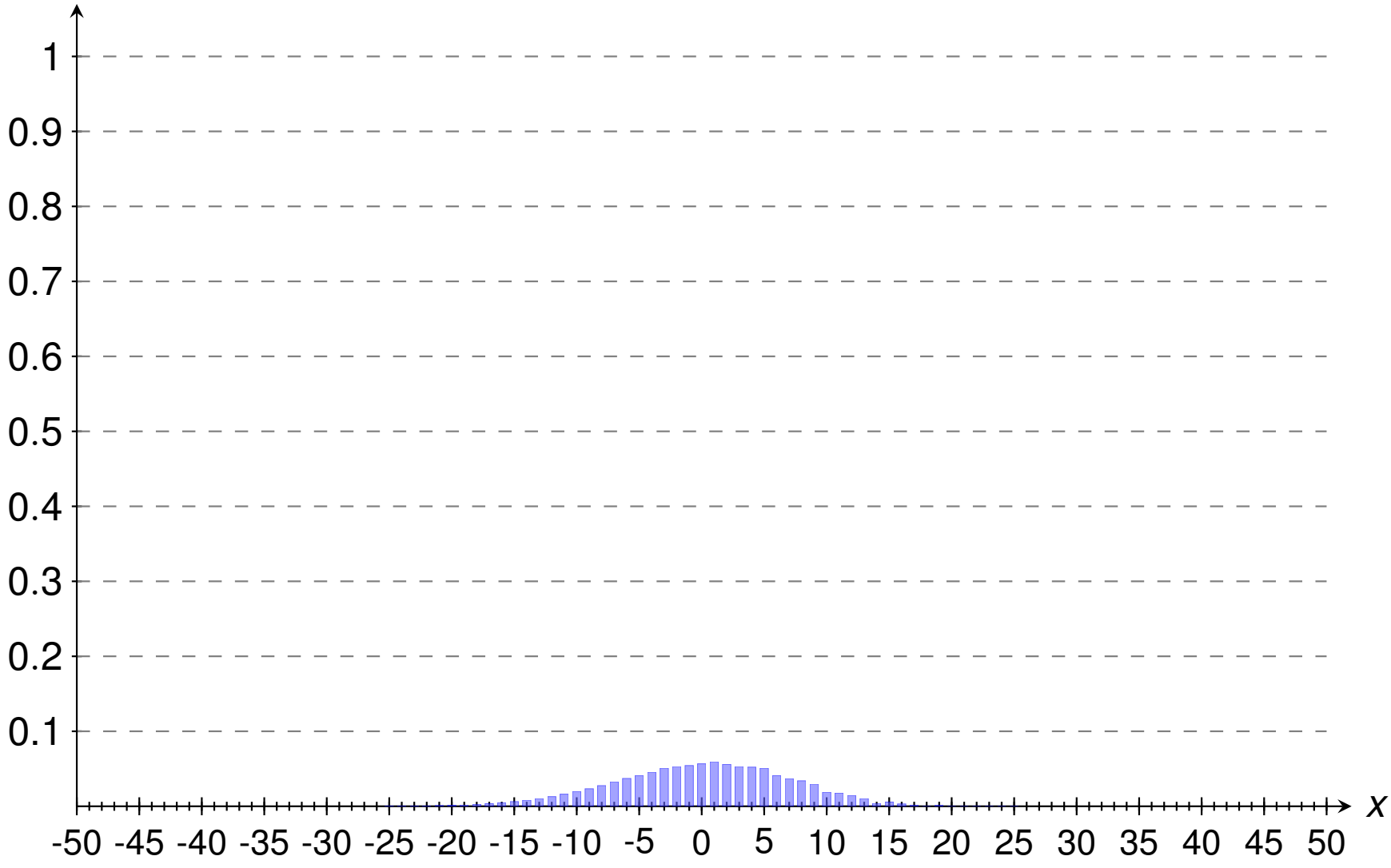

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{20} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

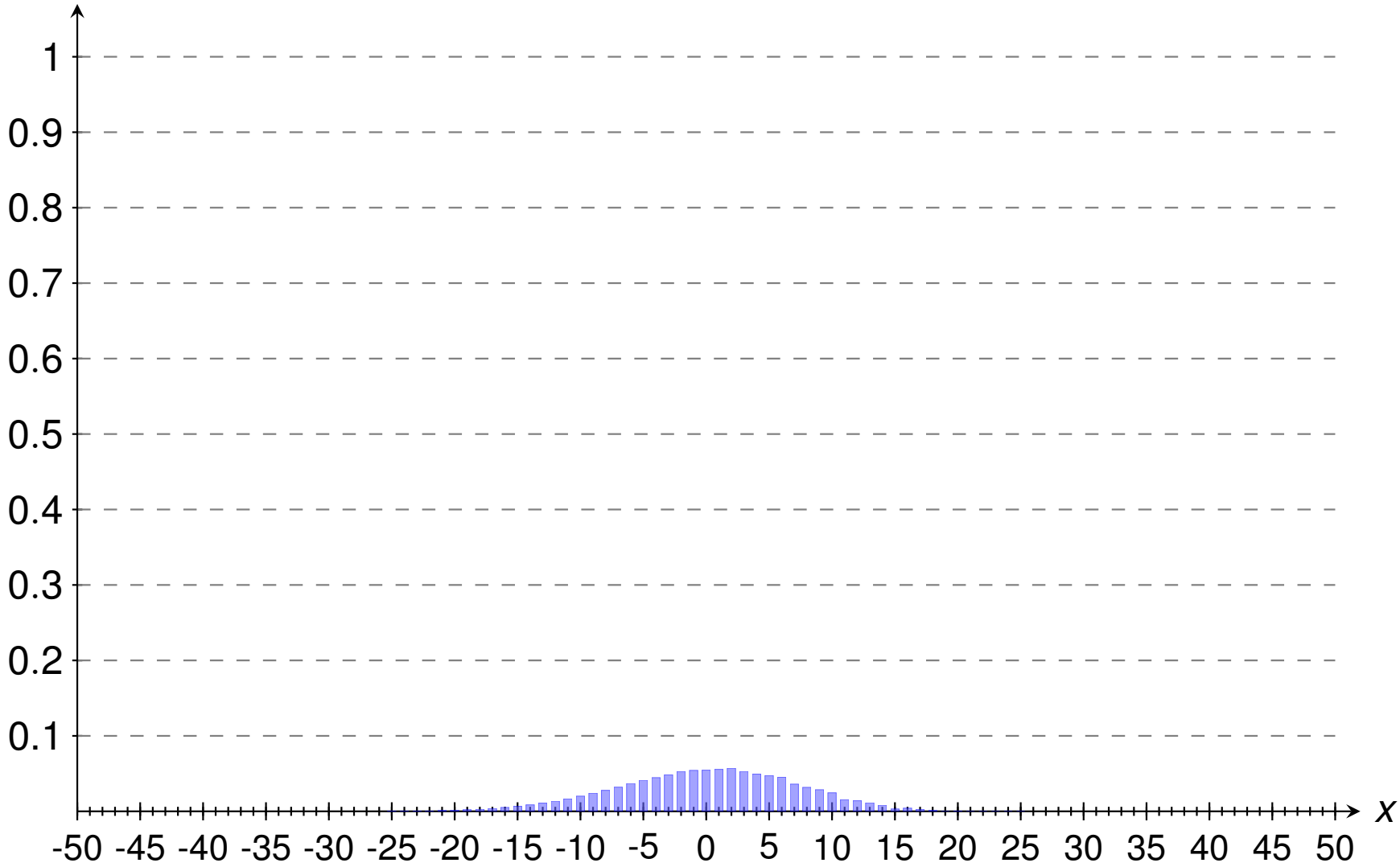

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{21} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

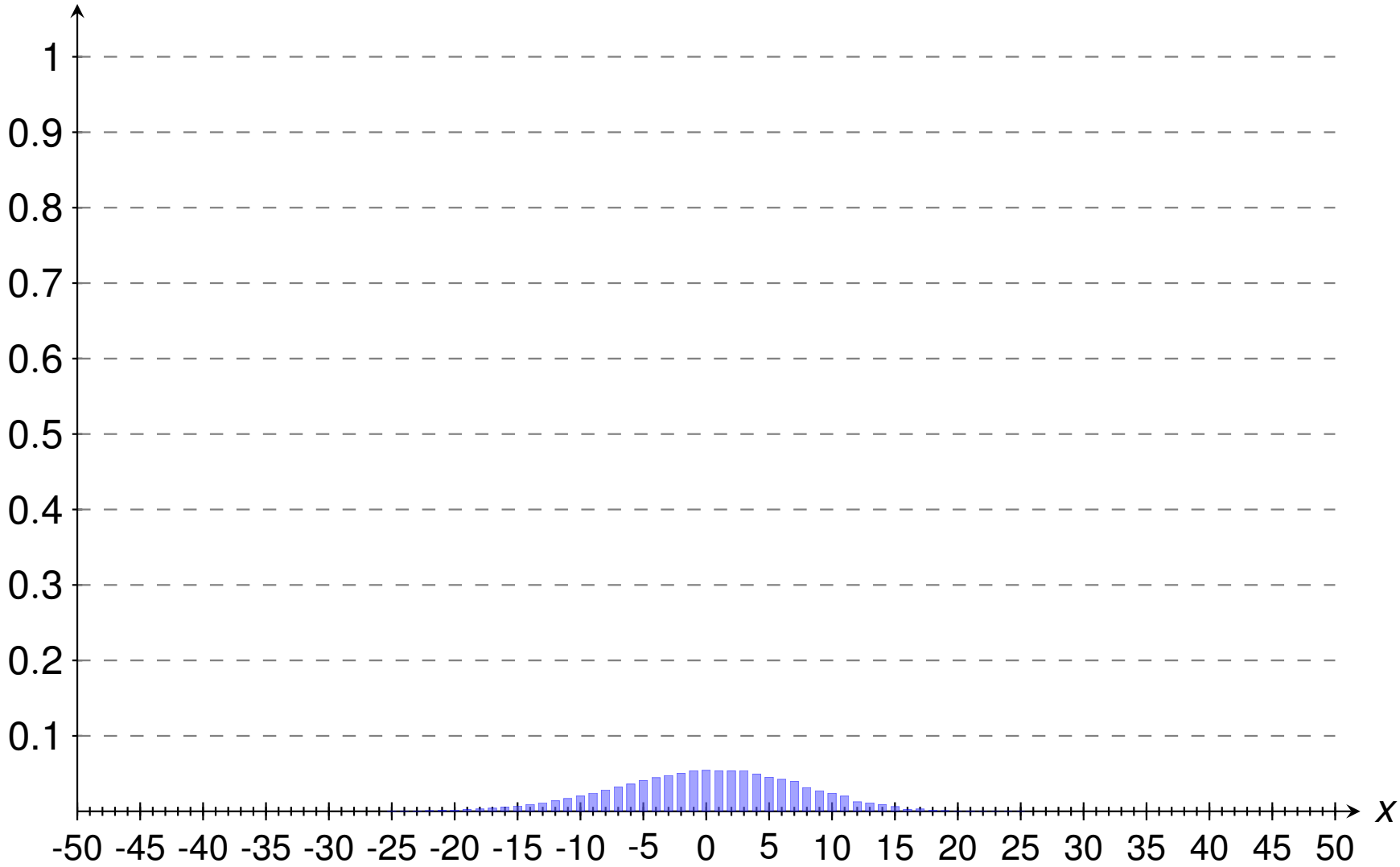

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{22} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

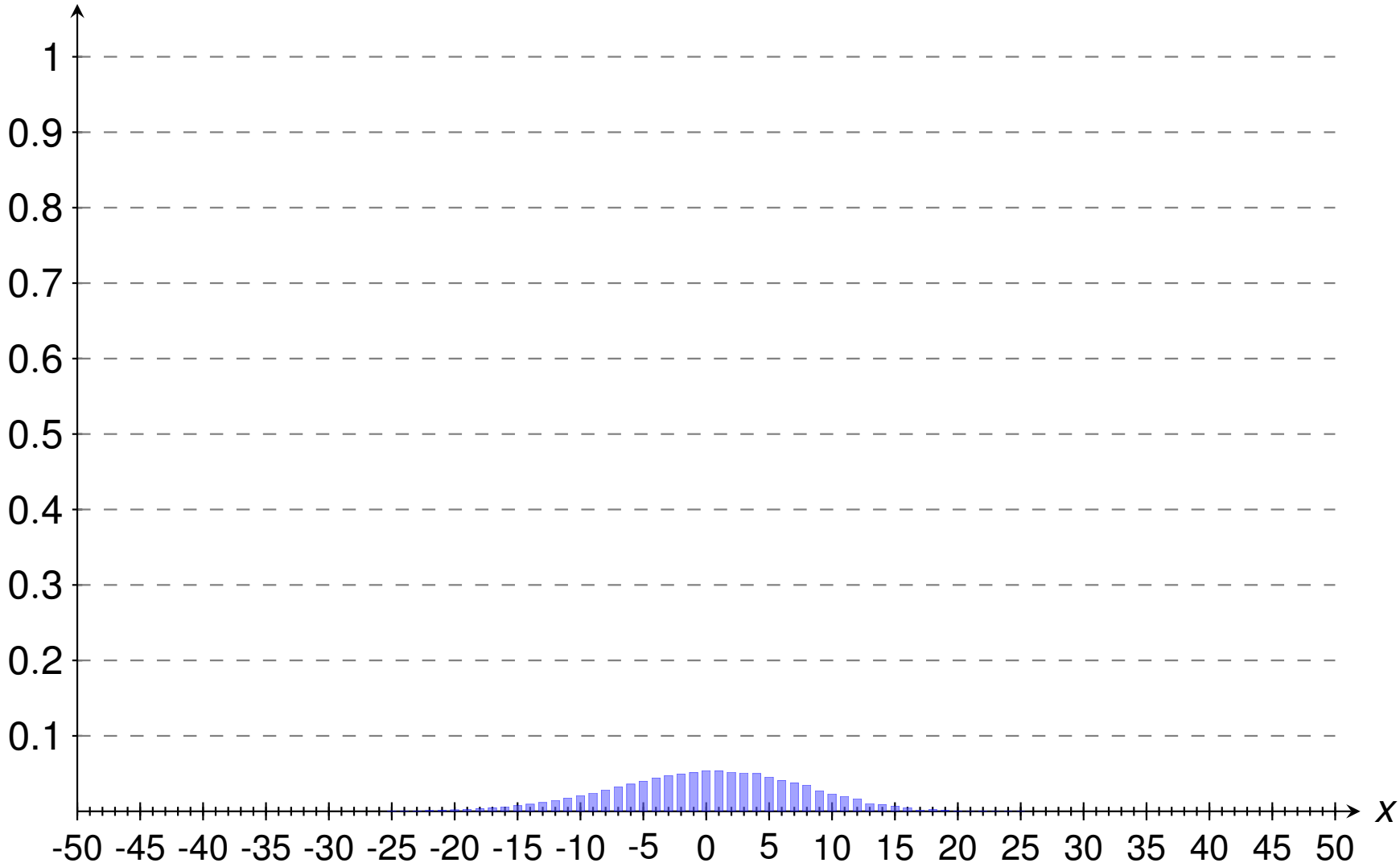

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{23} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

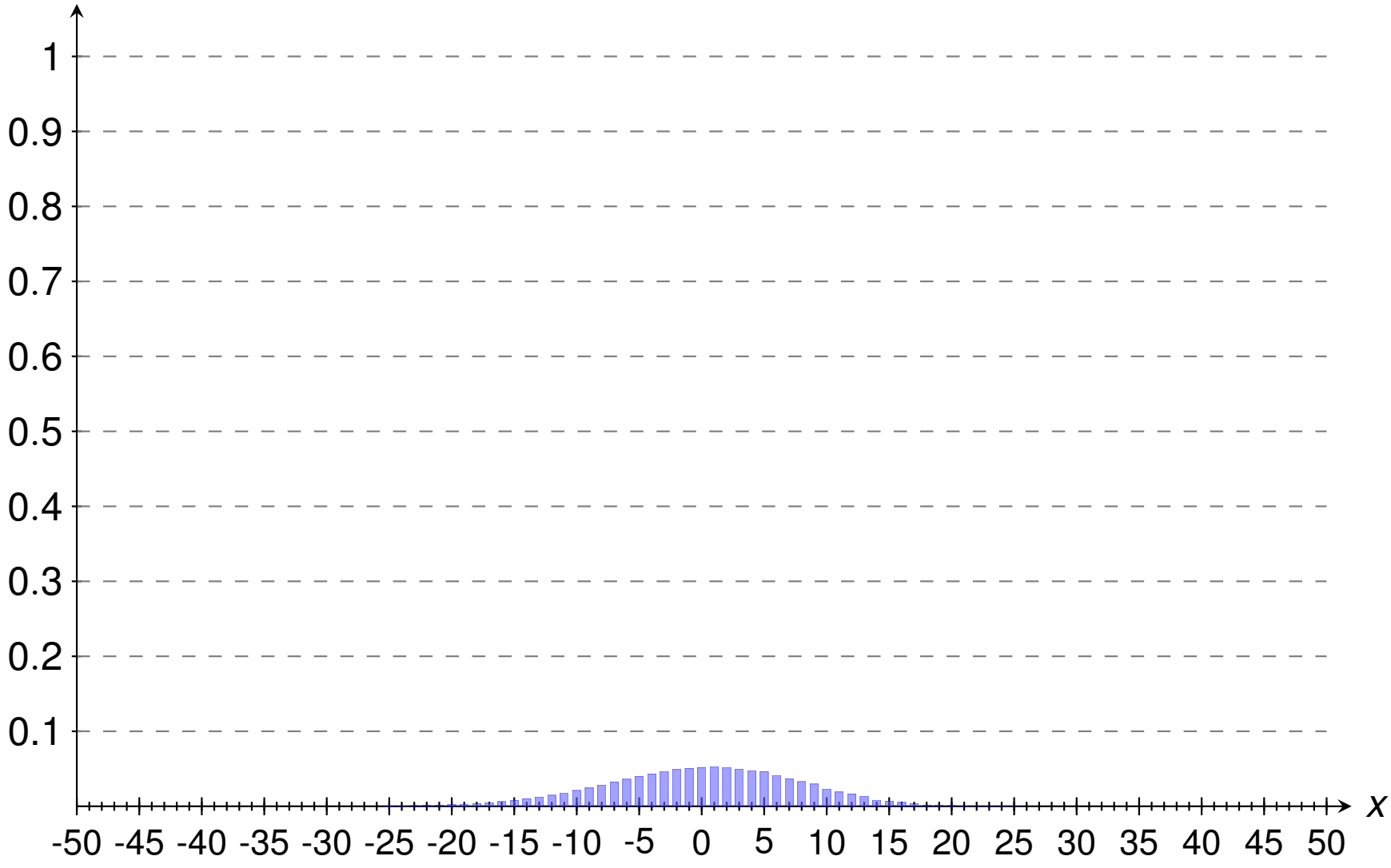

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{24} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

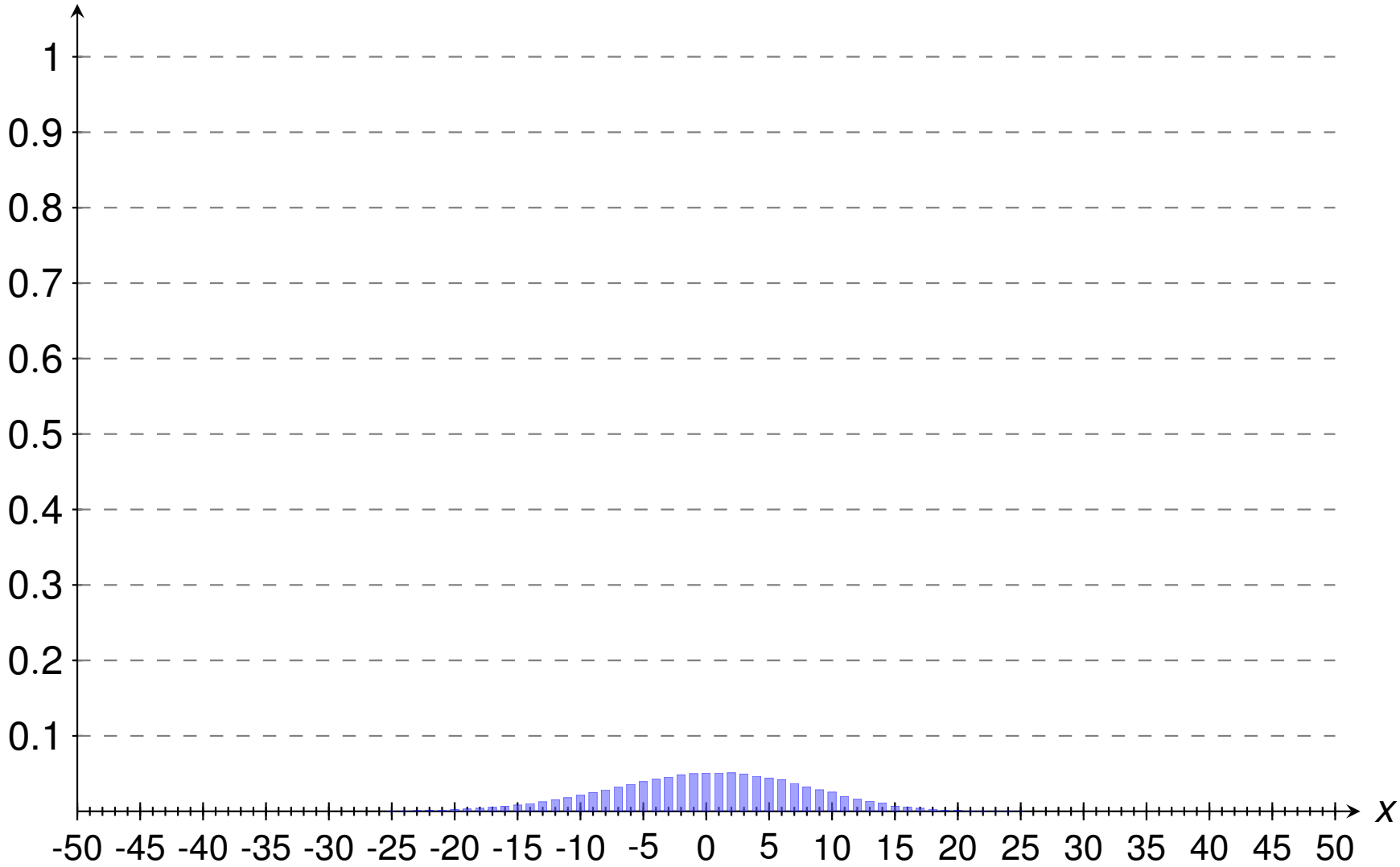

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{25} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

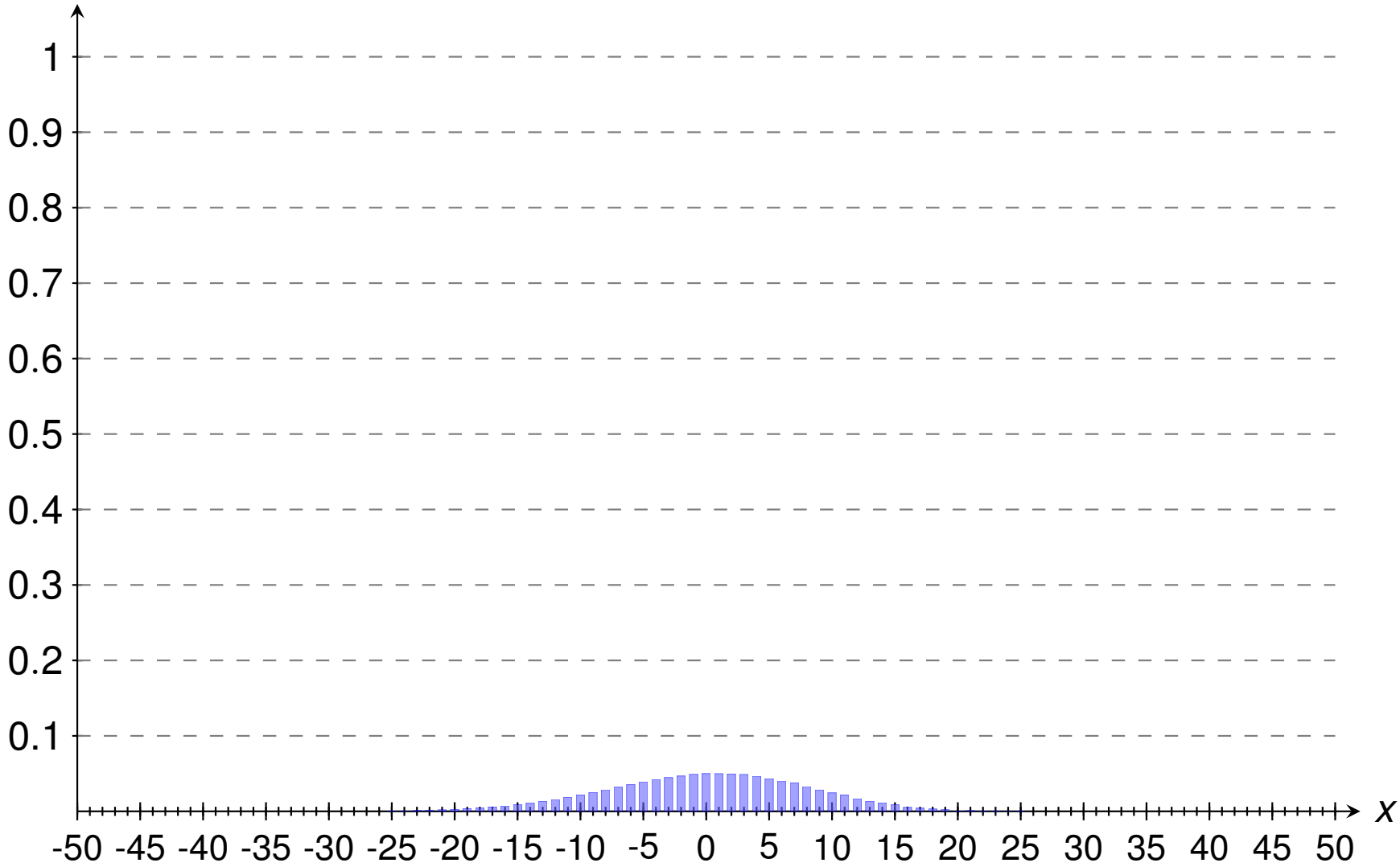

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{26} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

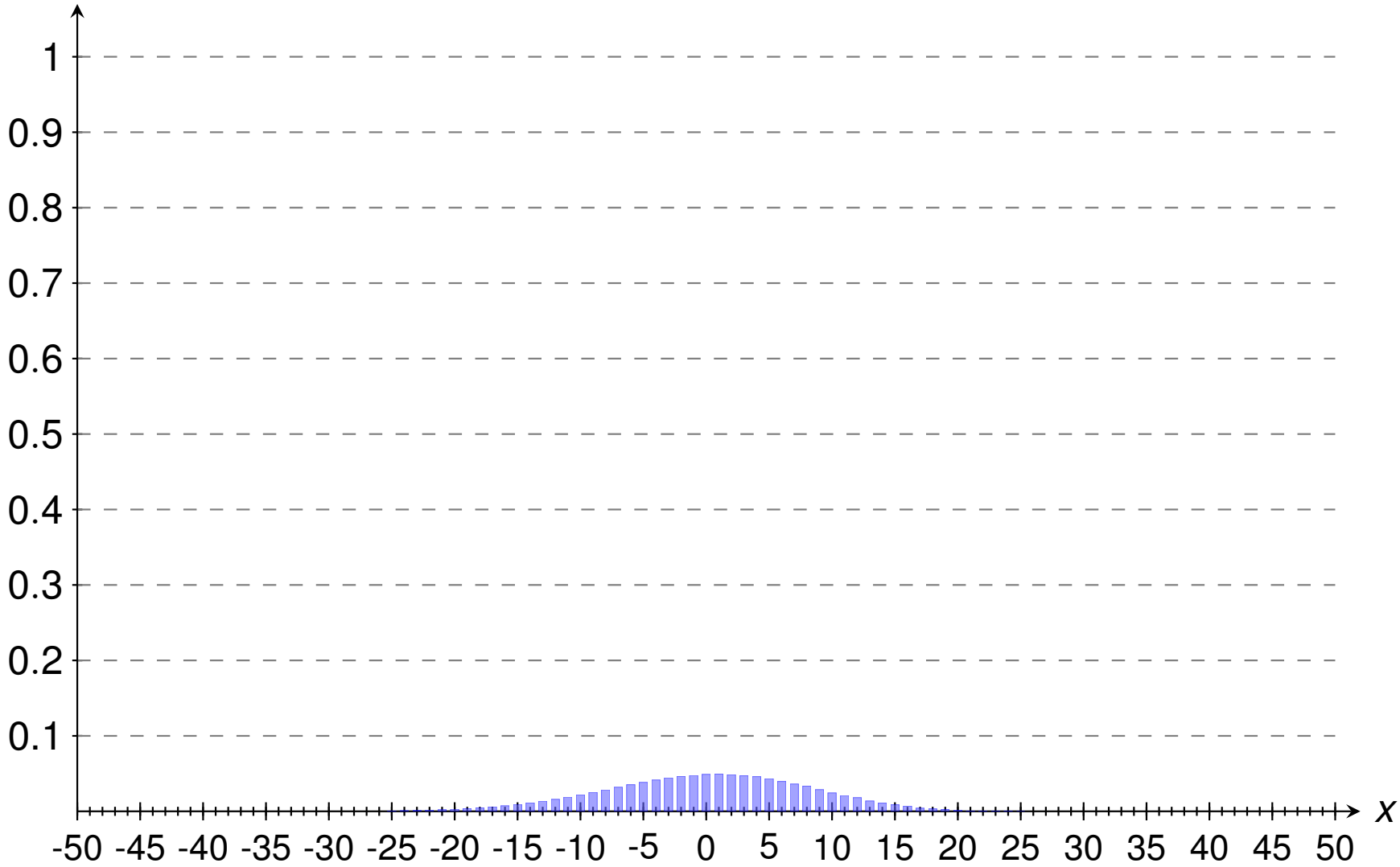

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{27} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

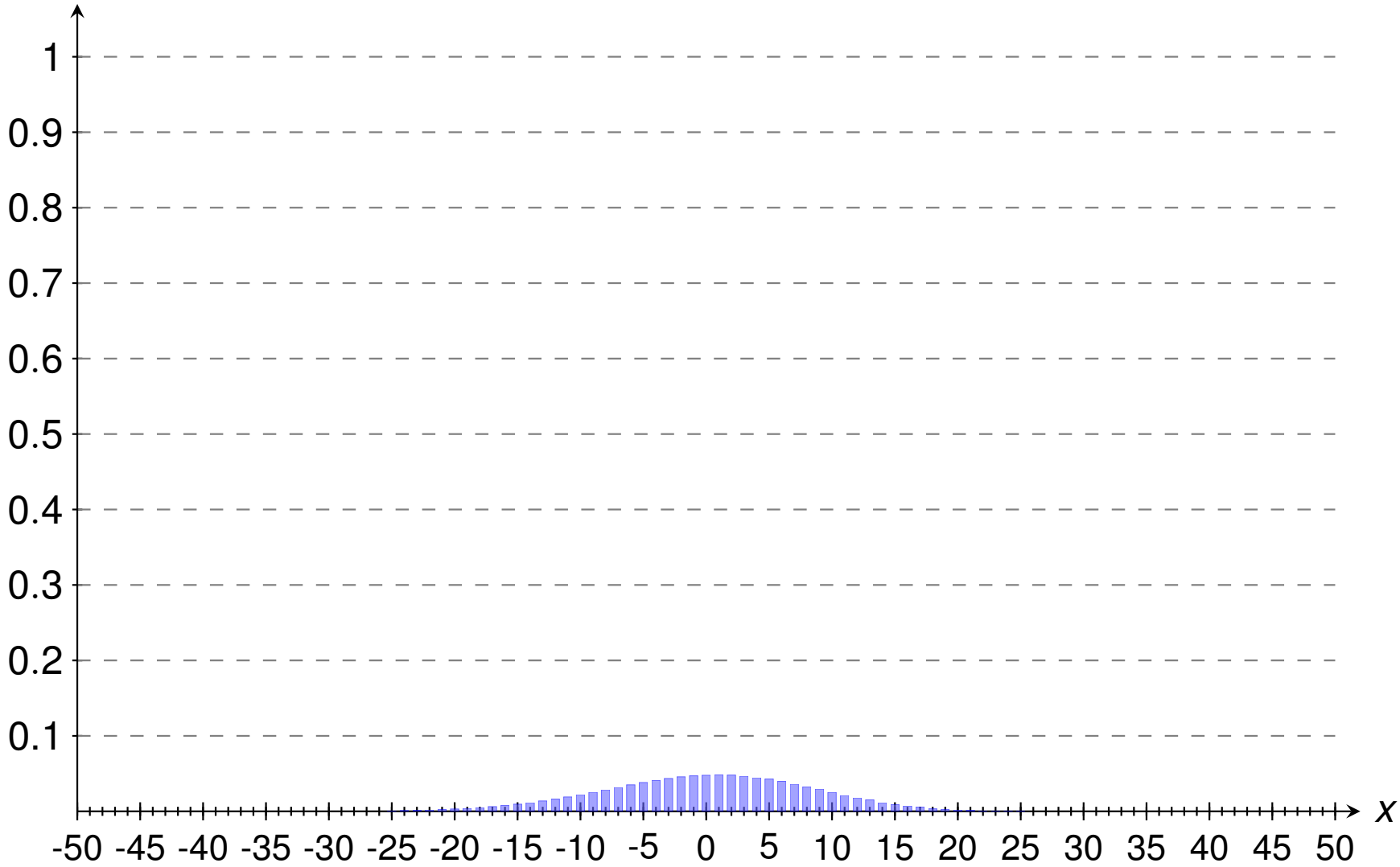

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{28} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

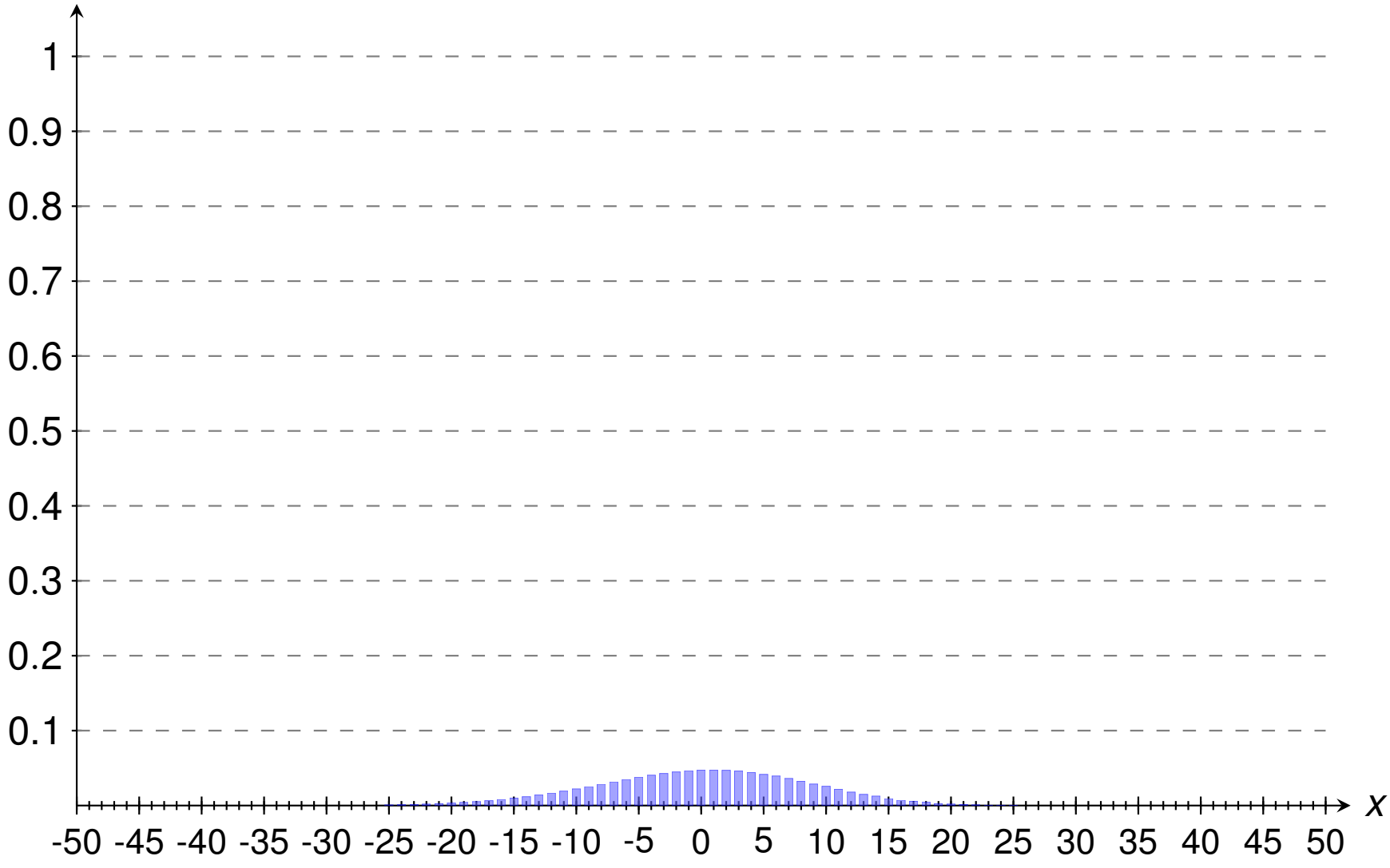

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{29} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

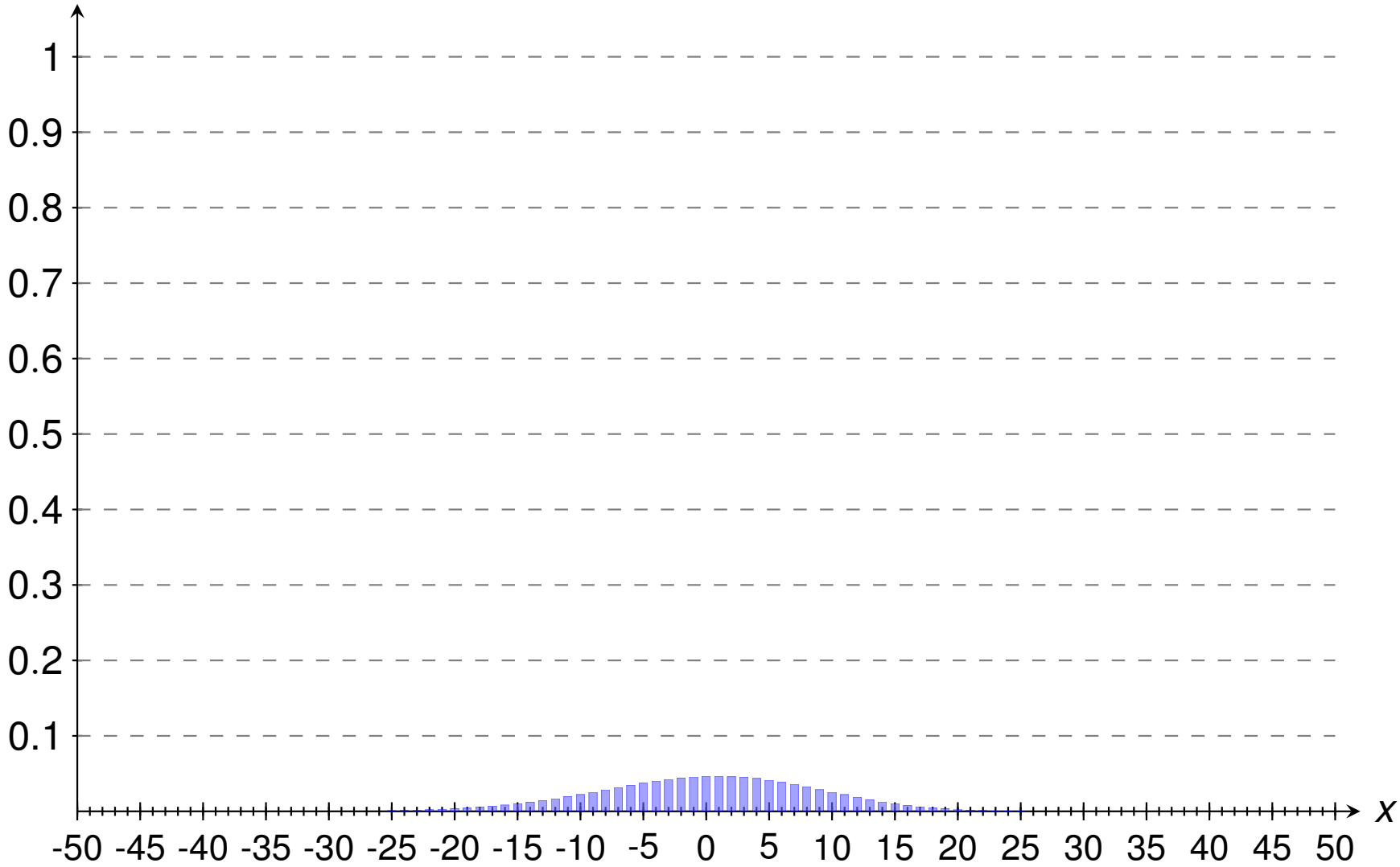

Illustration of CLT (2/4)

$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$

- $\mu = 0.15 \cdot (-3) + 0.1 \cdot (-2) + 0.05 \cdot (-1) + 0.7 \cdot 1 = 0$
- $\sigma^2 = 0.15 \cdot 9 + 0.1 \cdot 4 + 0.05 \cdot 1 + 0.7 \cdot 1 = 2.5$

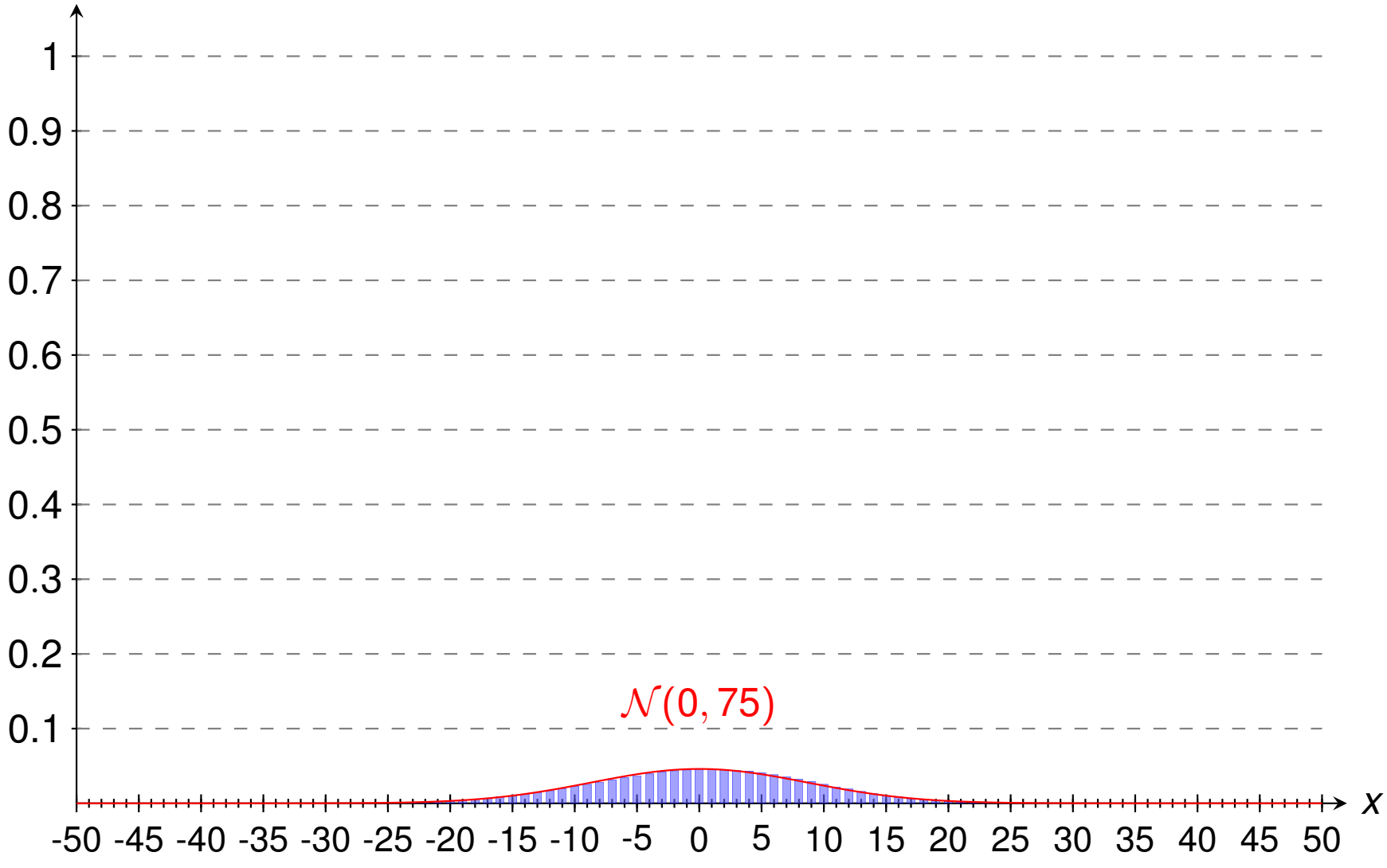

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{1} X_j = x\right]$

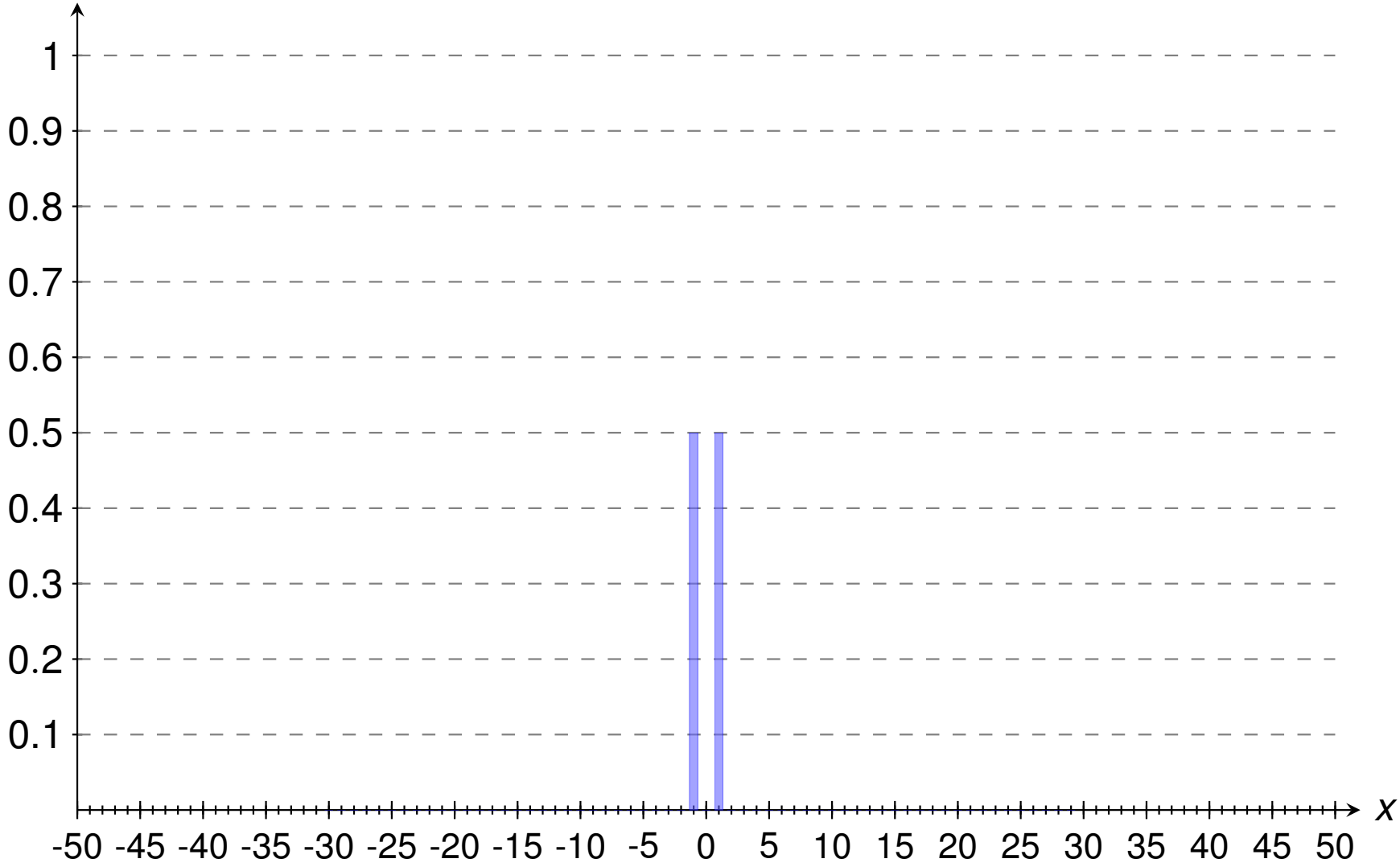

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{2} X_j = x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

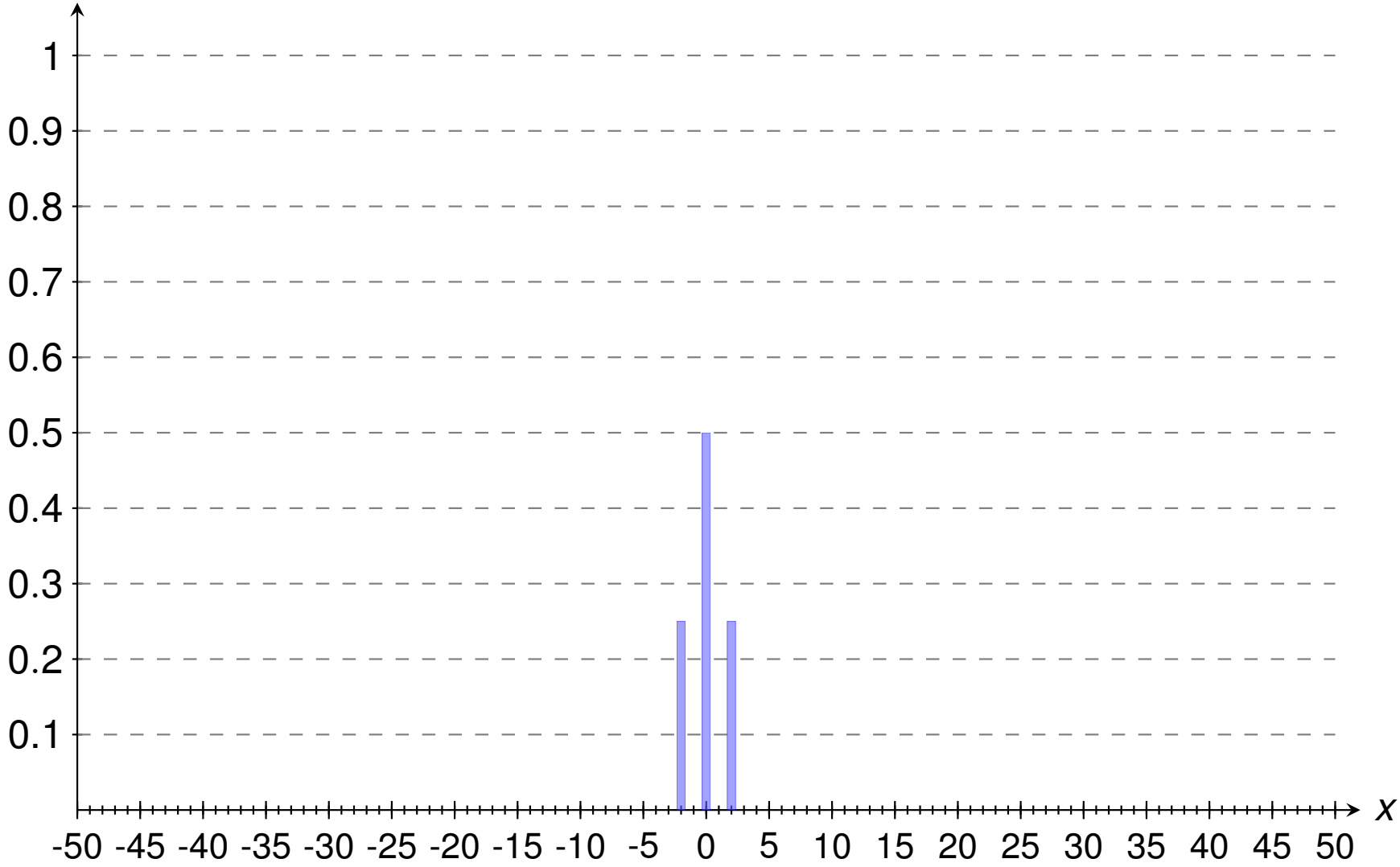

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{3} X_j = x\right]$

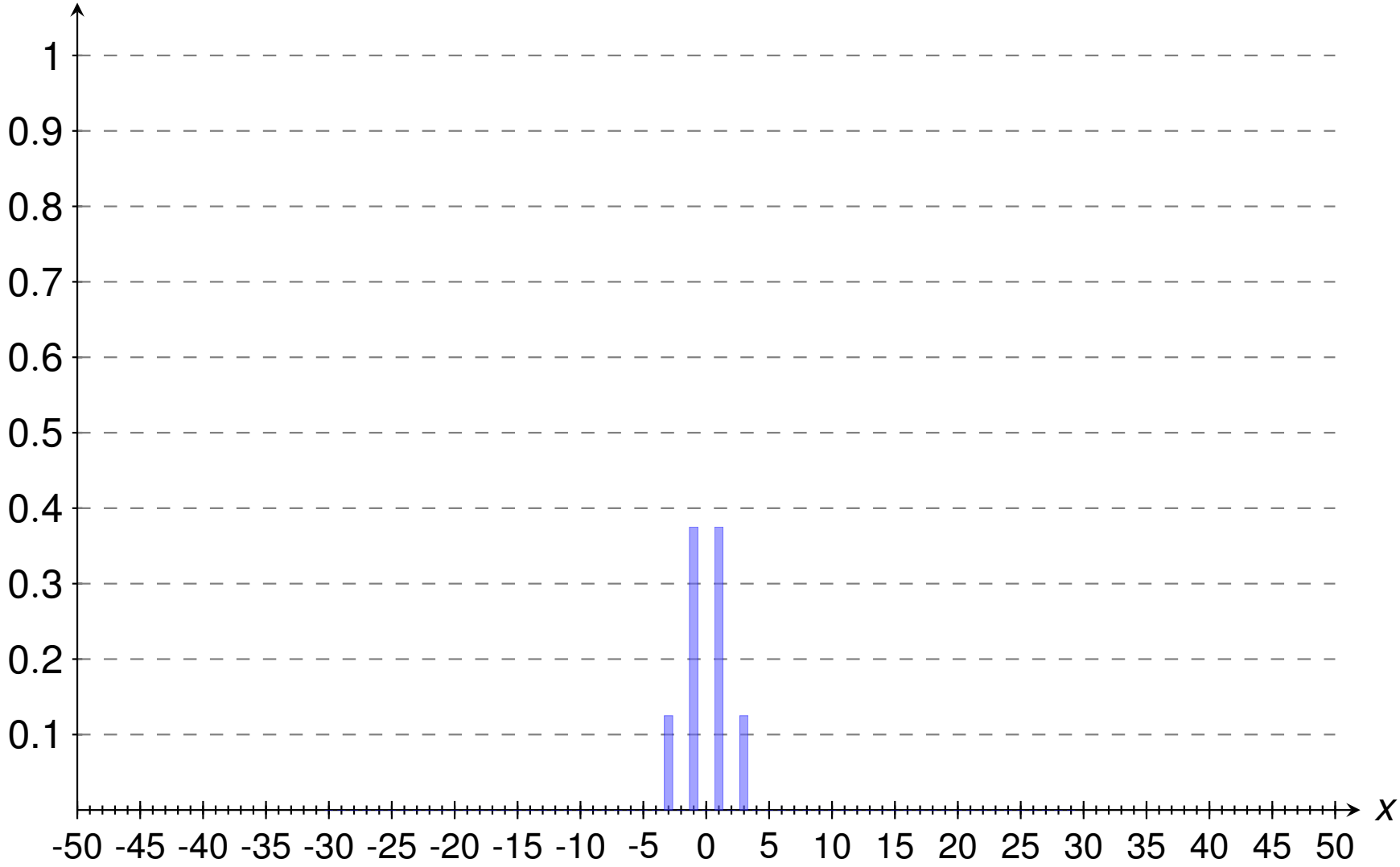

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{4} X_j = x\right]$

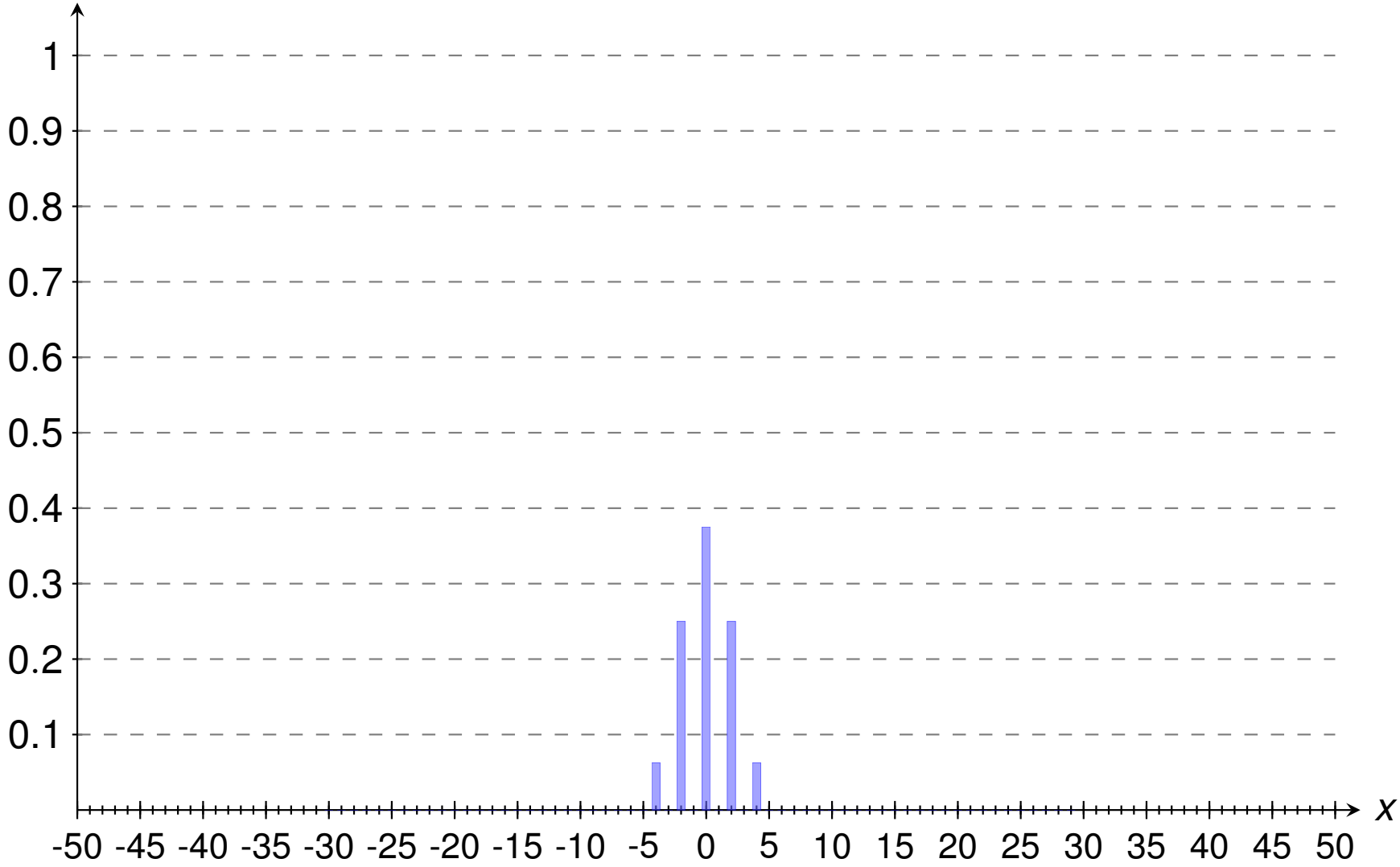

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{5} X_j = x\right]$

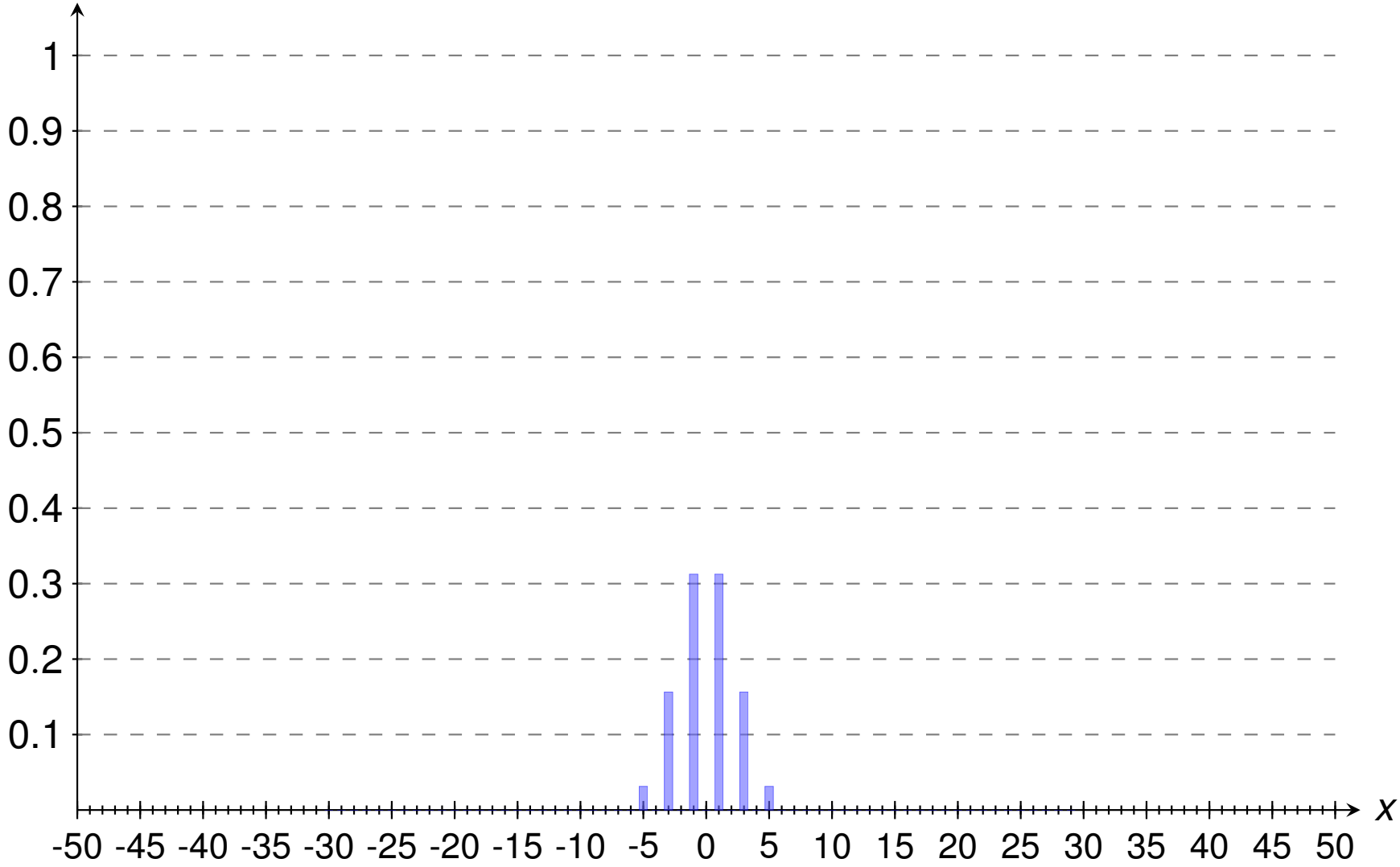

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{6} X_j = x\right]$

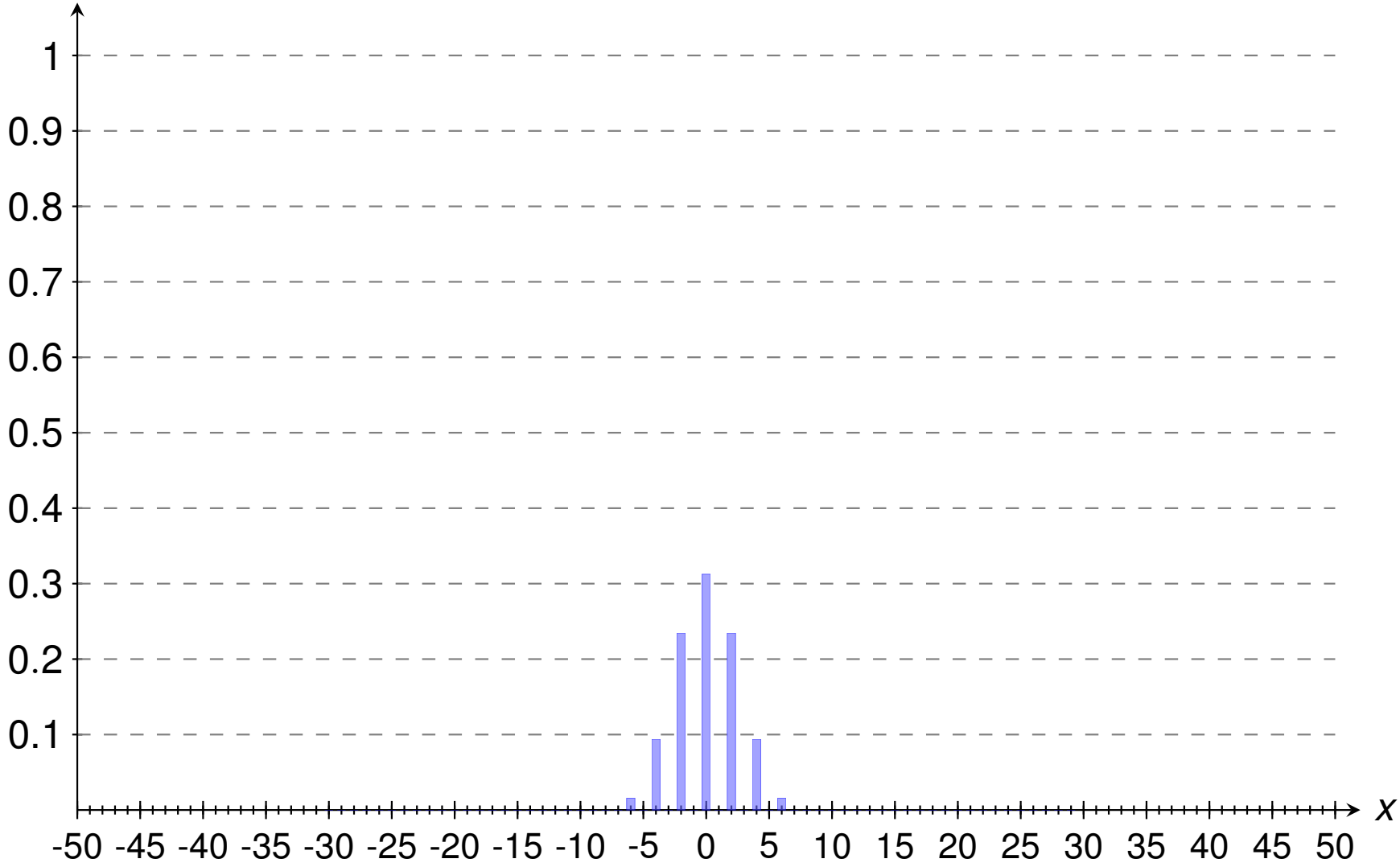

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{7} X_j = x\right]$

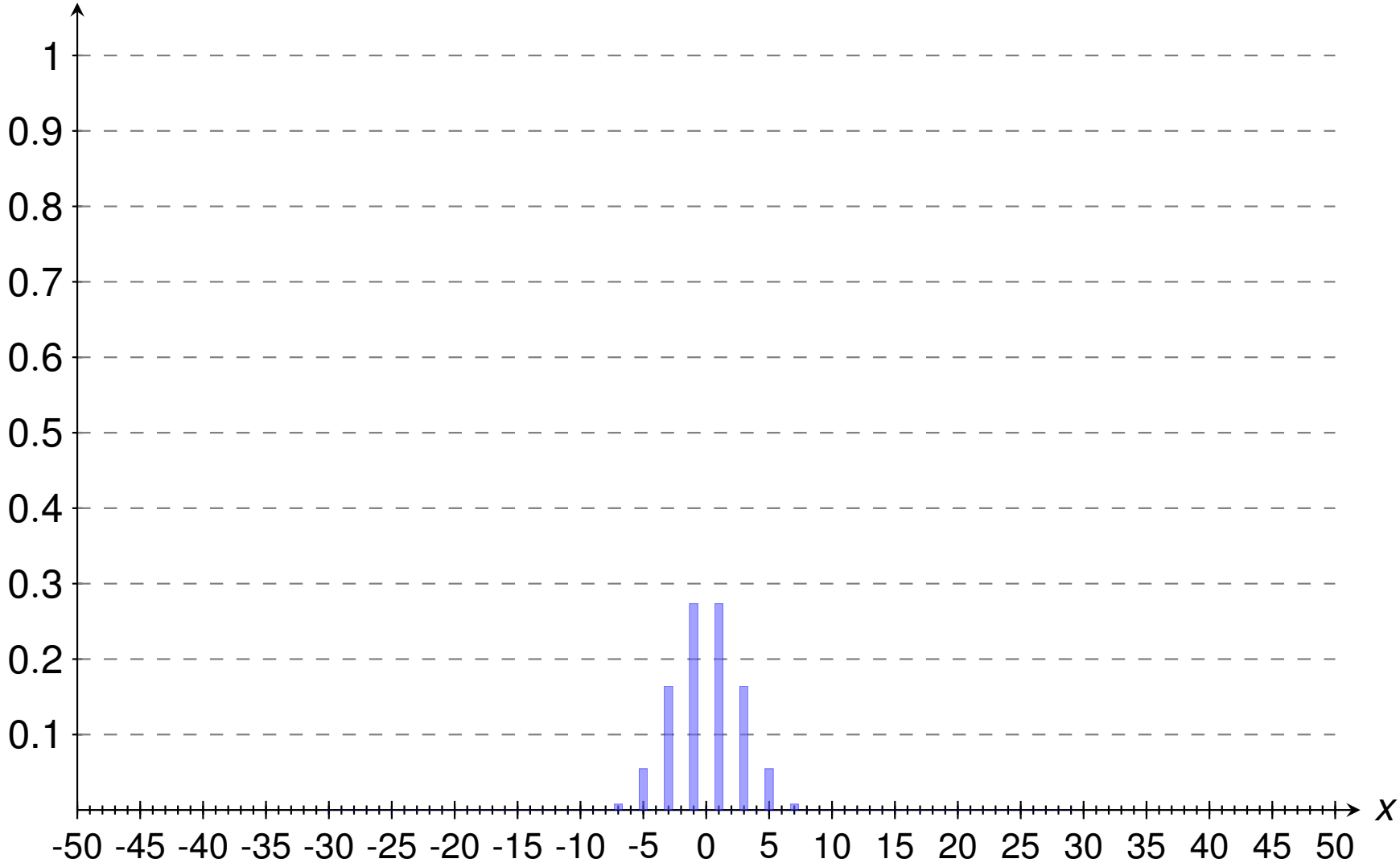

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{8} X_j = x\right]$

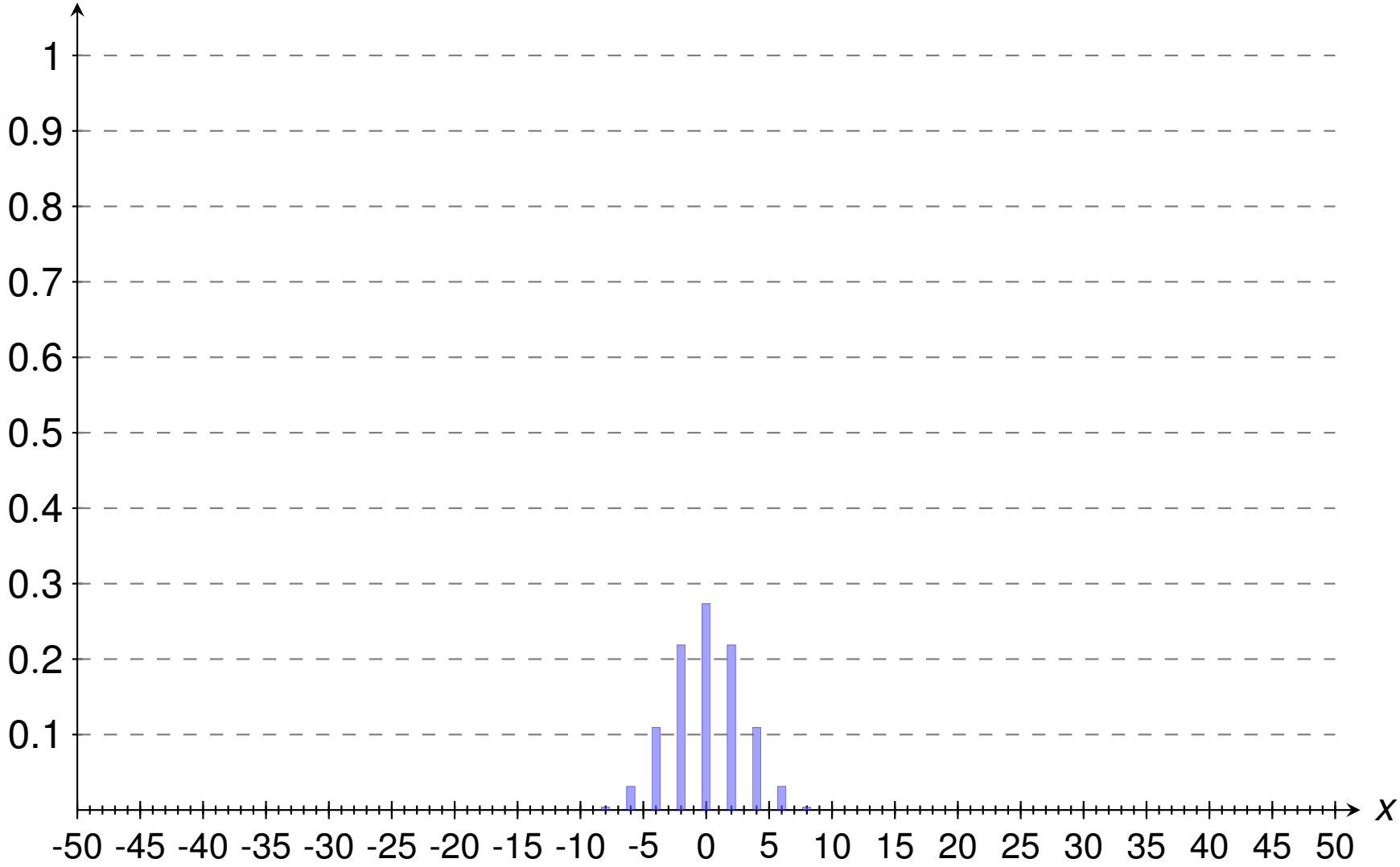

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{9} X_j = x\right]$

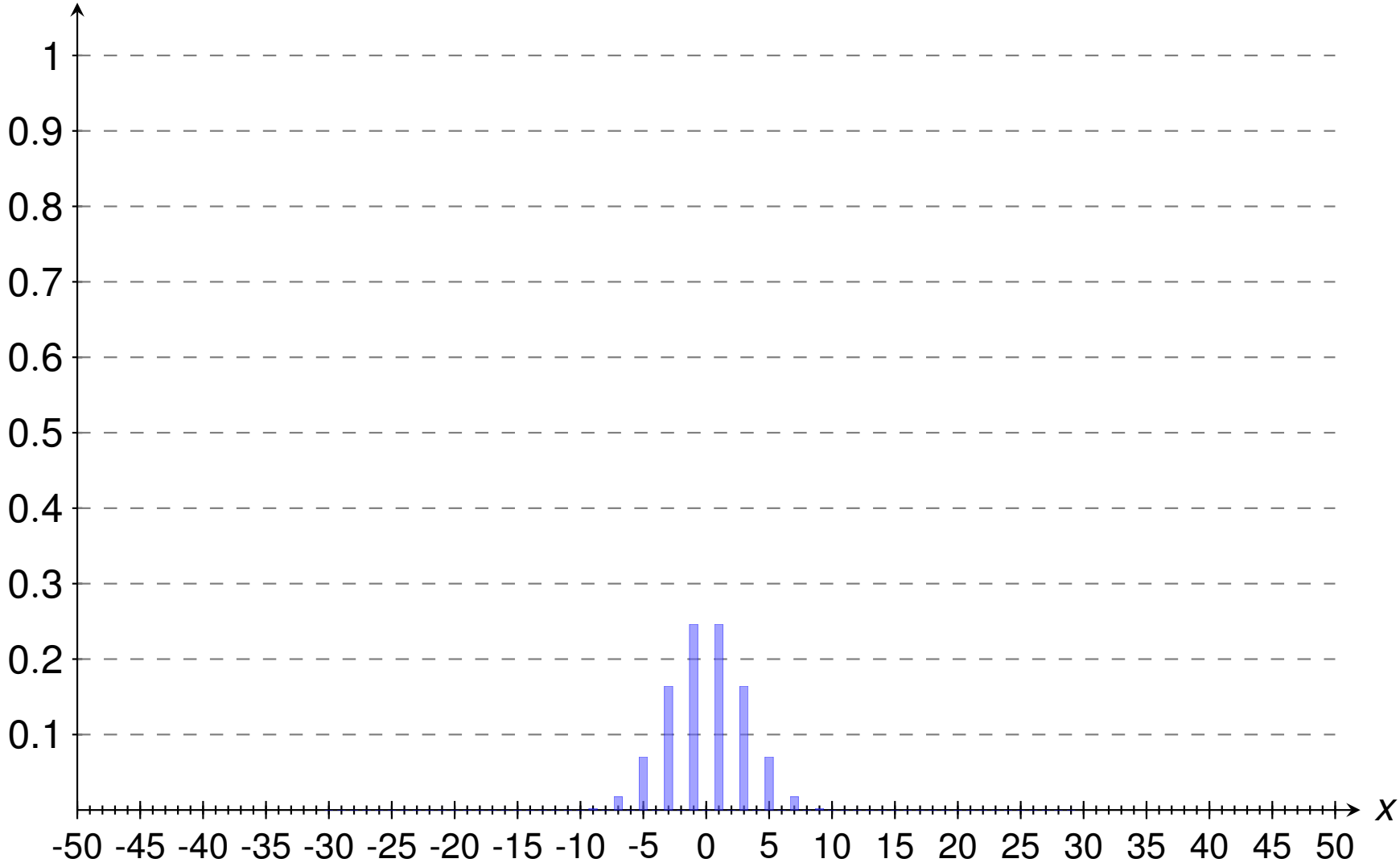

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{10} X_j = x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

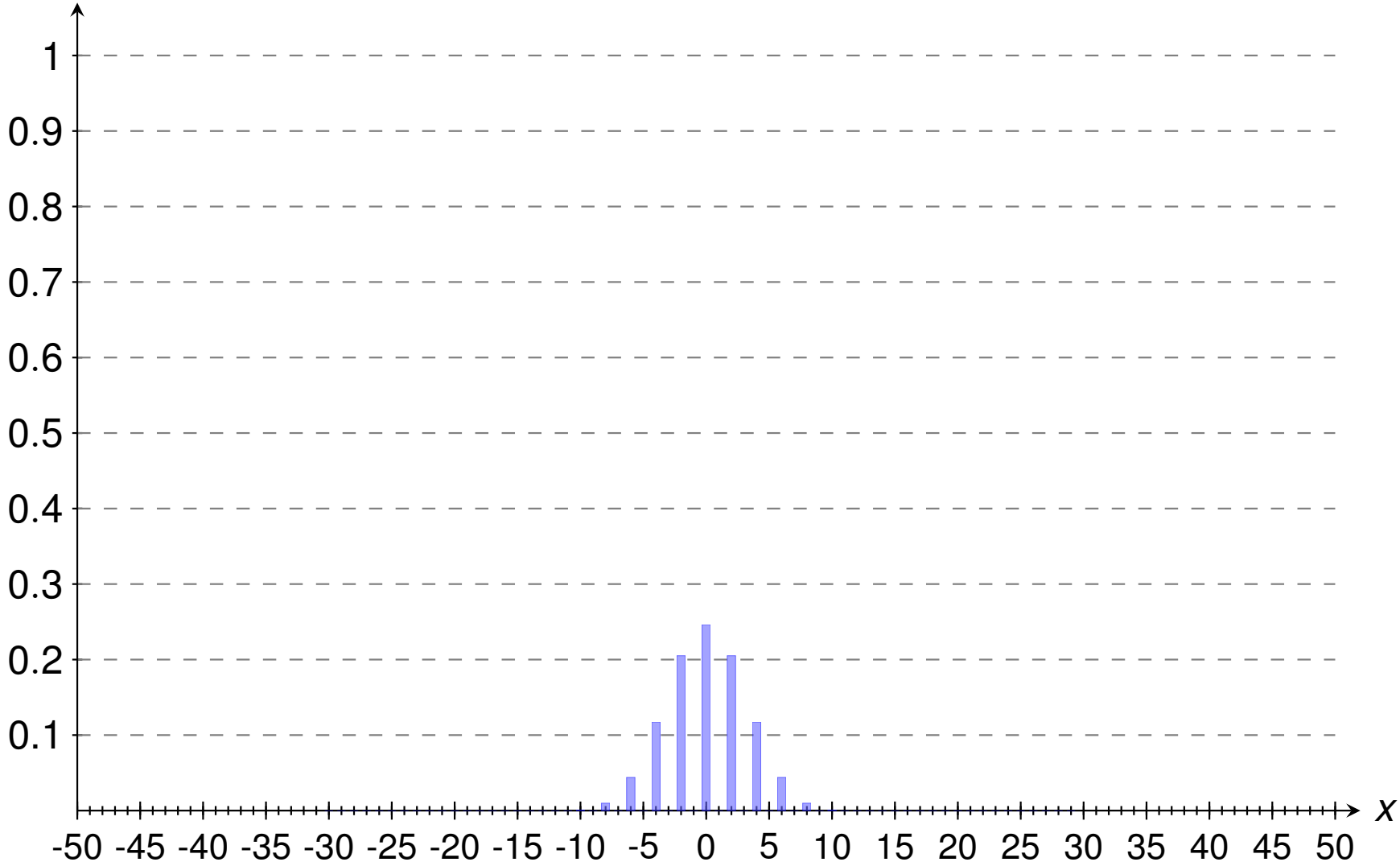

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{11} X_j = x\right]$

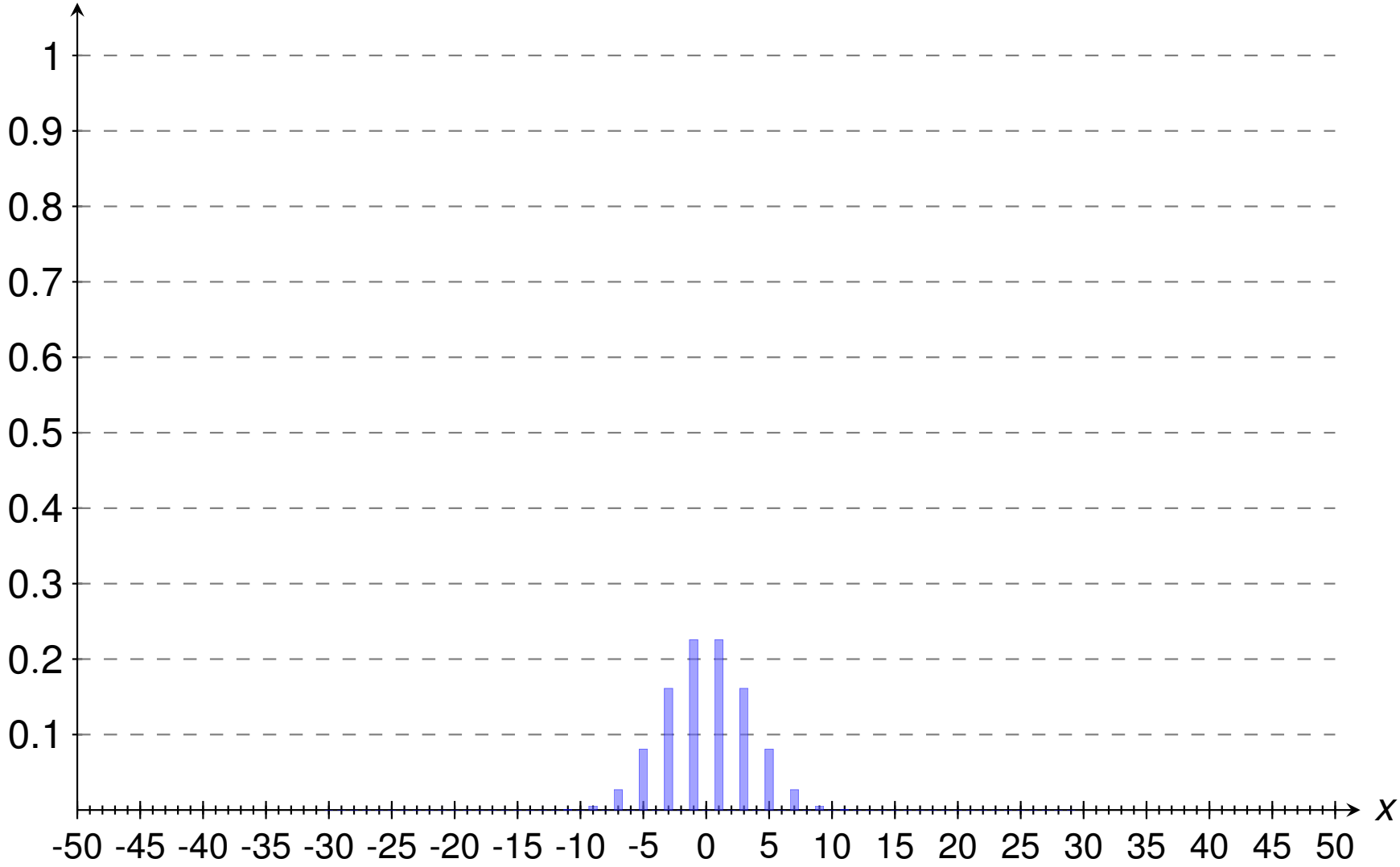

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{12} X_j = x\right]$

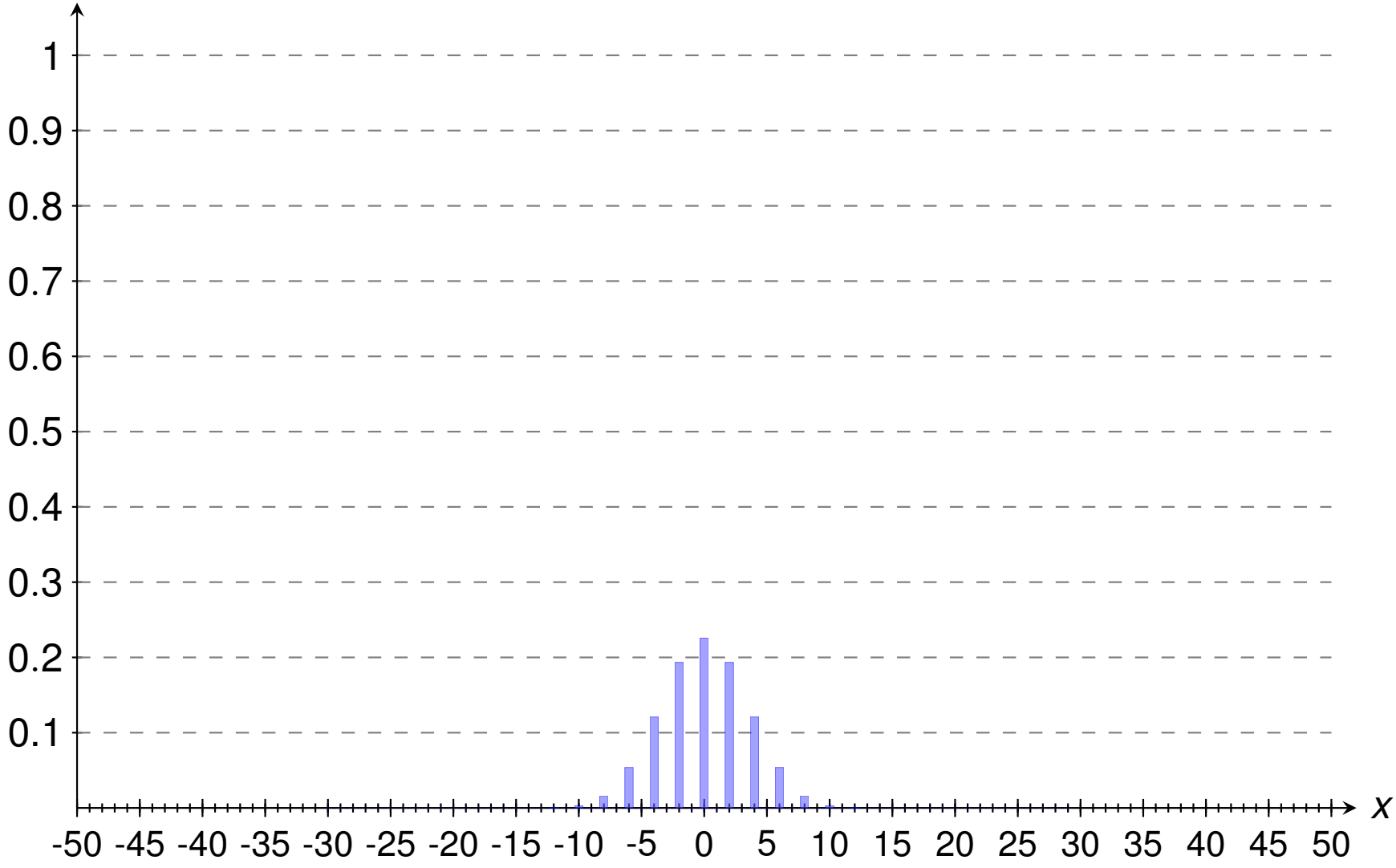

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{13} X_j = x\right]$

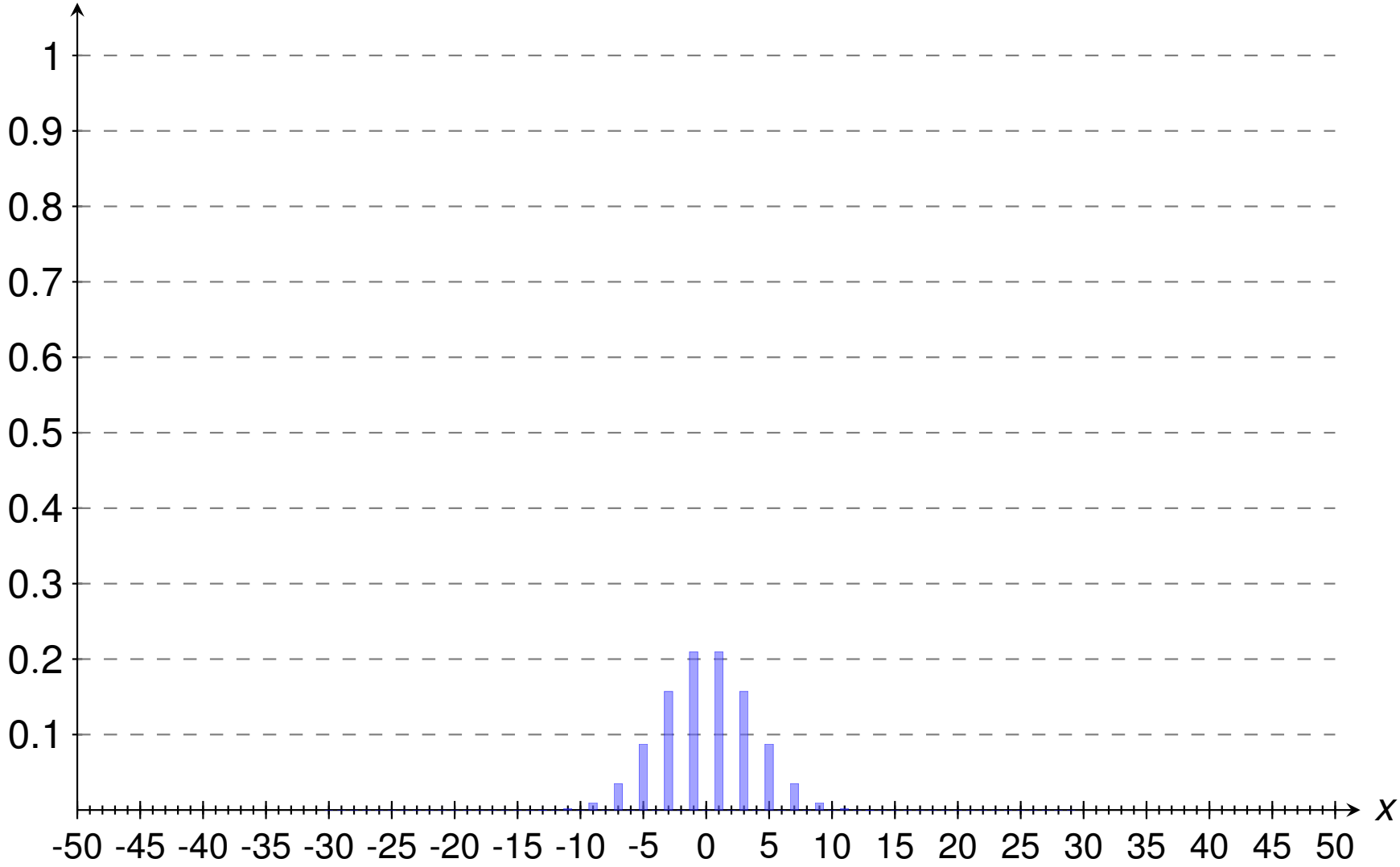

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{14} X_j = x\right]$

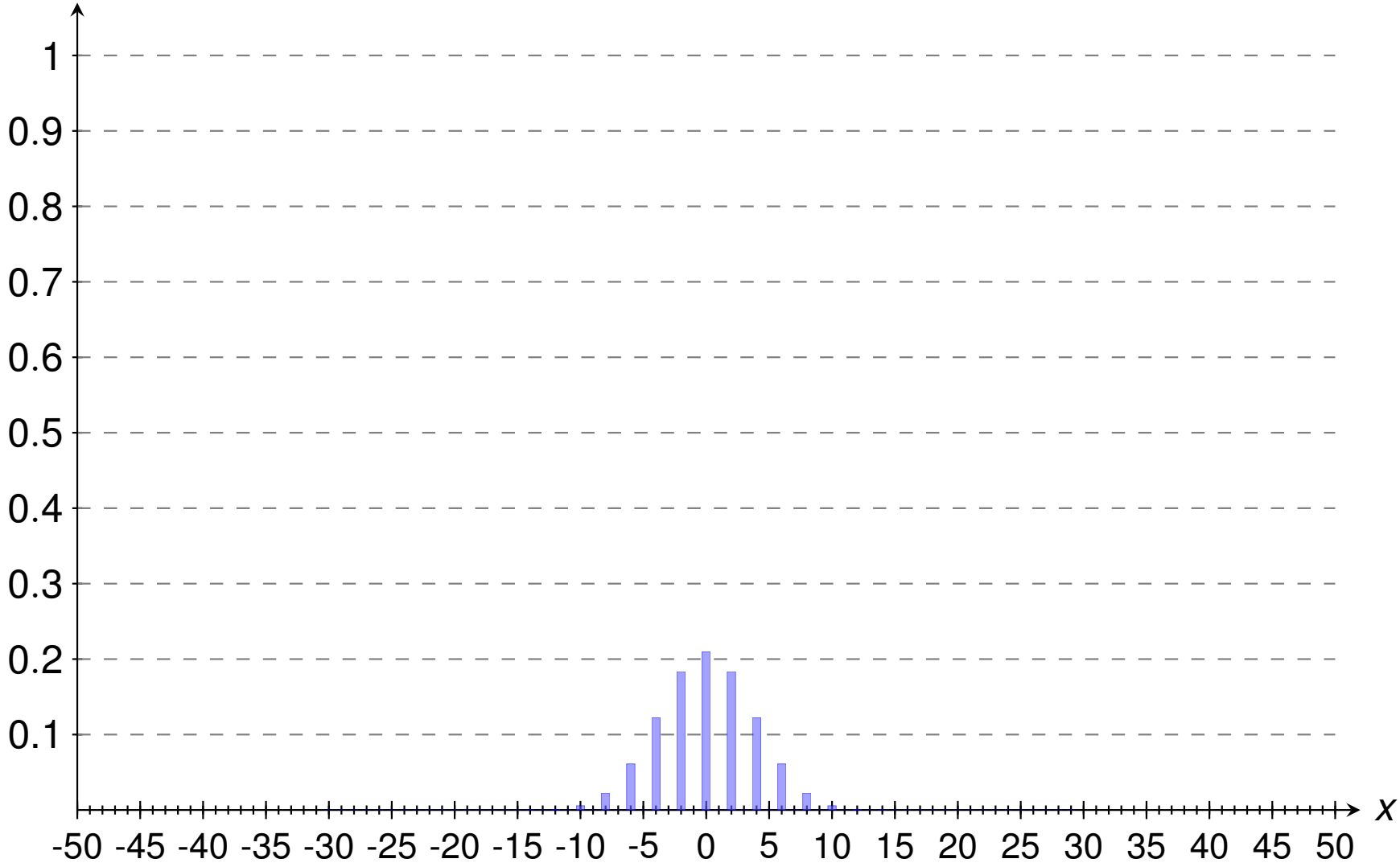

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{15} X_j = x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

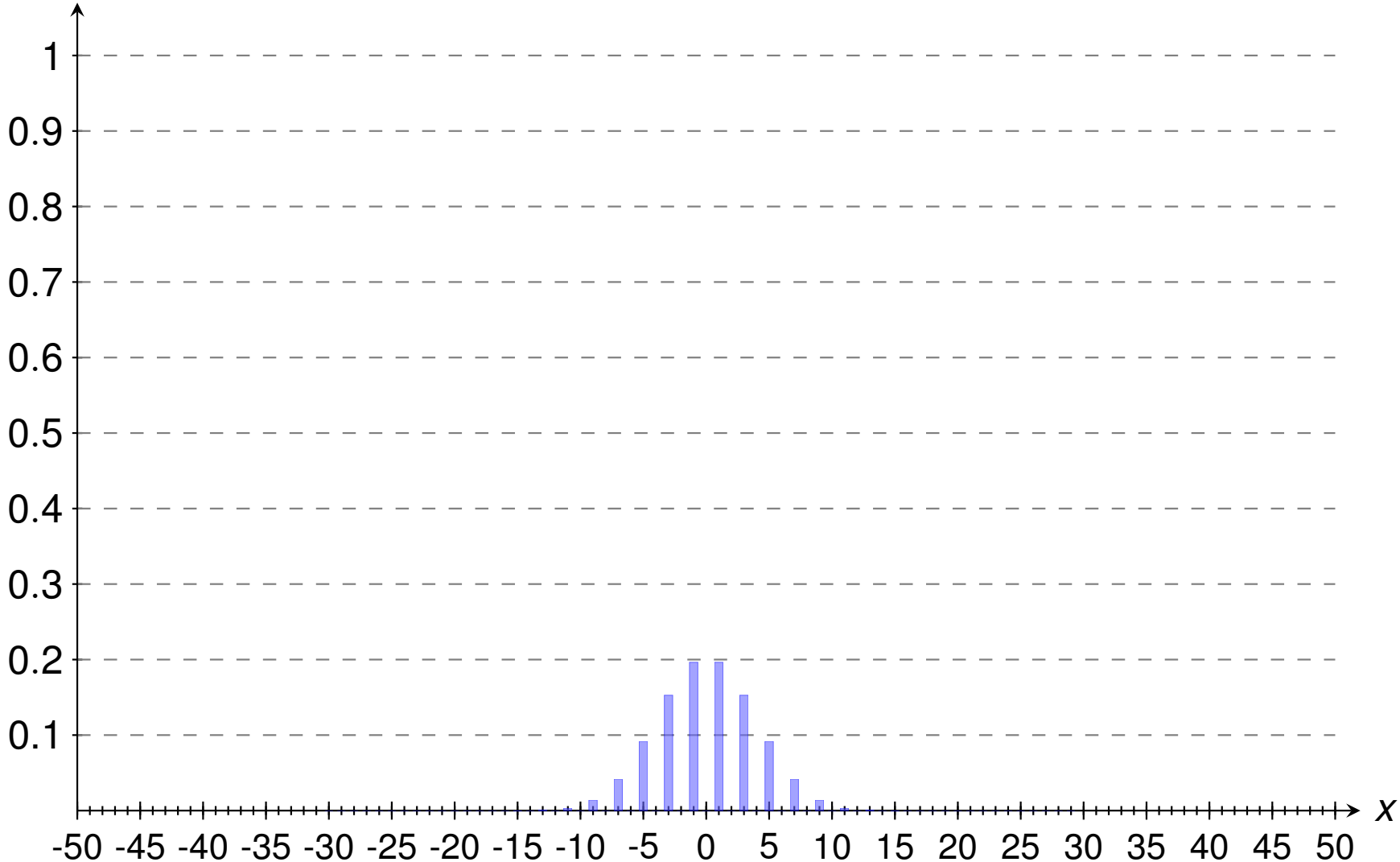

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{16} X_j = x\right]$

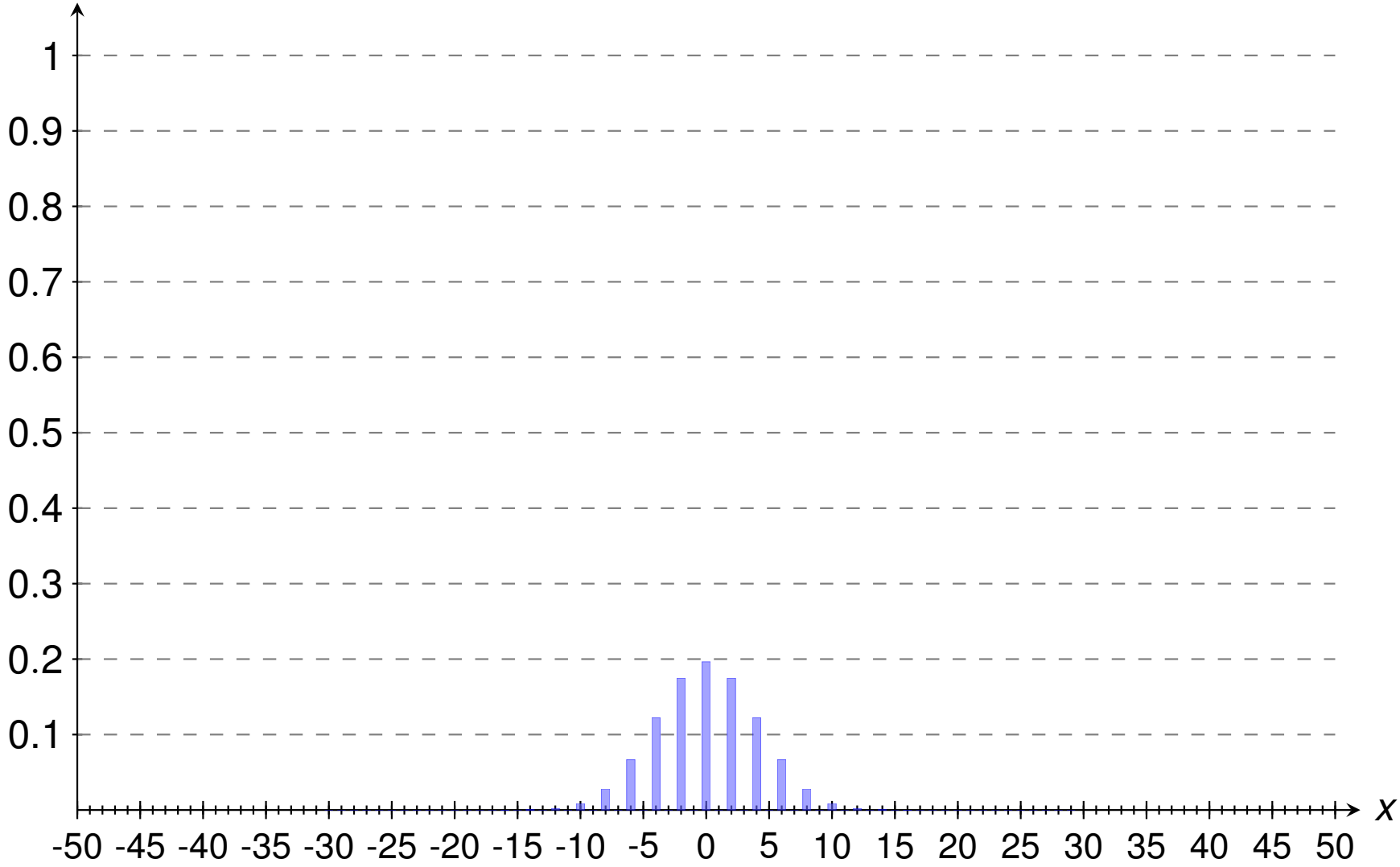

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{17} X_j = x\right]$

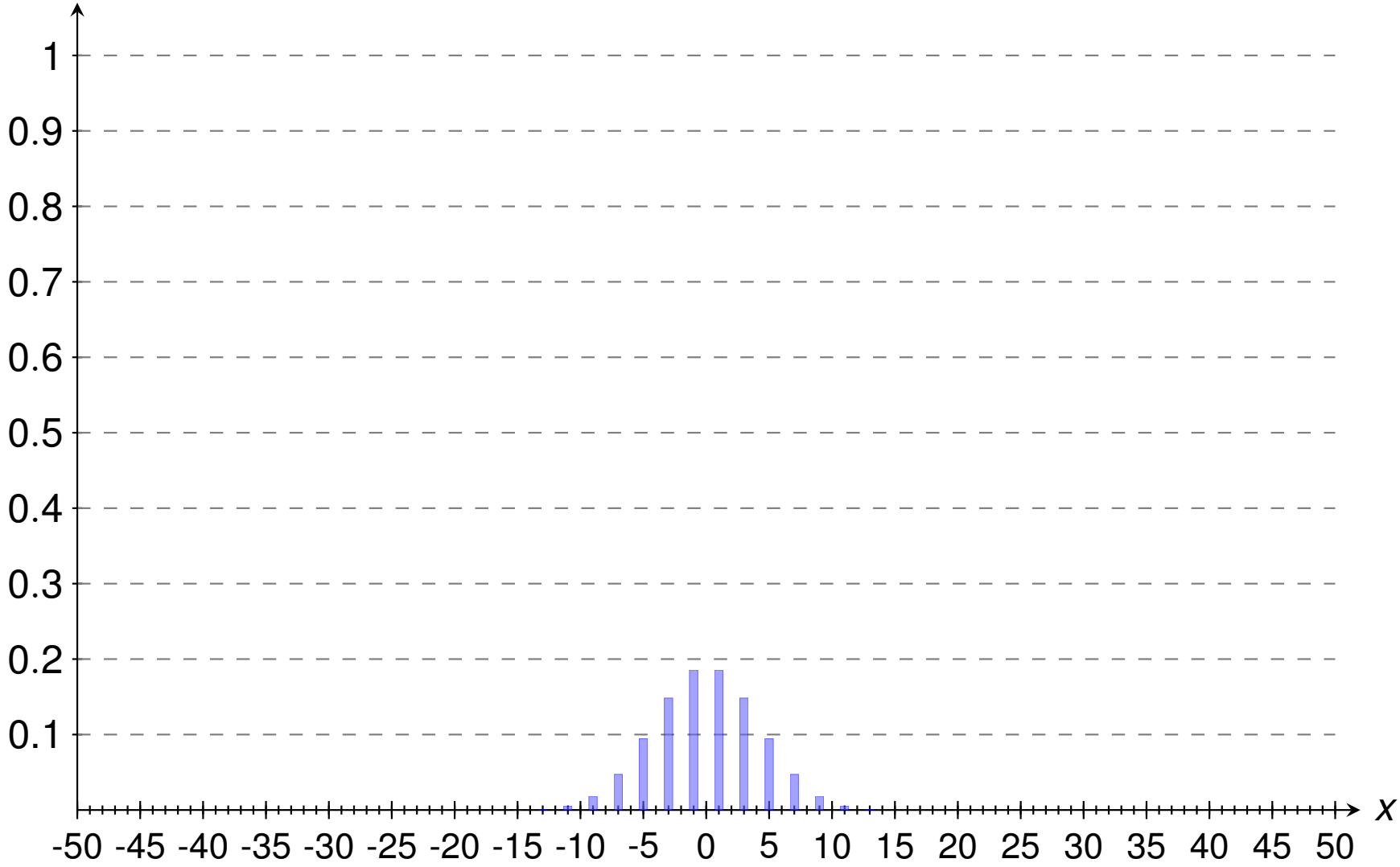

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{18} X_j = x\right]$

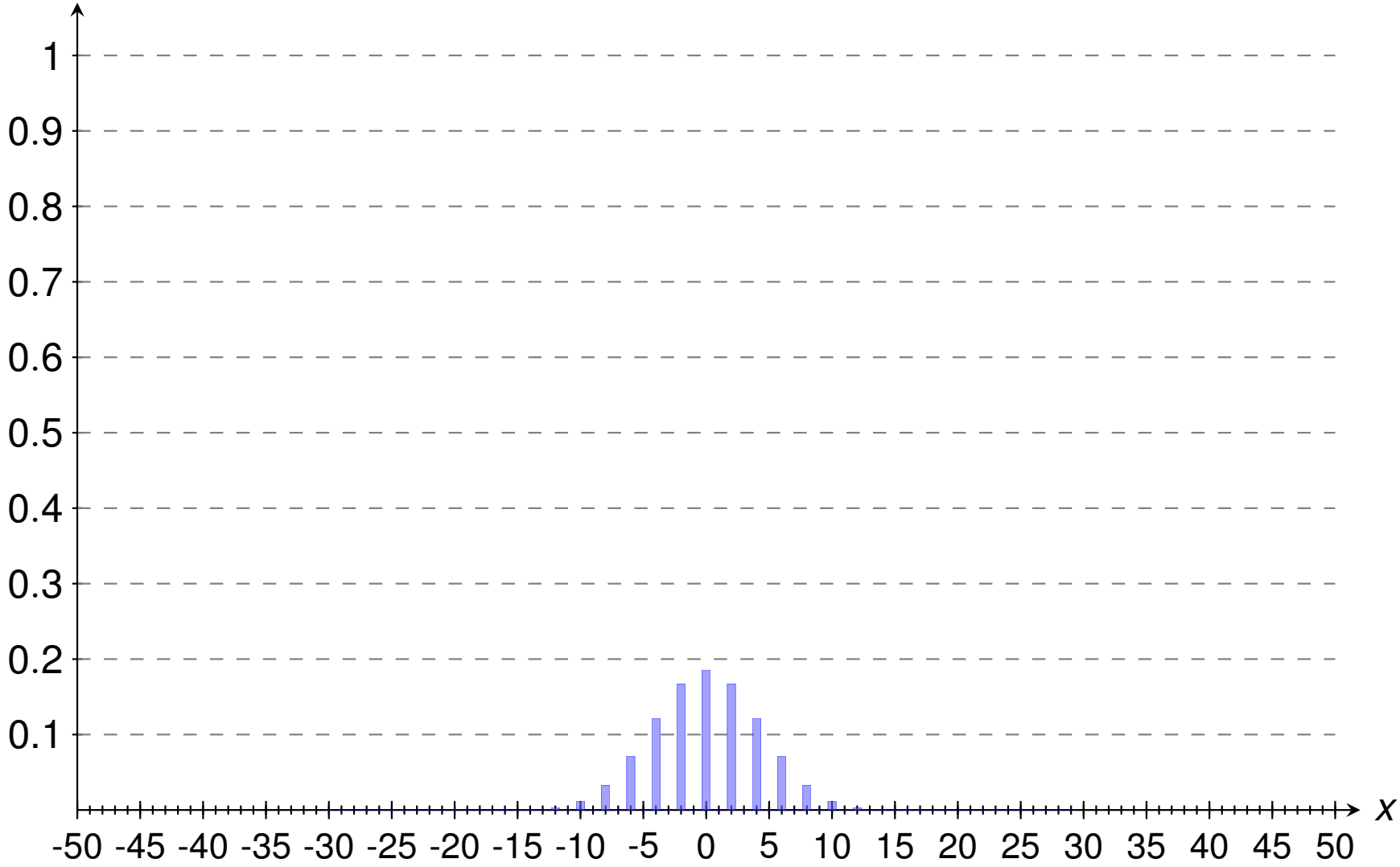

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{19} X_j = x\right]$

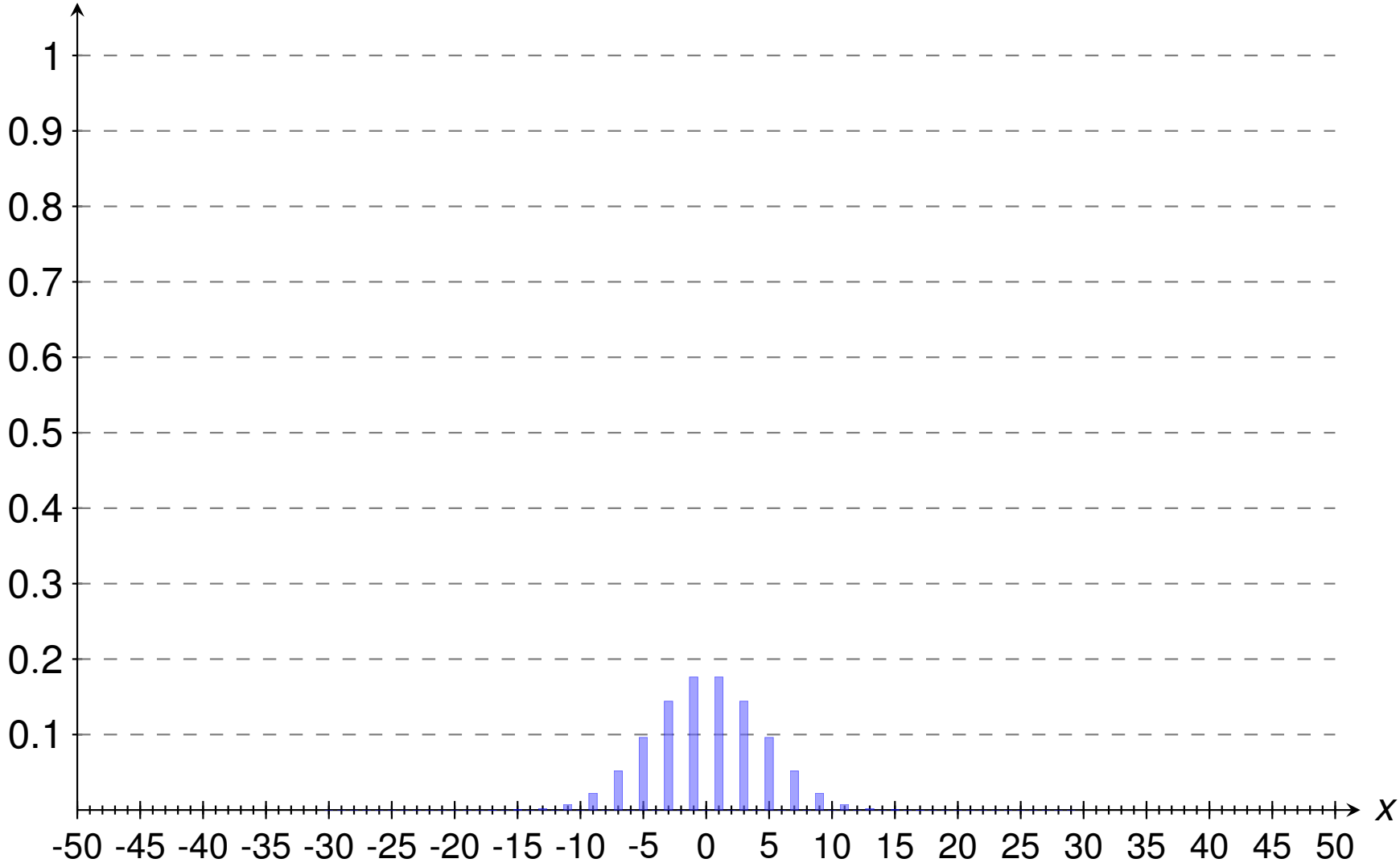

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{20} X_j = x\right]$

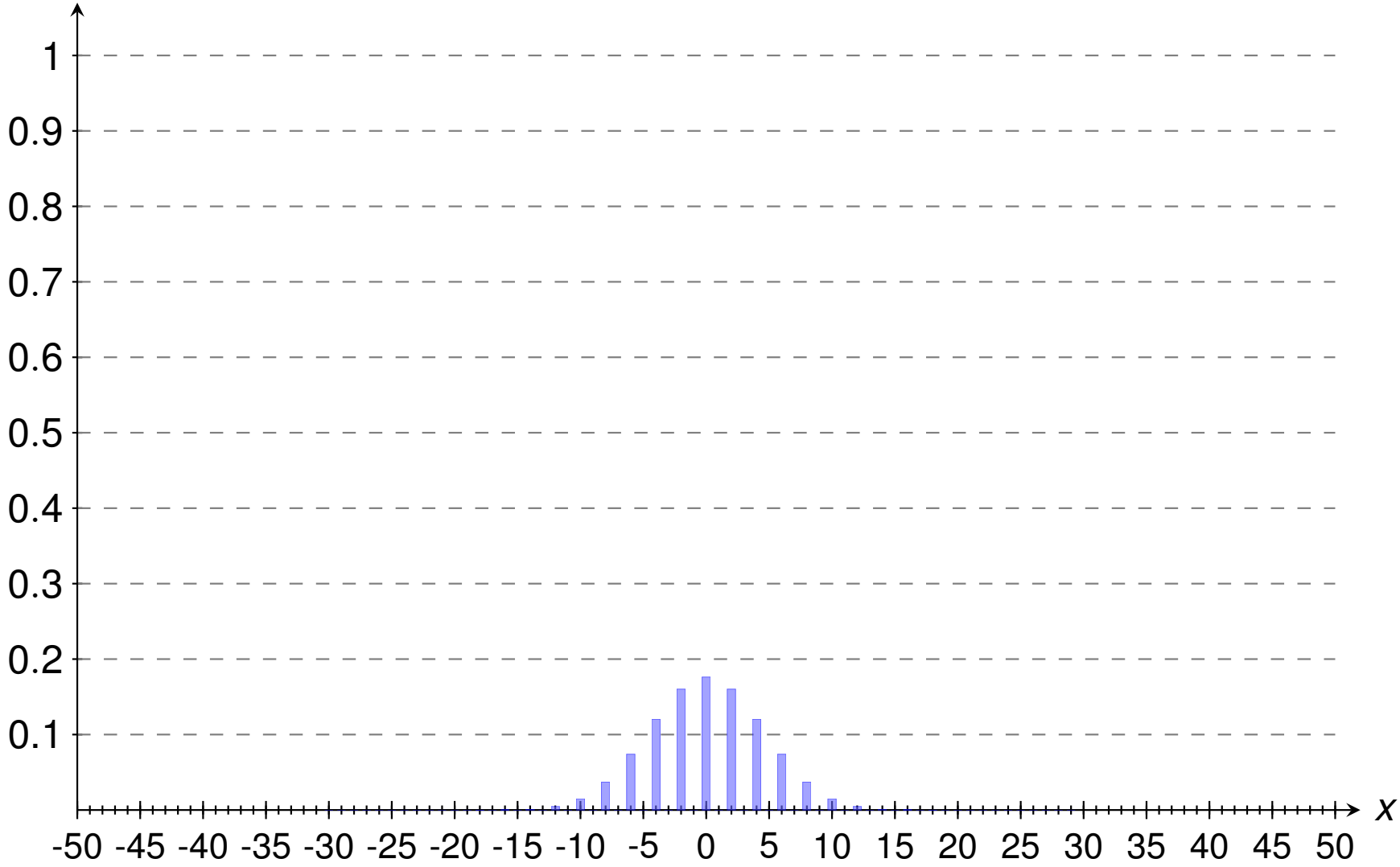

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{21} X_j = x\right]$

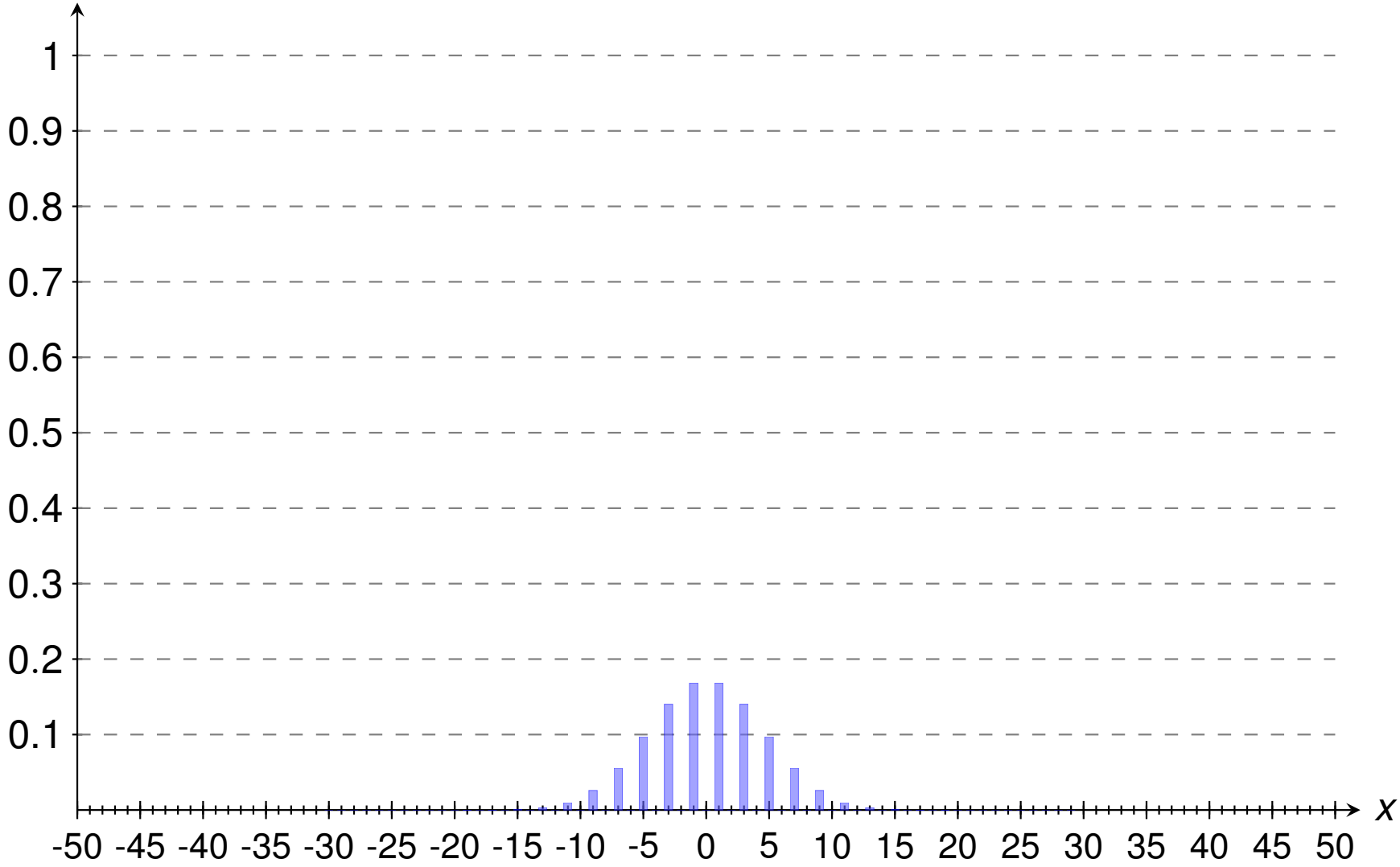

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{22} X_j = x\right]$

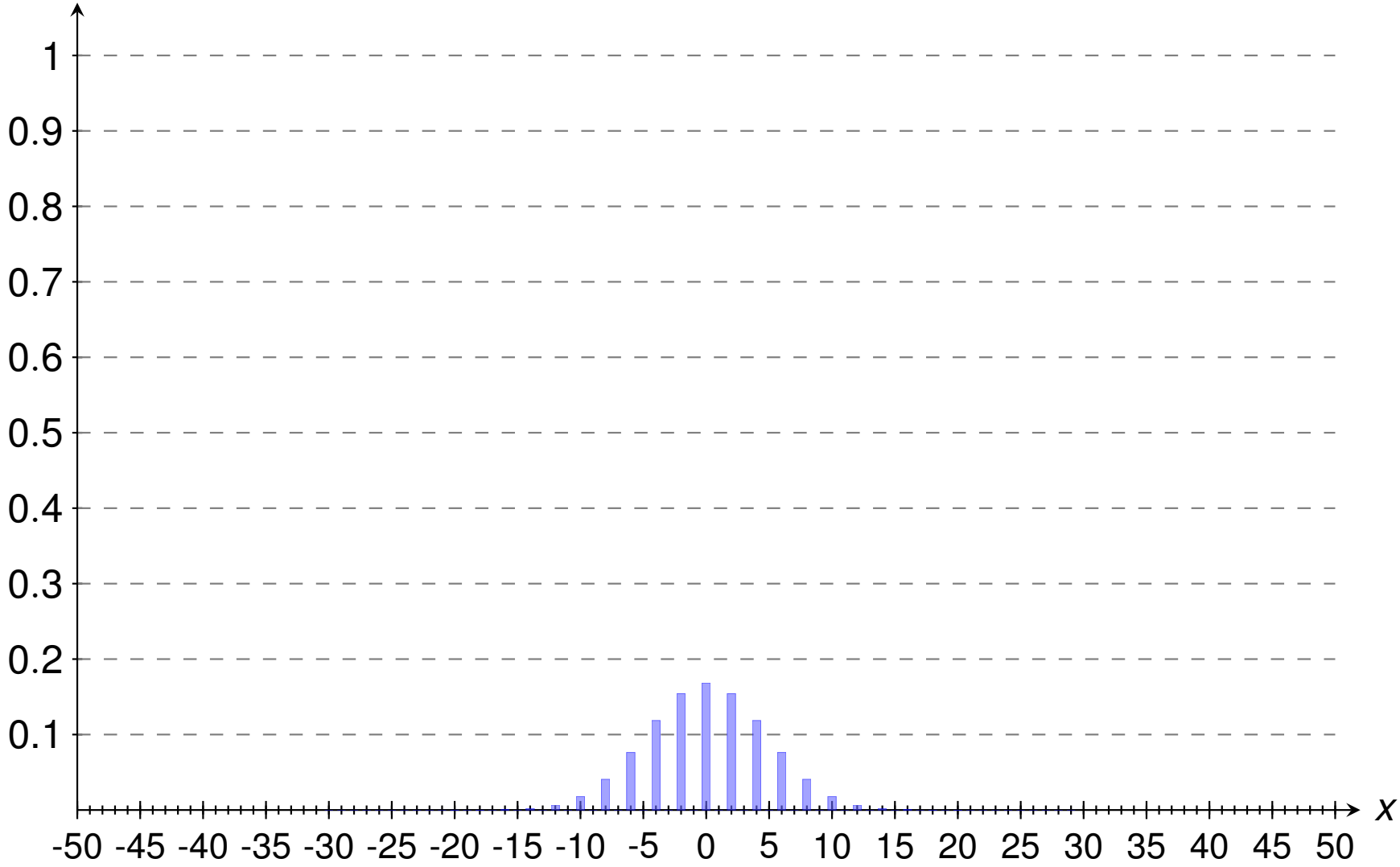

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{23} X_j = x\right]$

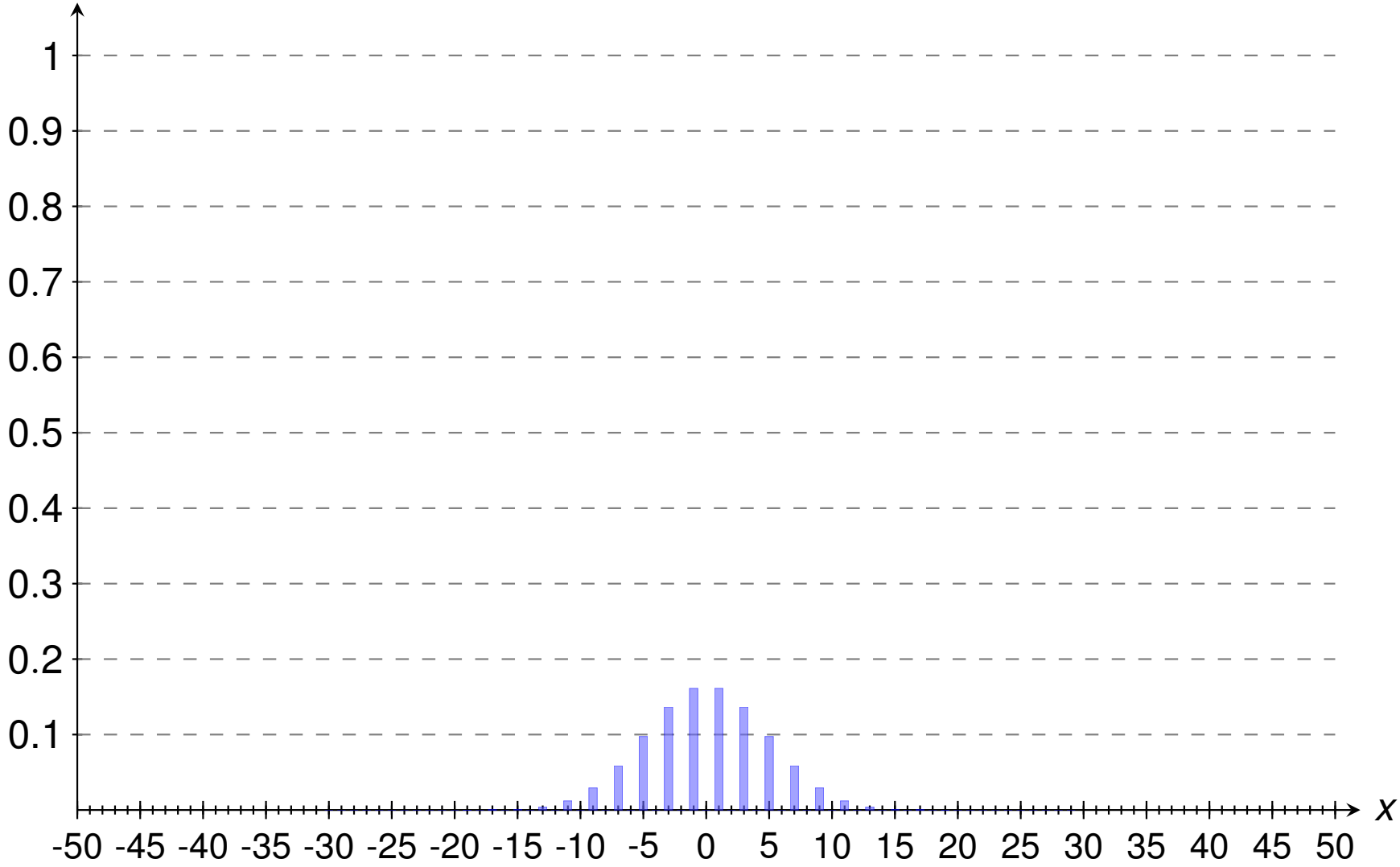

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{24} X_j = x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

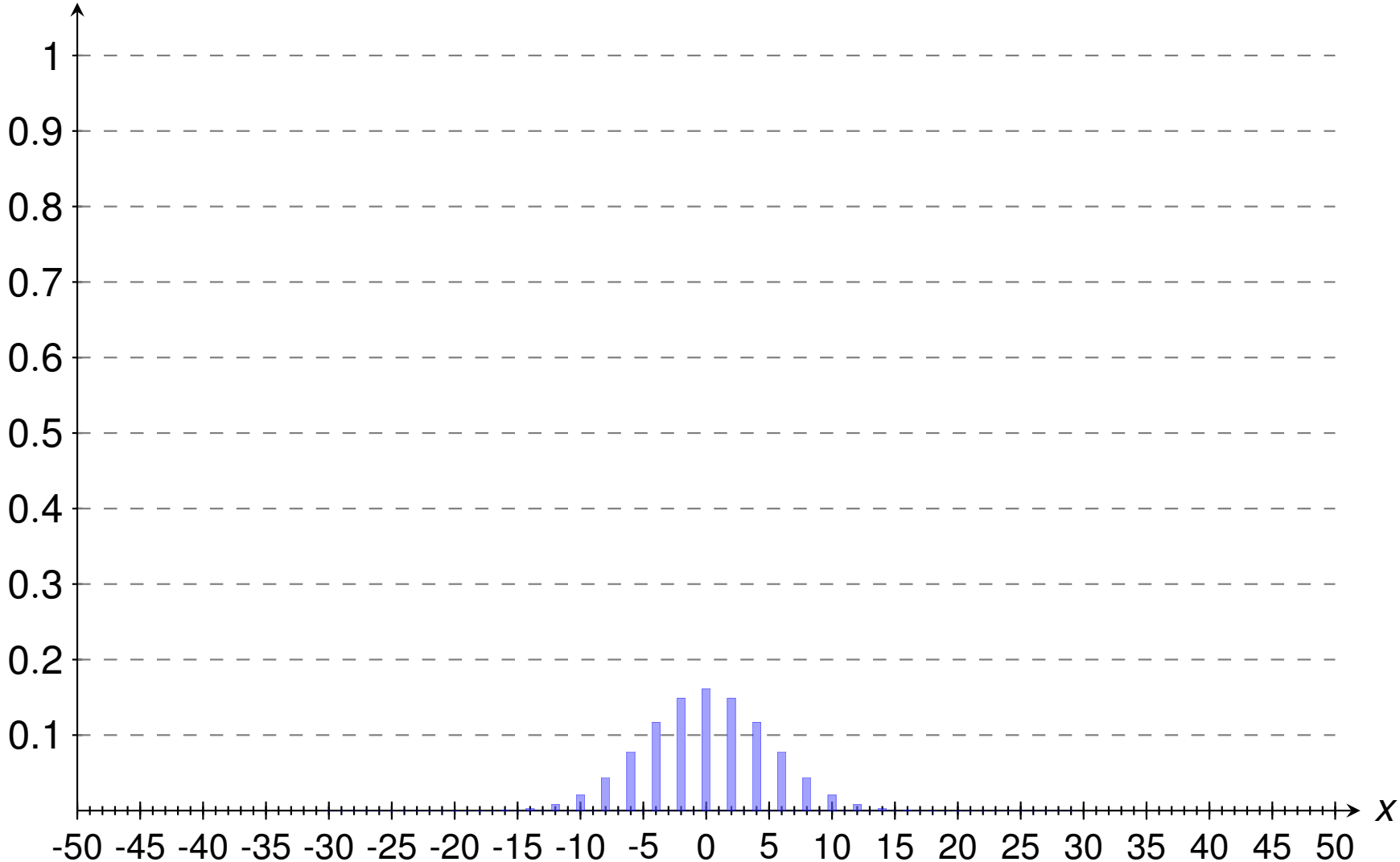

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{25} X_j = x\right]$

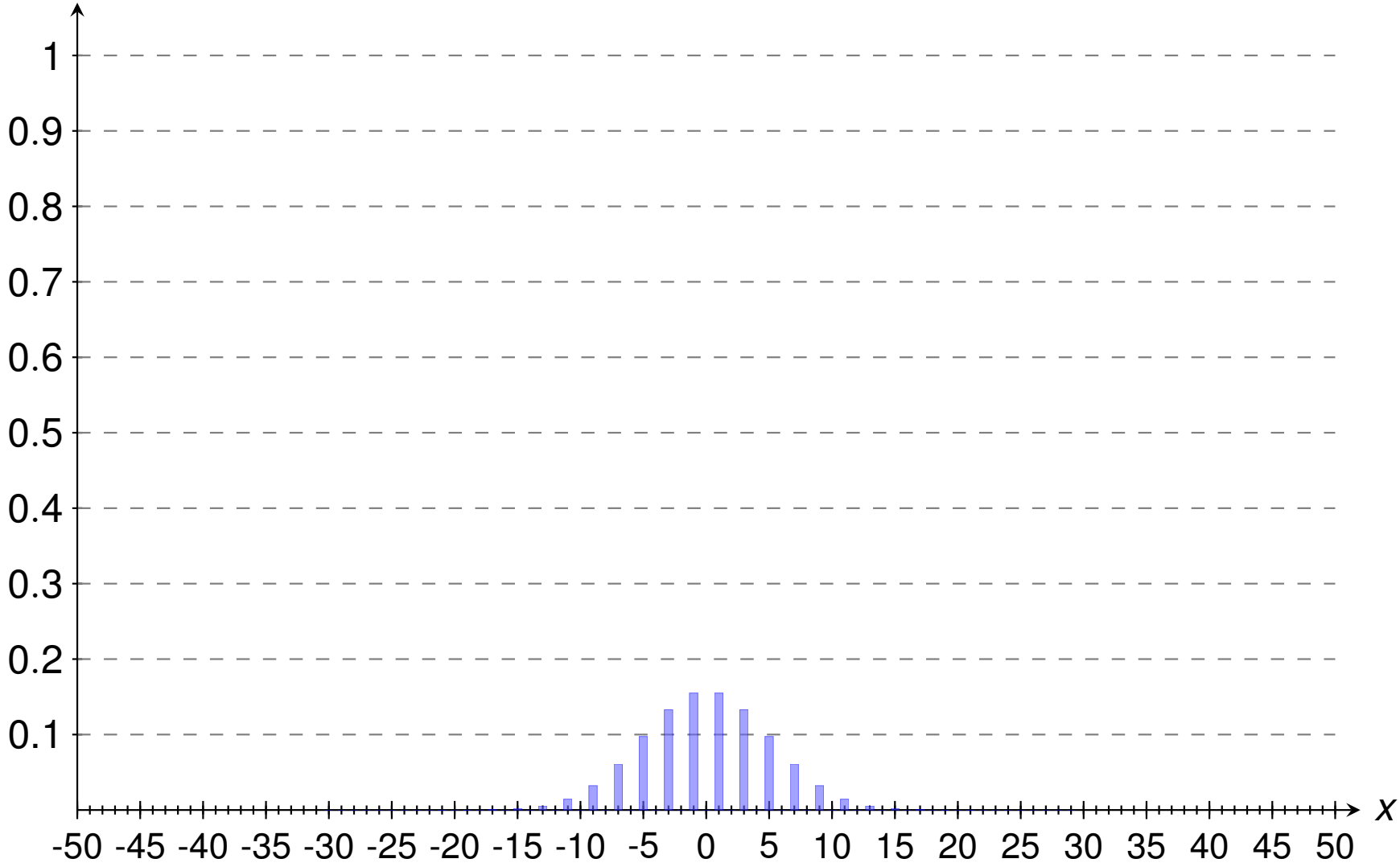

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{26} X_j = x\right]$

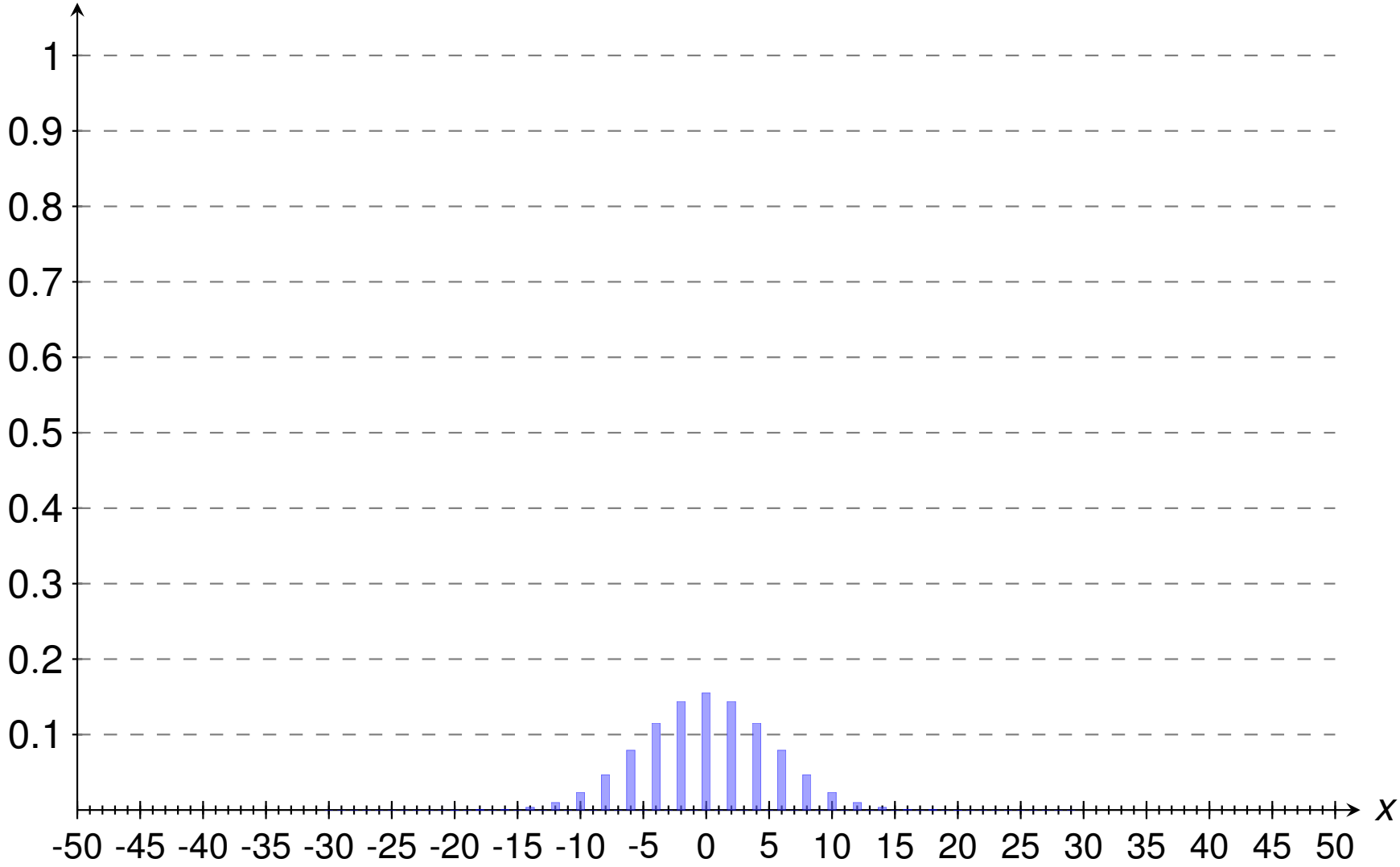

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{27} X_j = x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

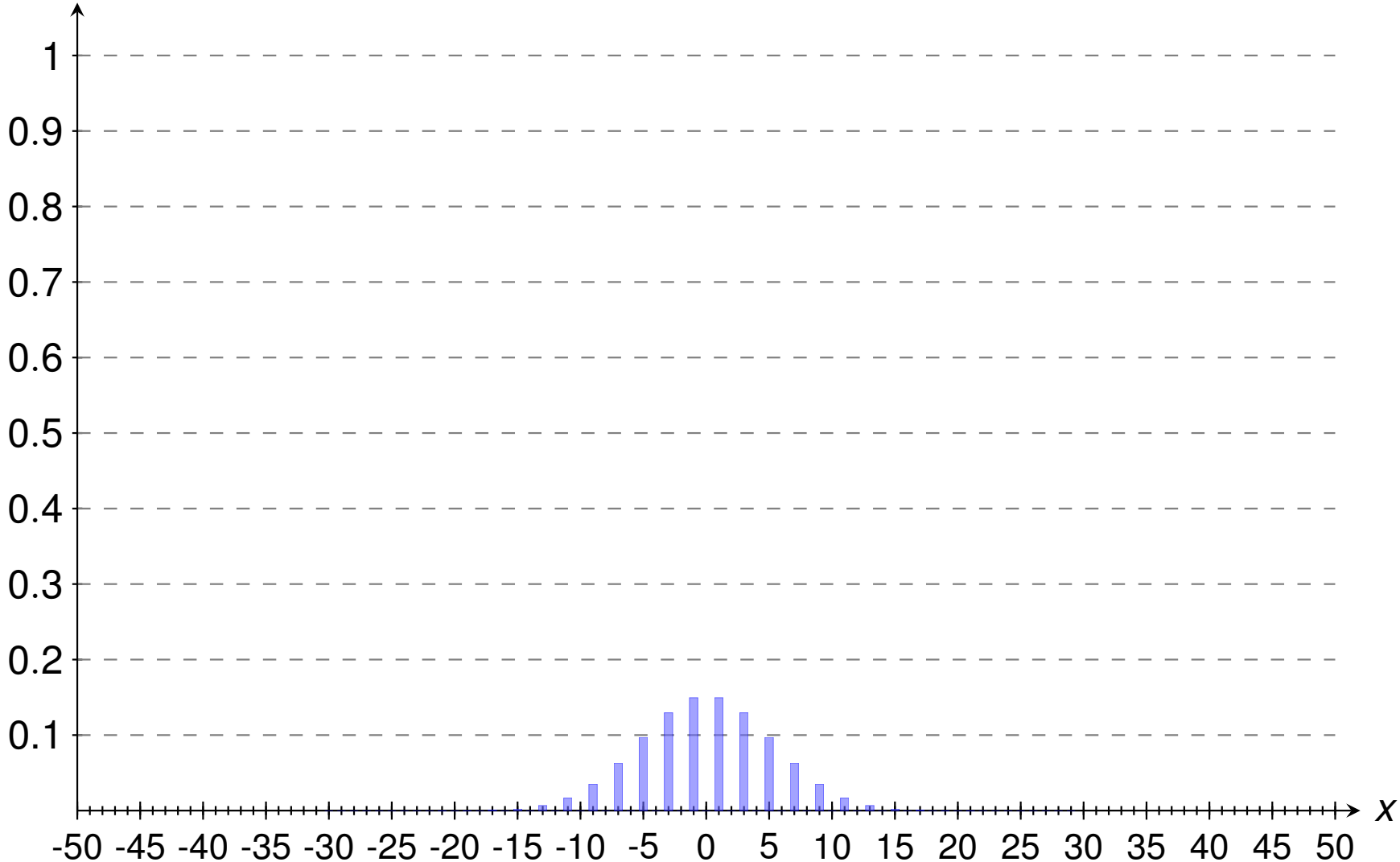

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{28} X_j = x\right]$

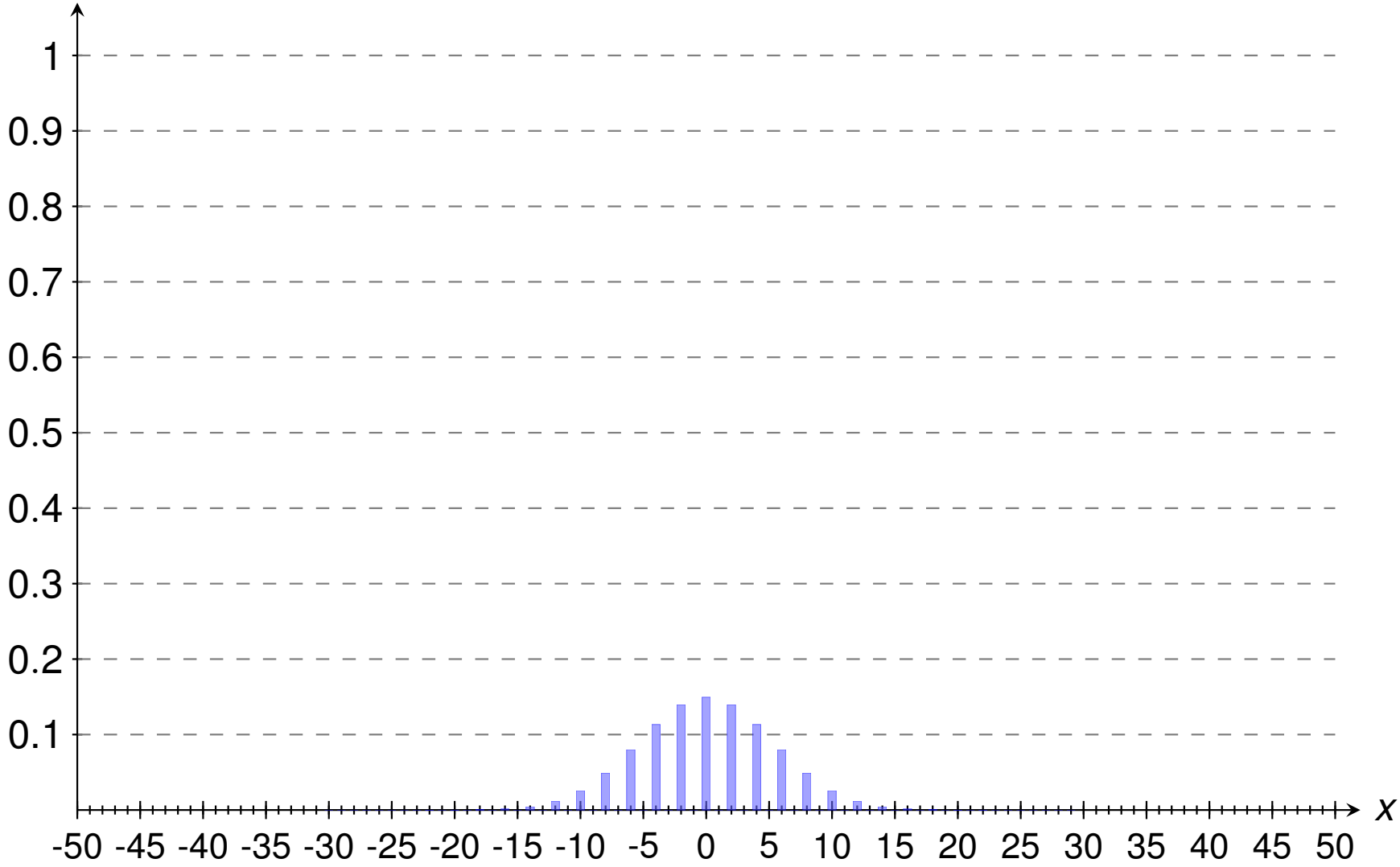

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{29} X_j = x\right]$

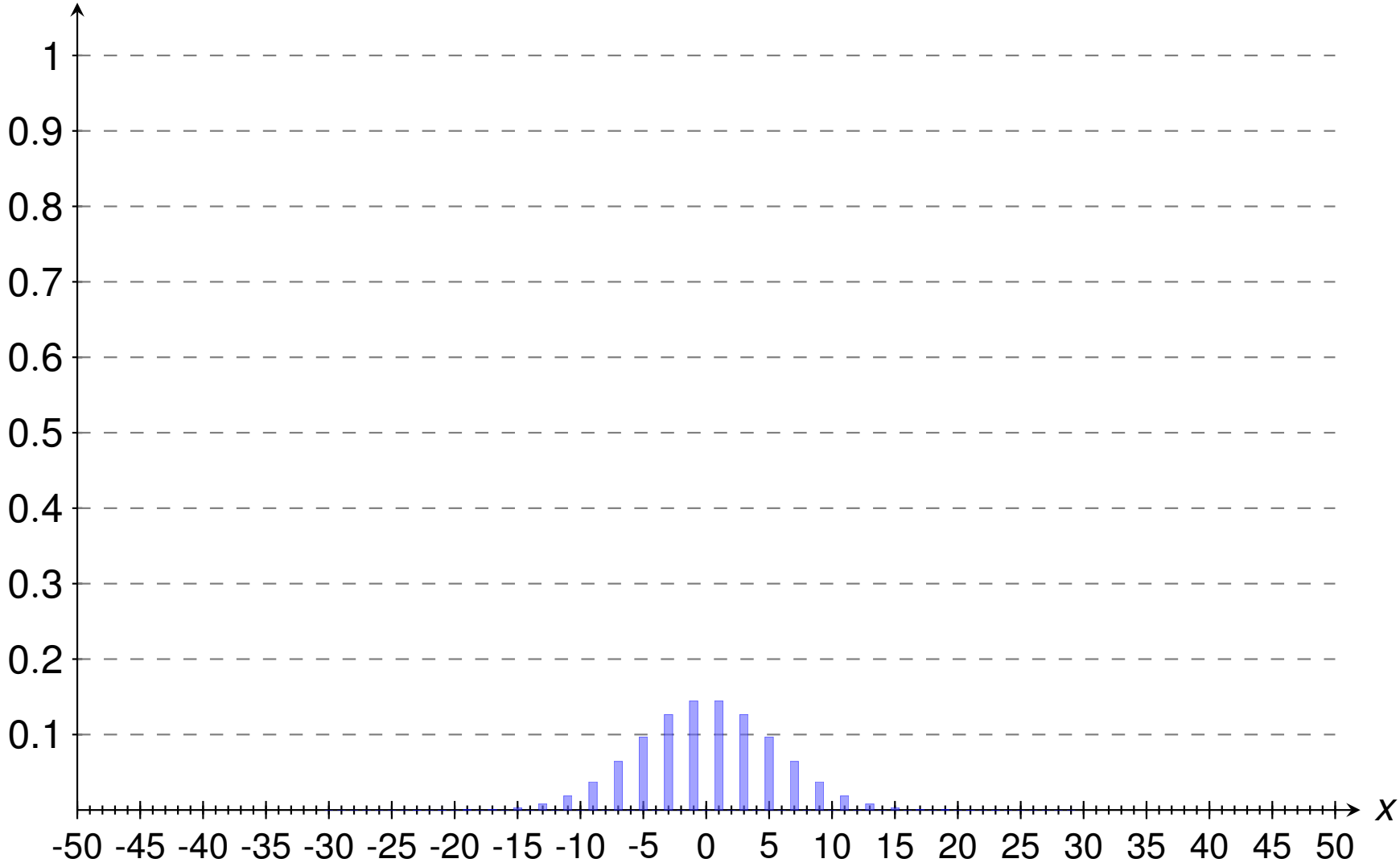

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$

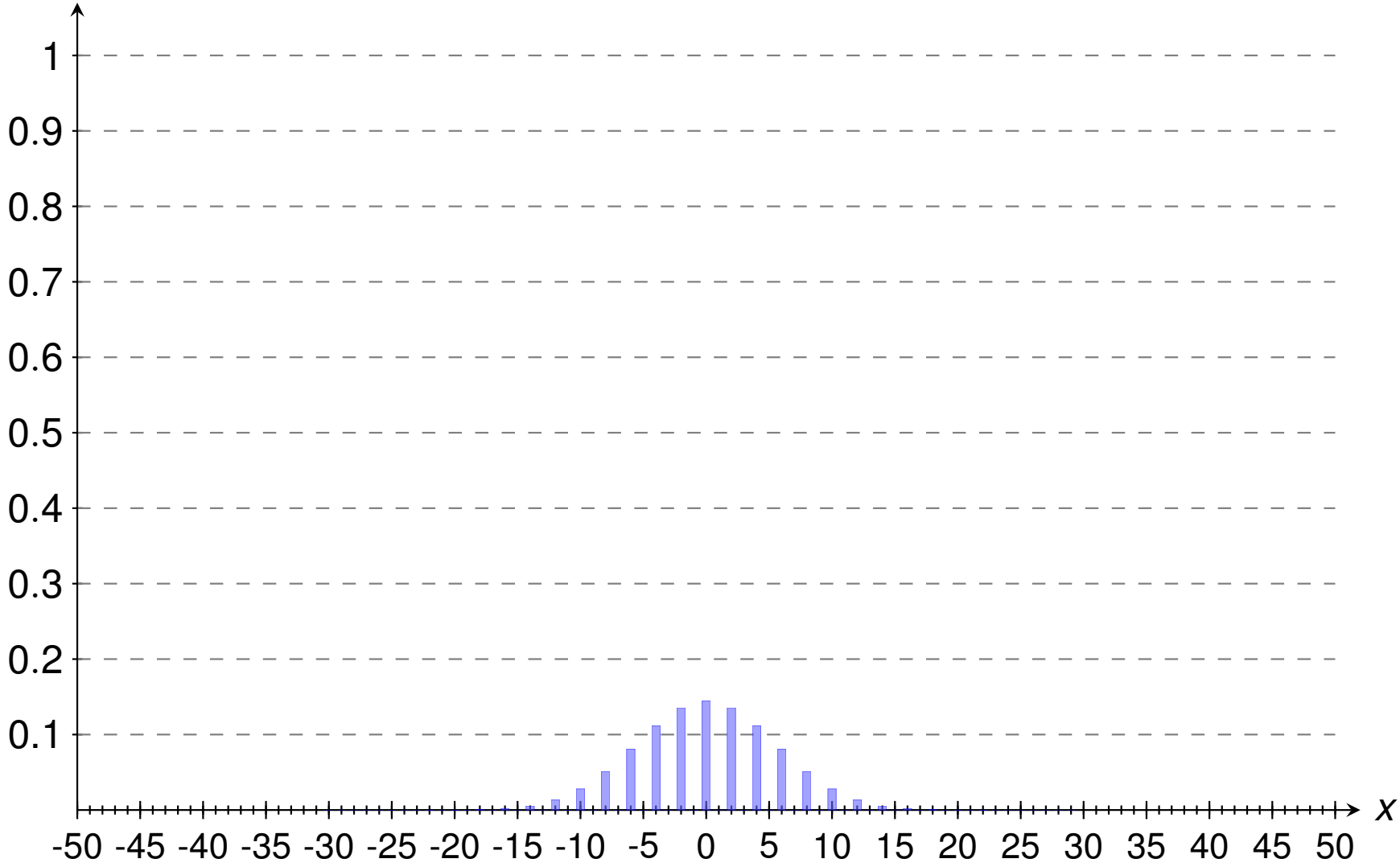

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$

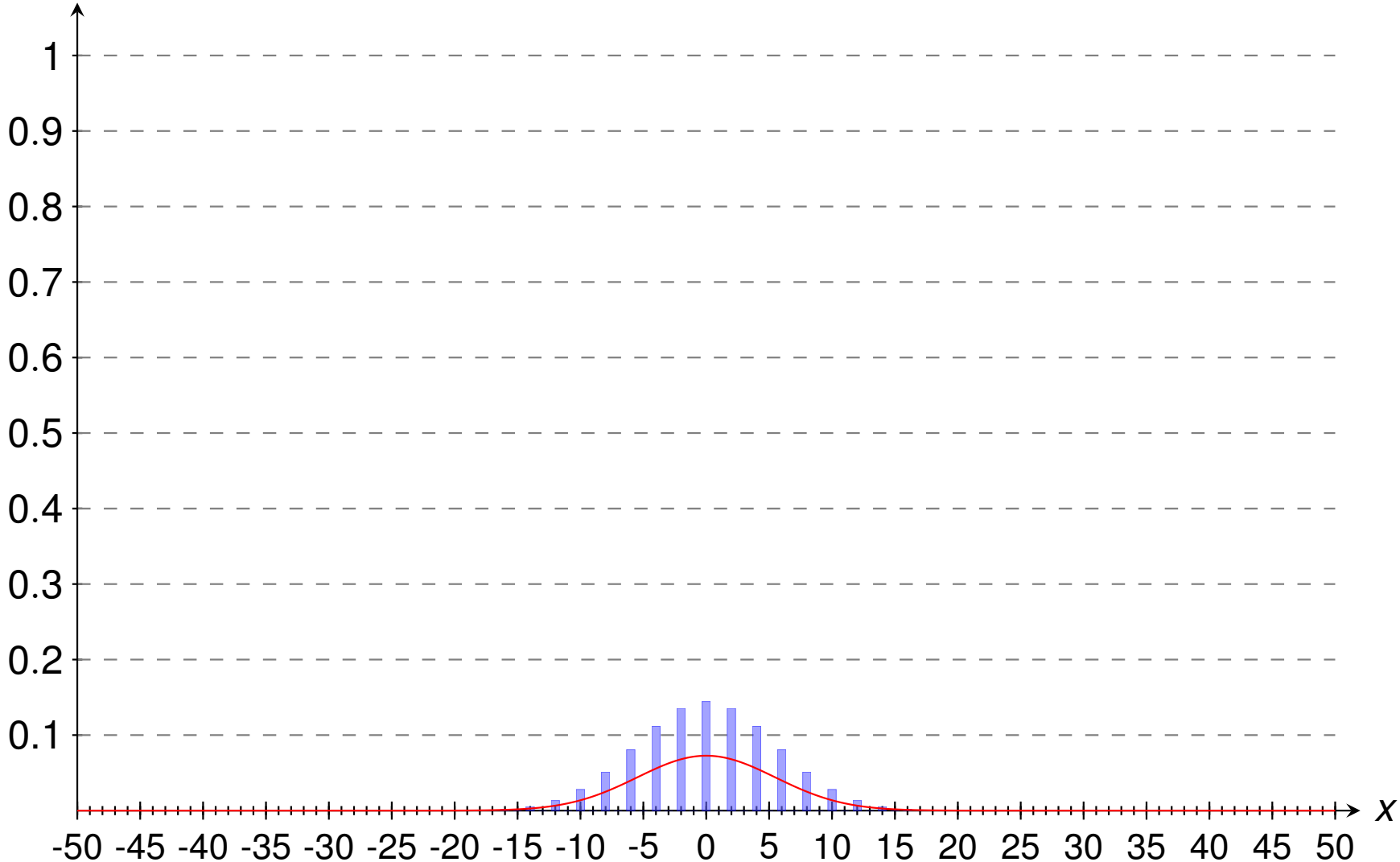

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{30} X_j = x\right]$

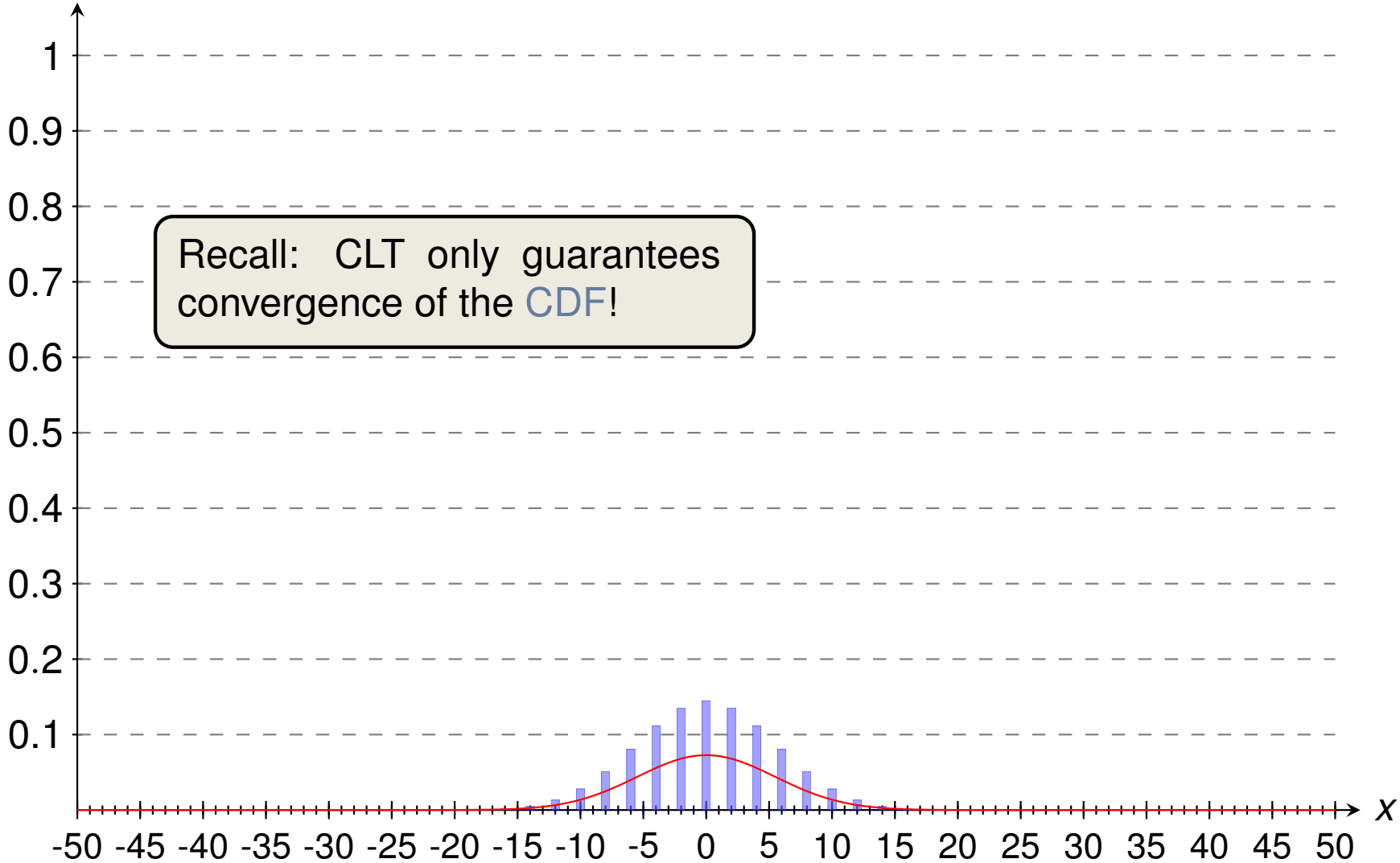

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{30} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

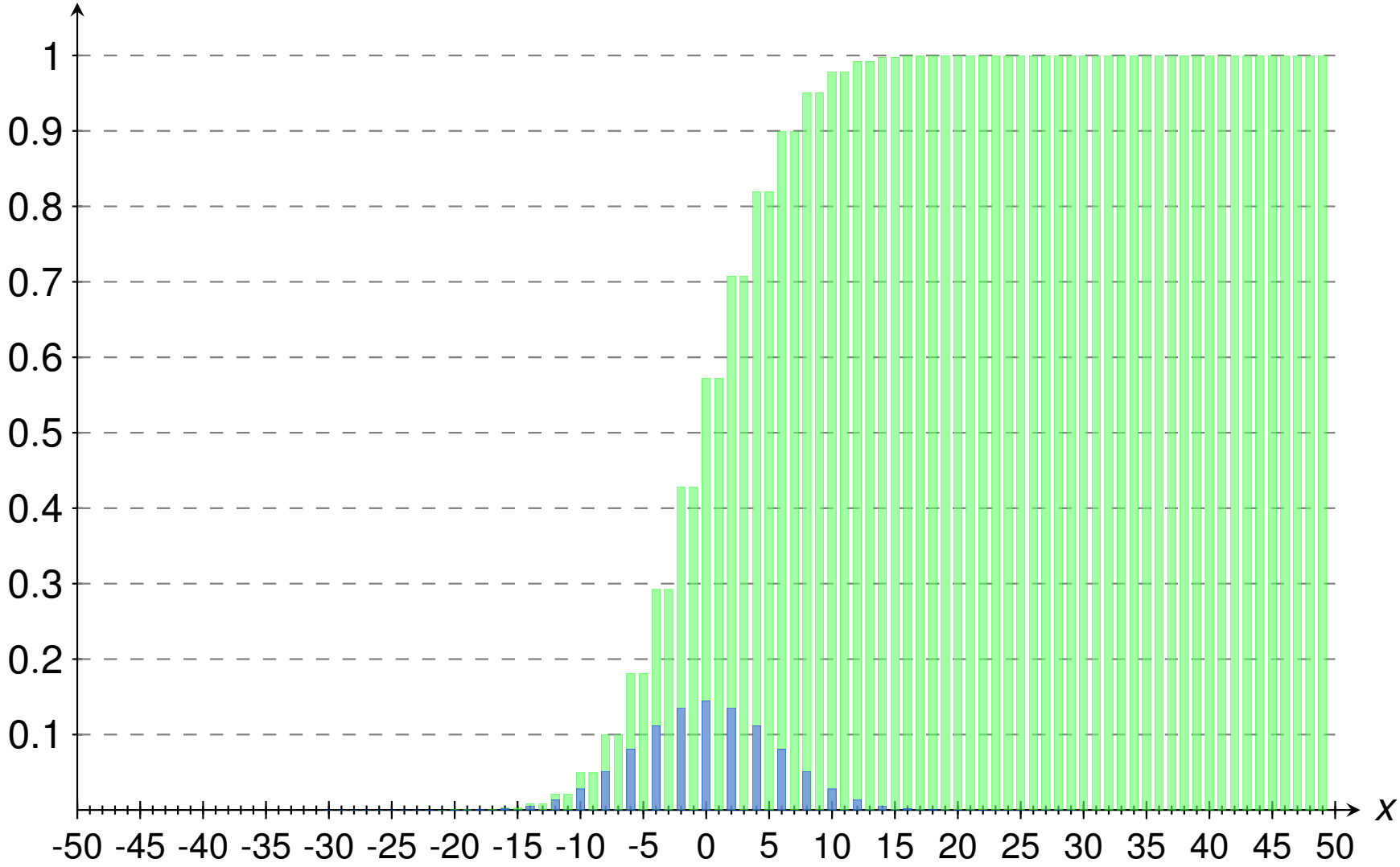

Illustration of CLT (3, Part I) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{30} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

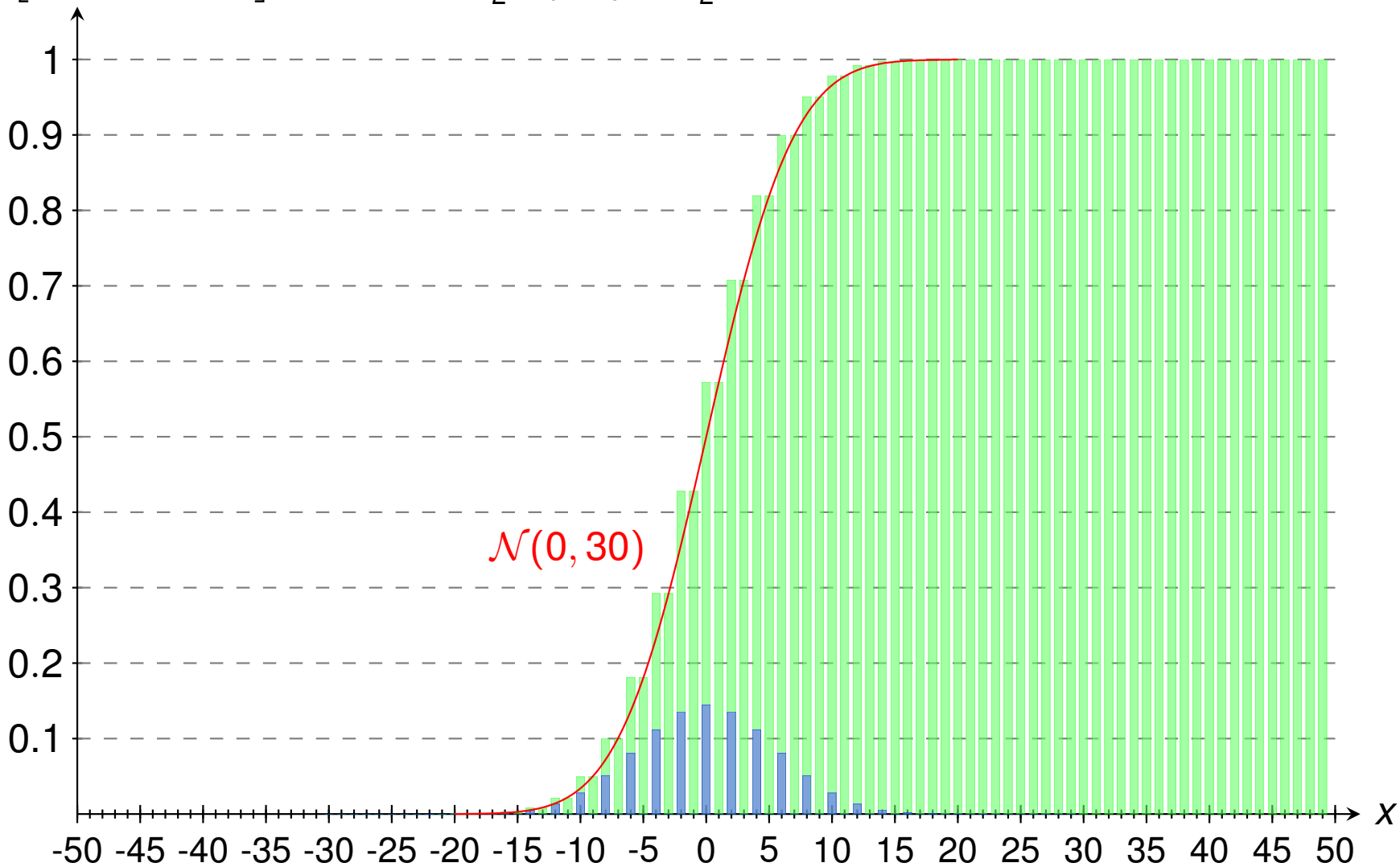

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{1} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

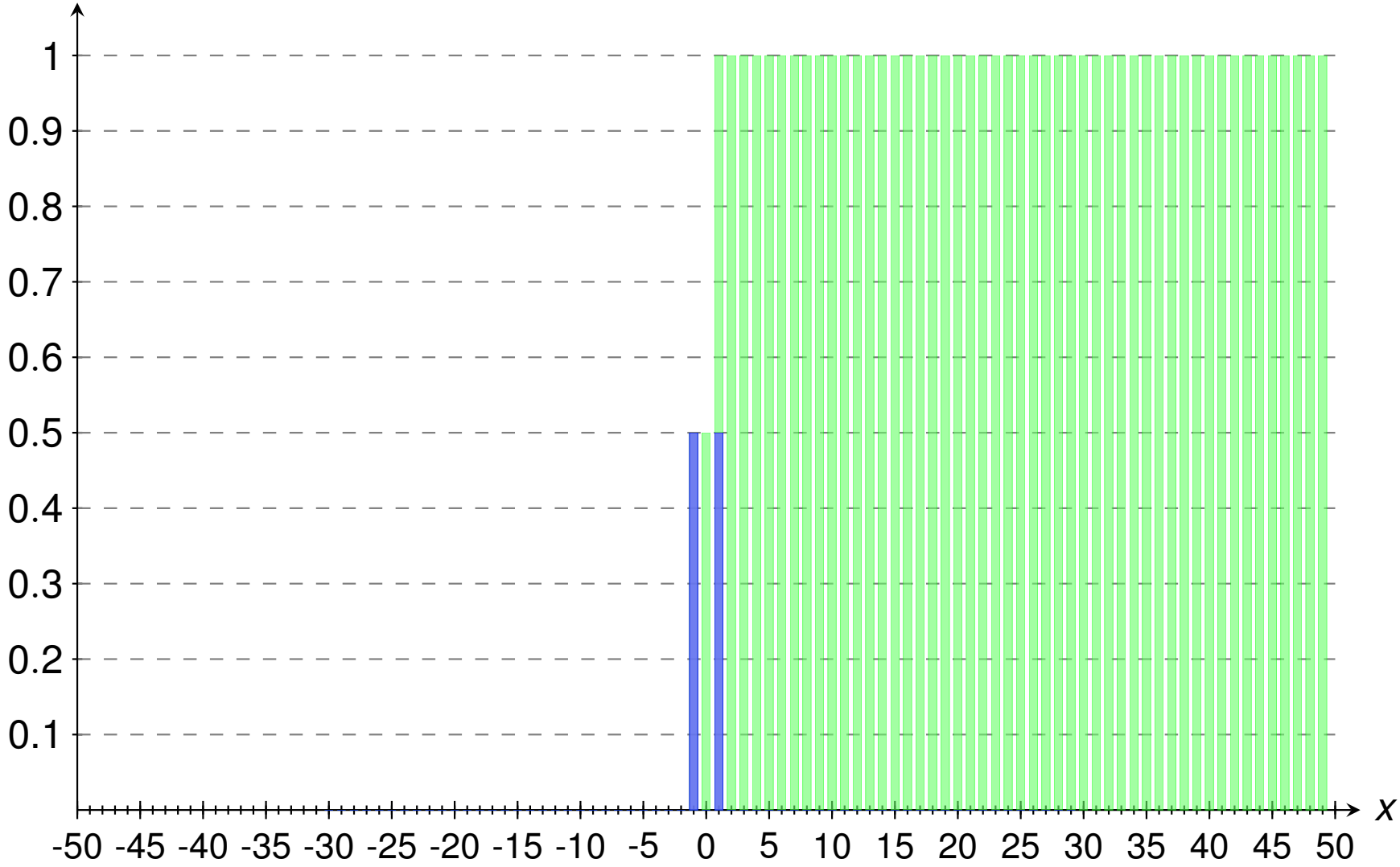

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{2} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

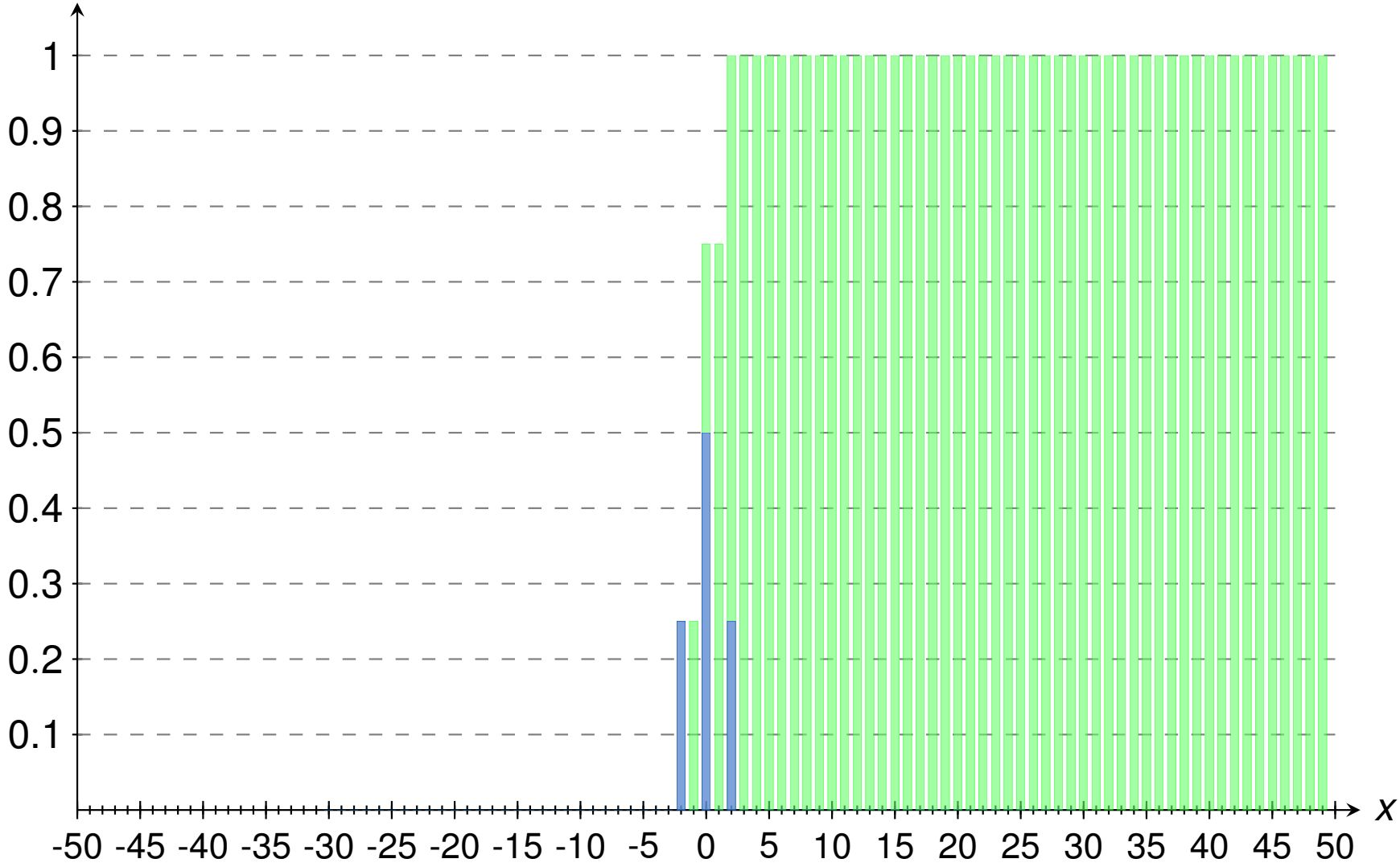

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{3} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

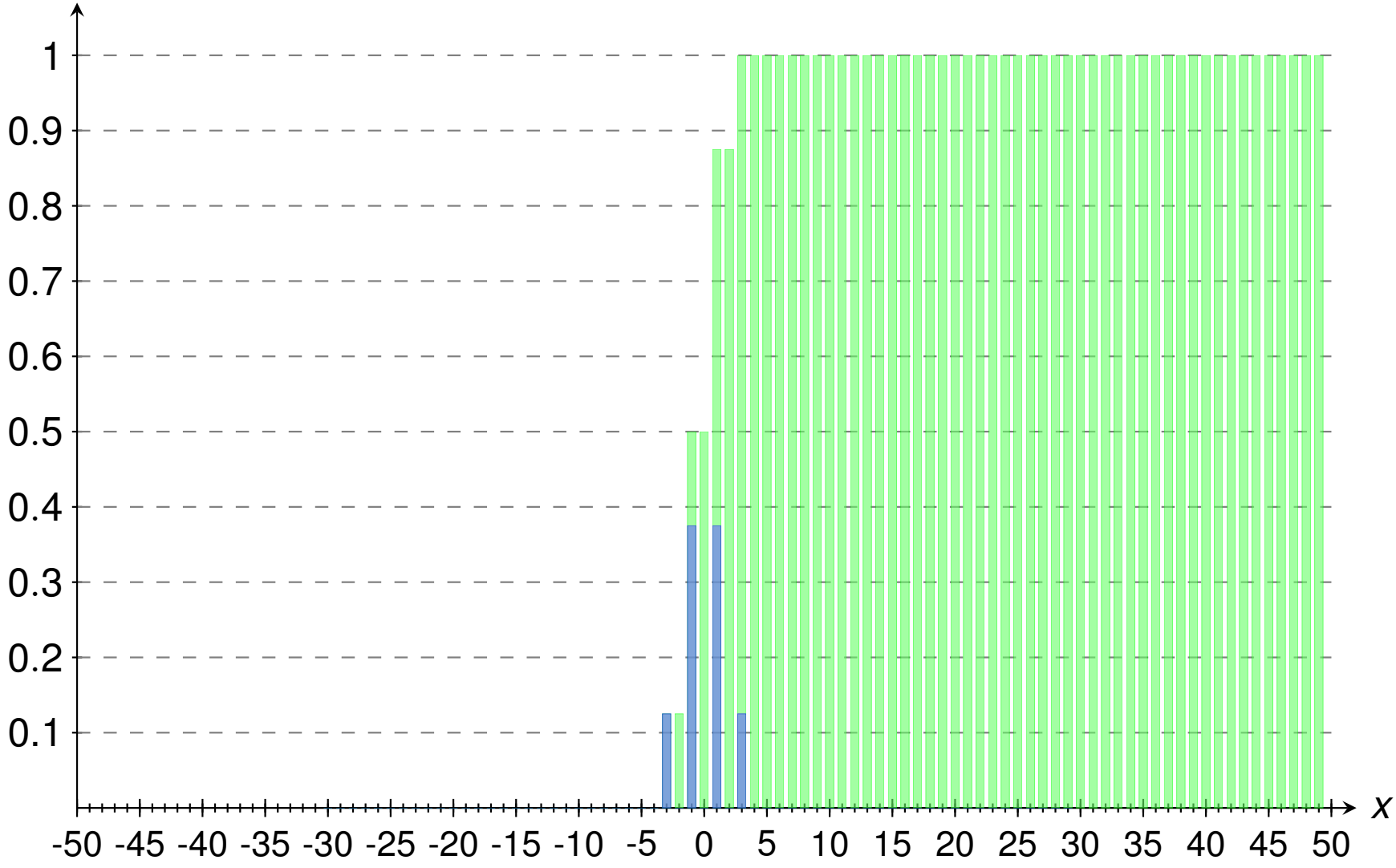

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{4} X_j \leq x\right]$

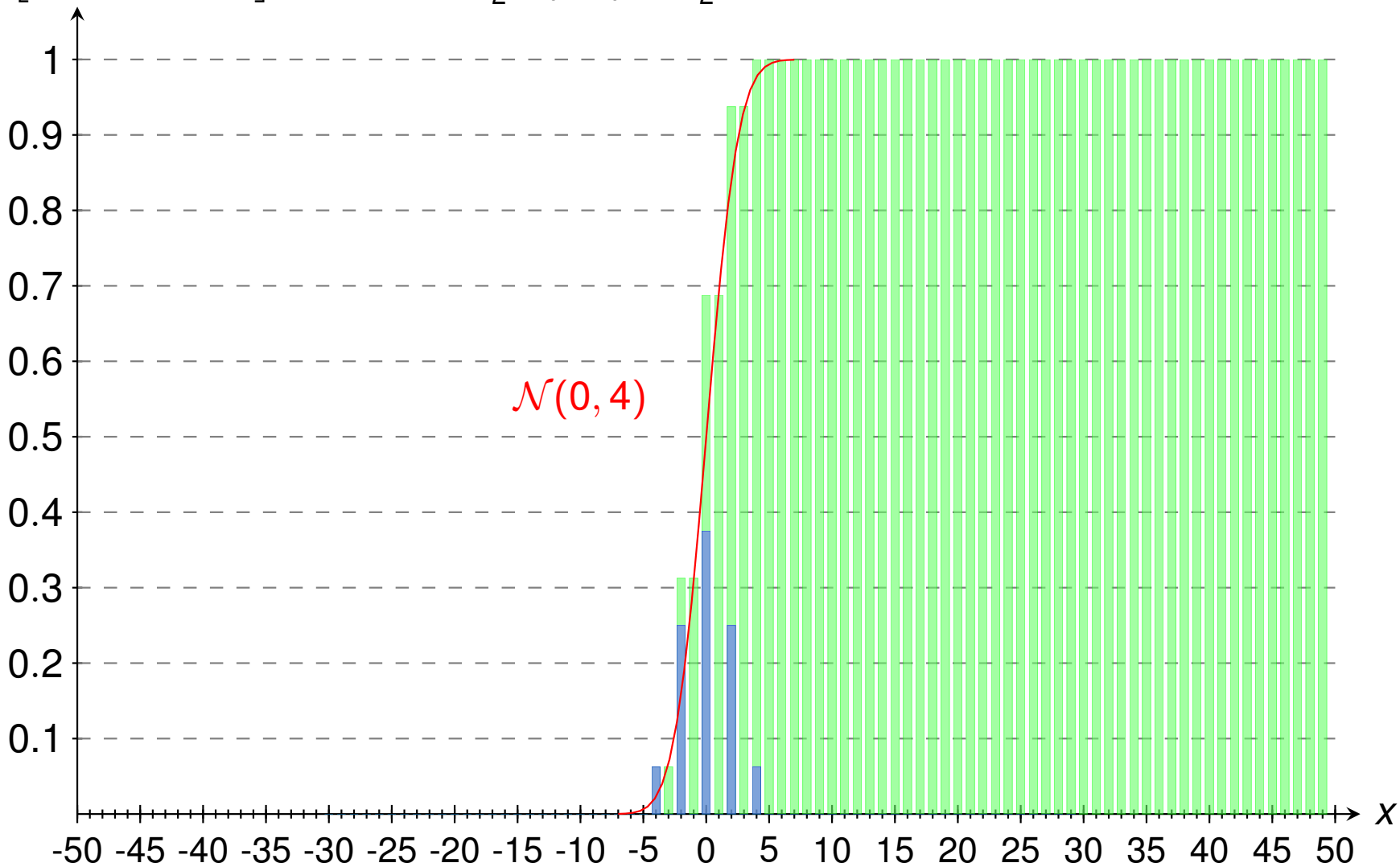

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{5} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

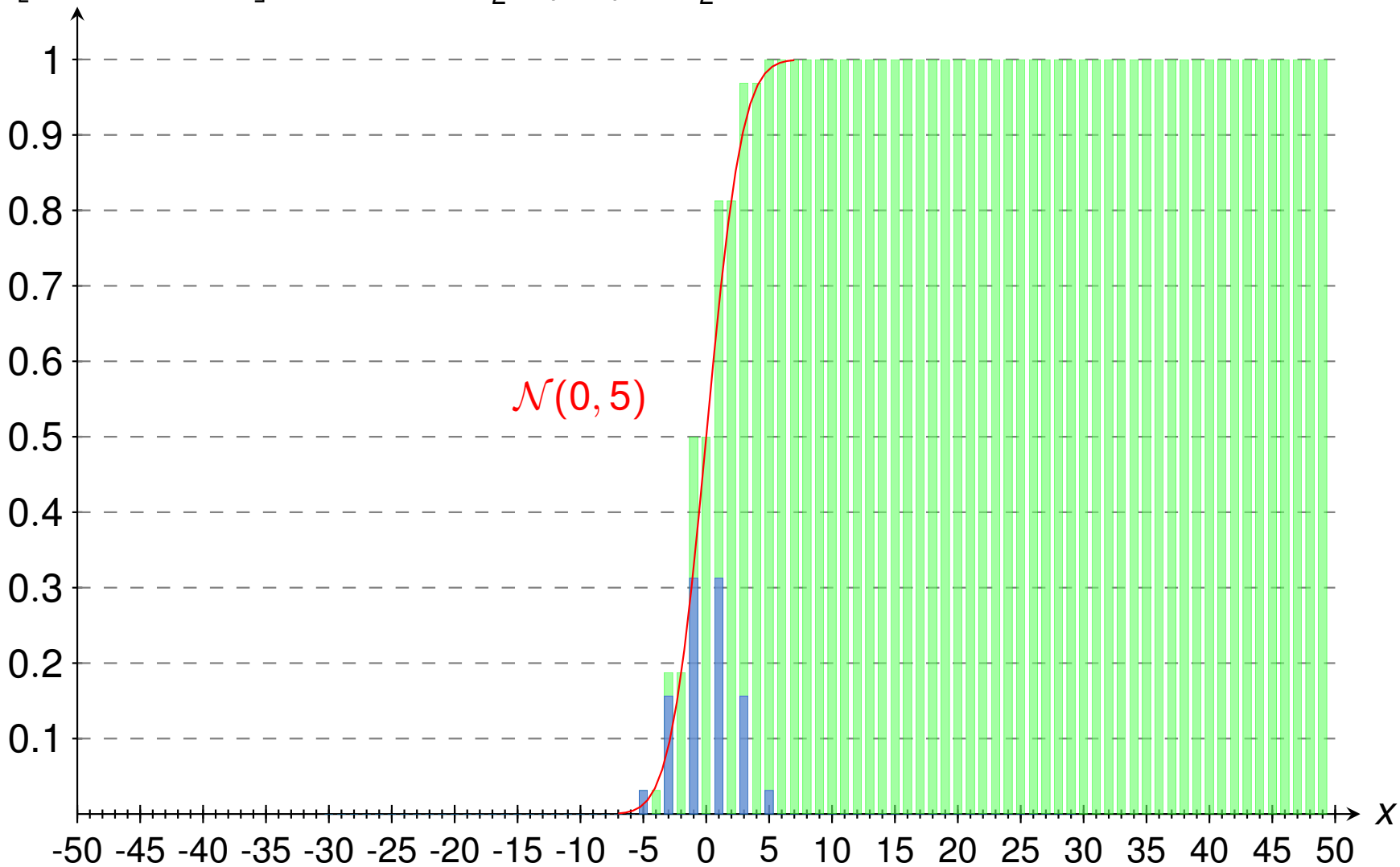

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{6} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

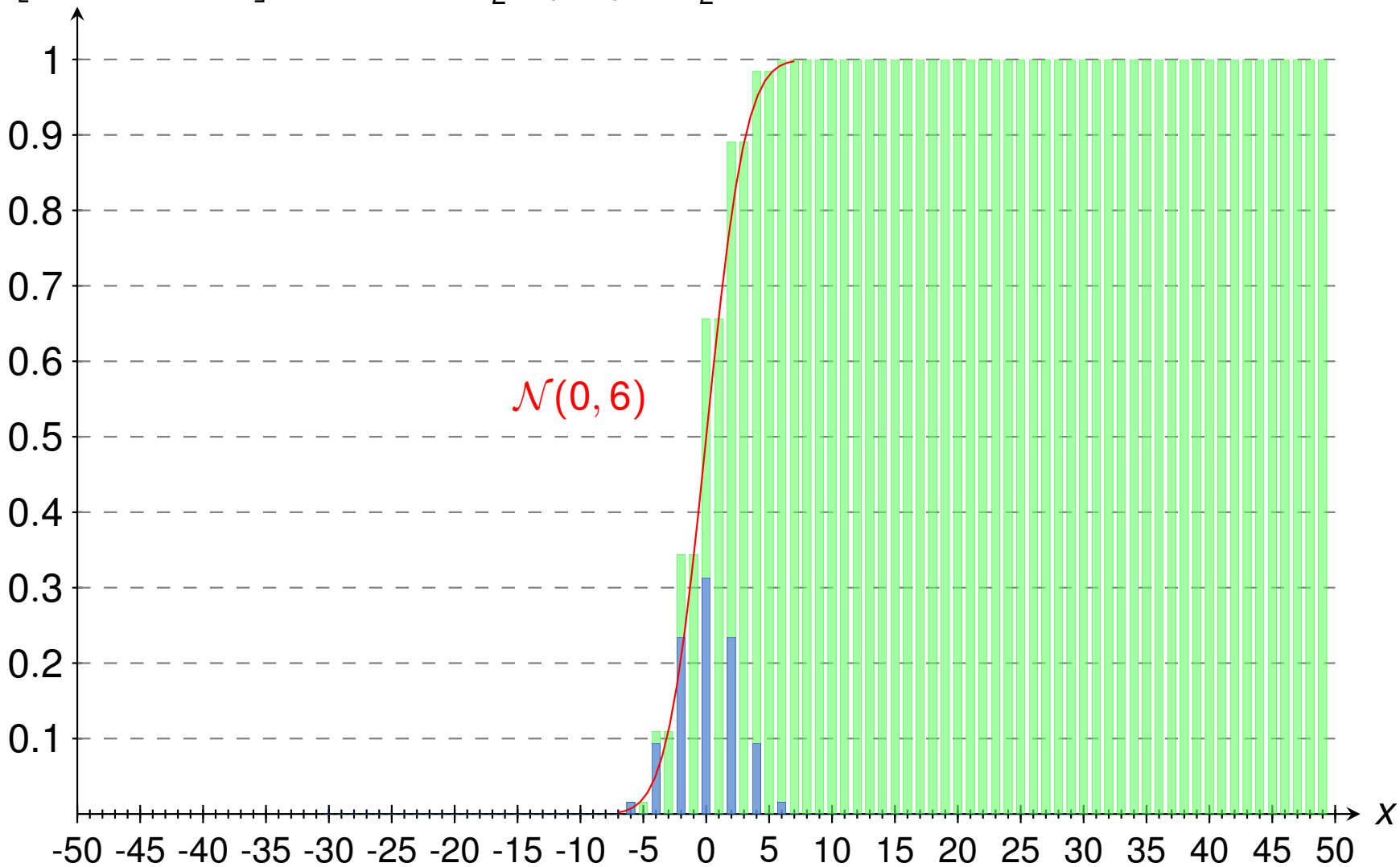

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{7} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

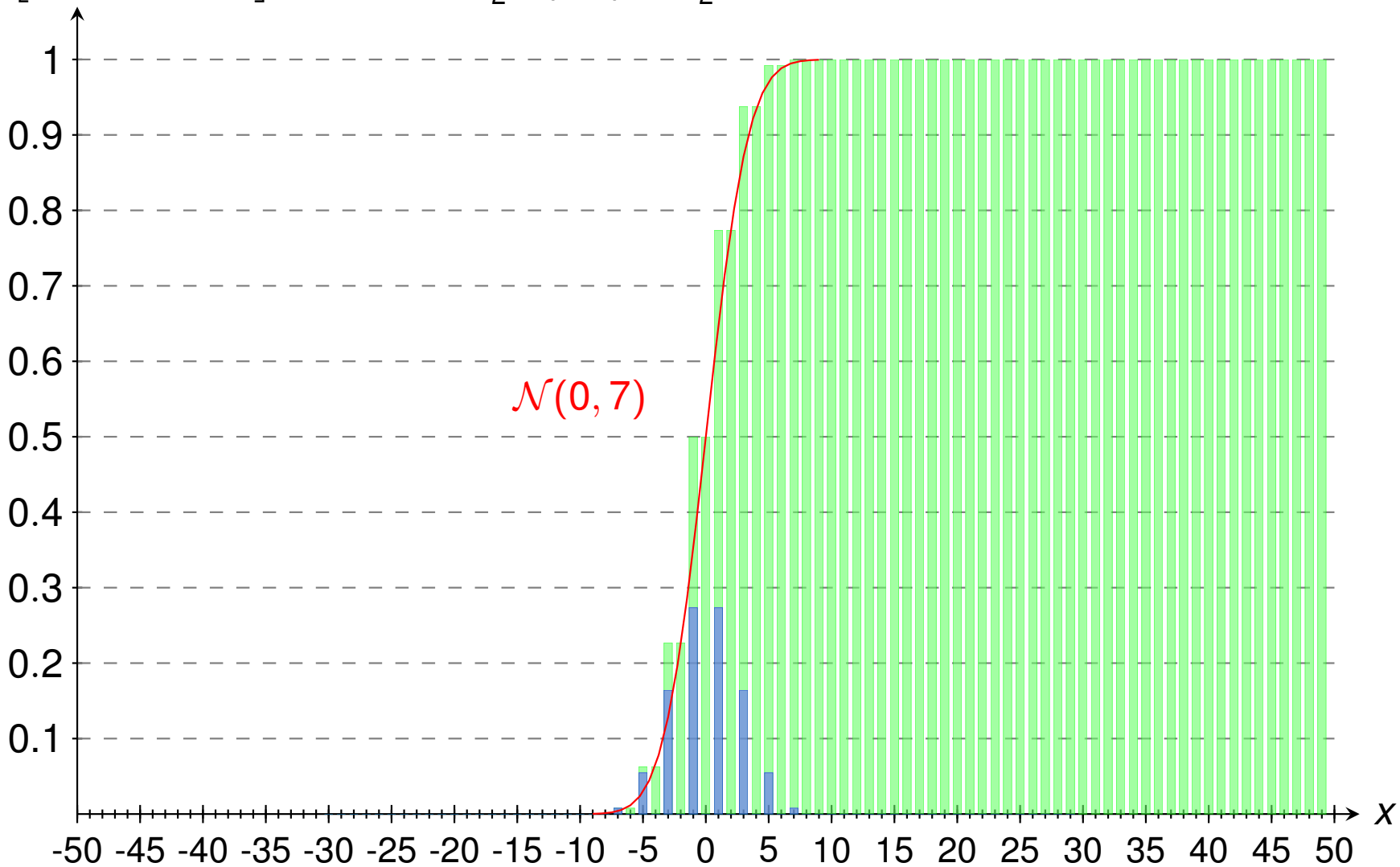

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{8} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

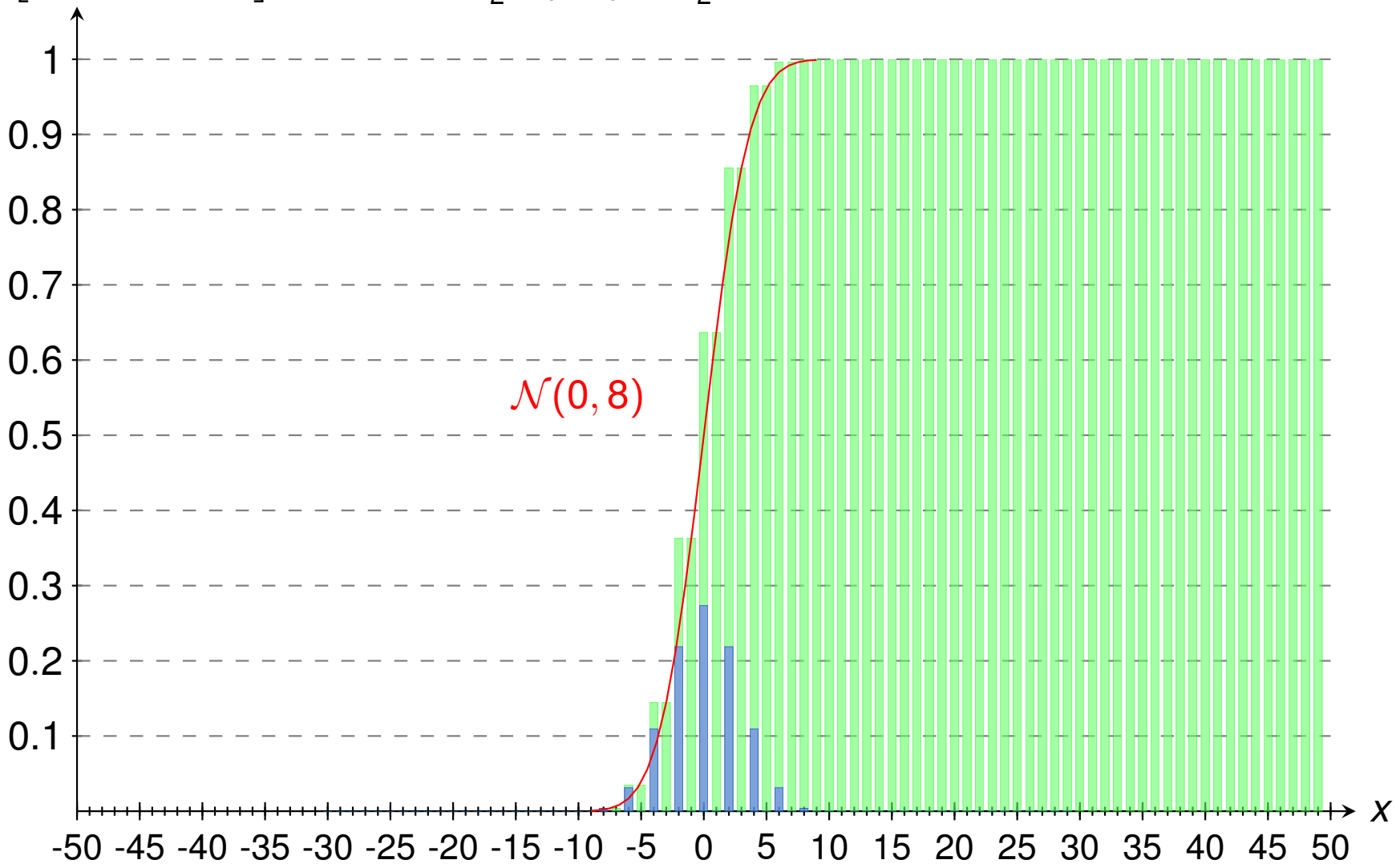

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{9} X_j \leq x\right]$

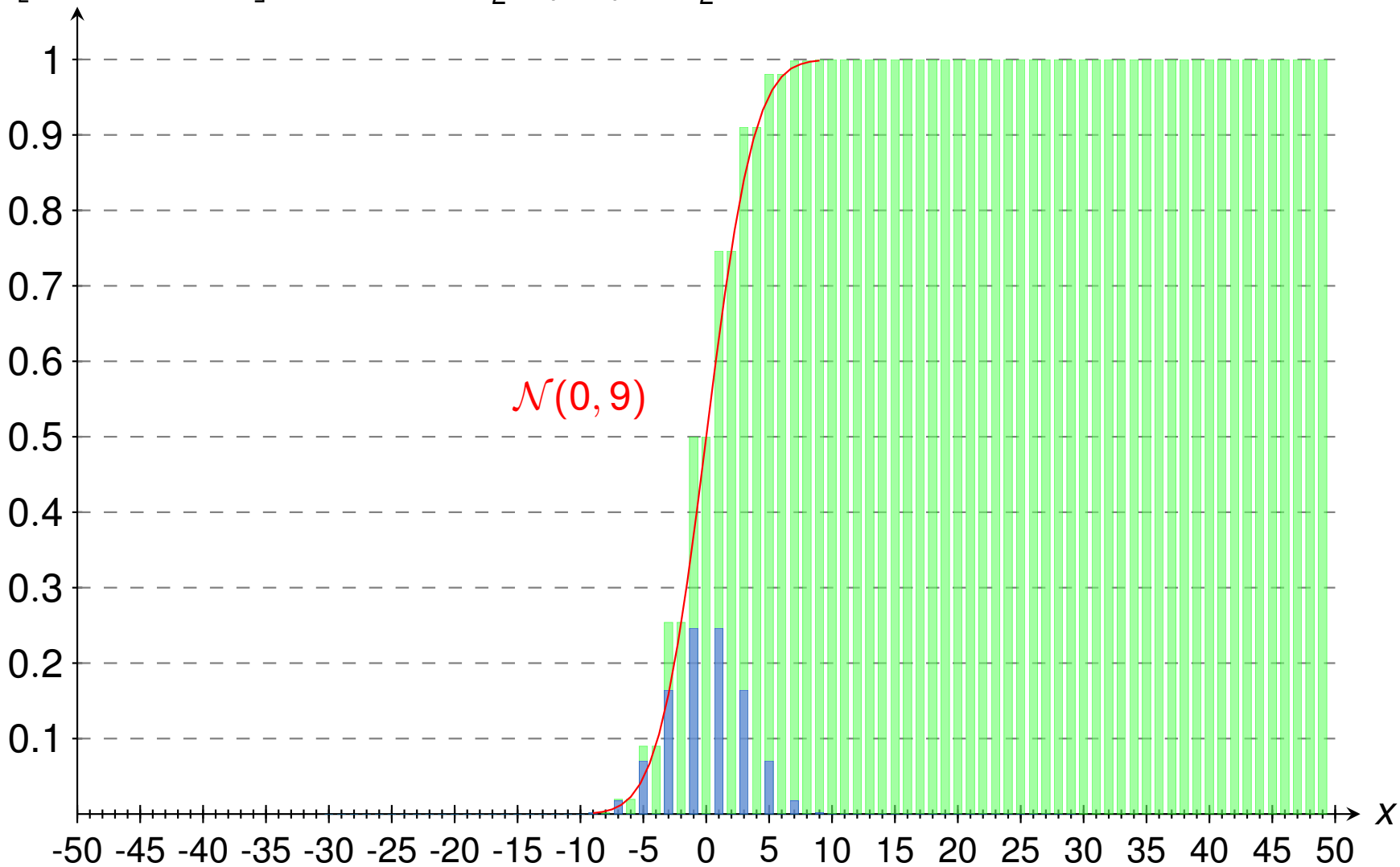

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{10} X_j \leq x\right]$

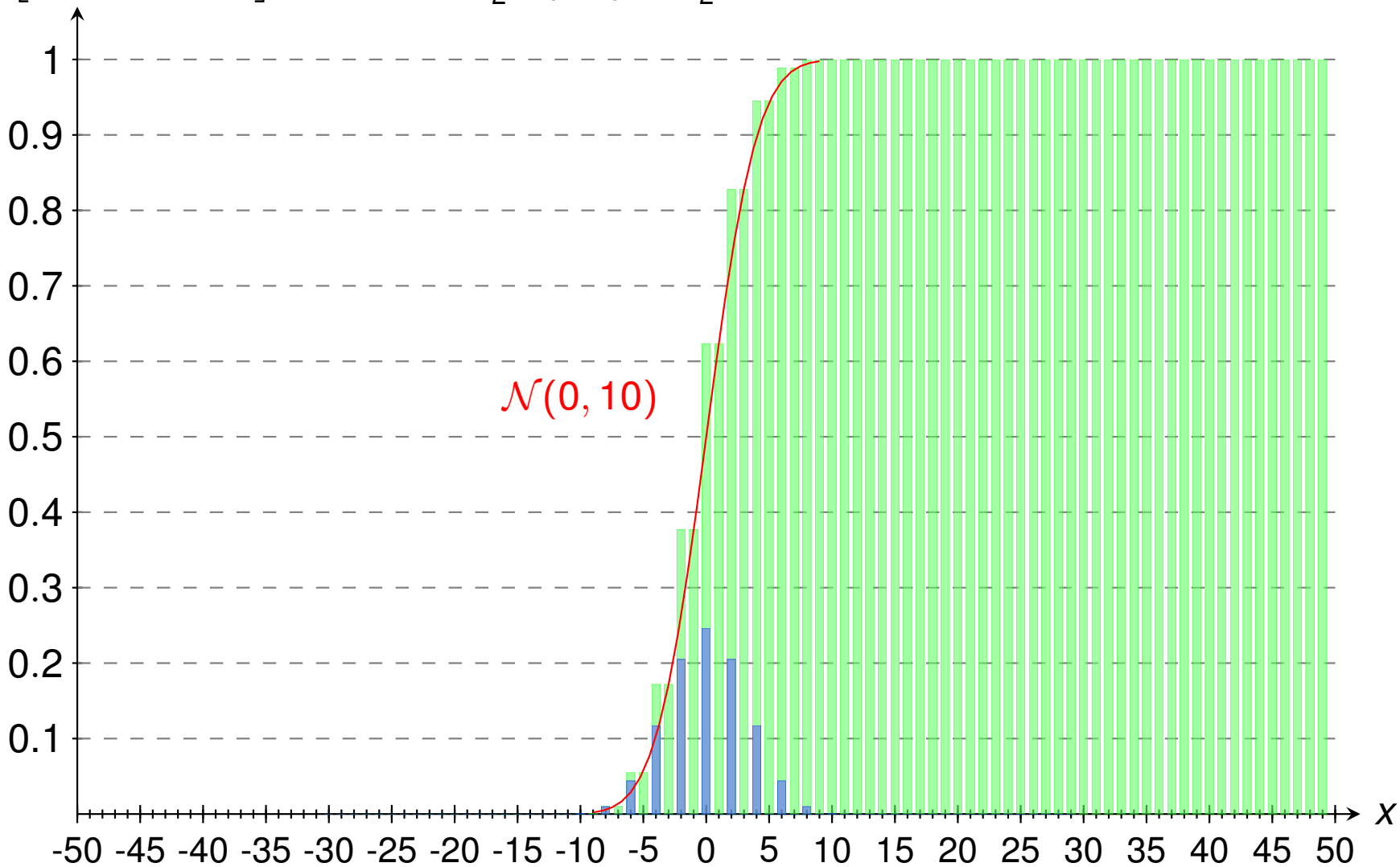

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{11} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

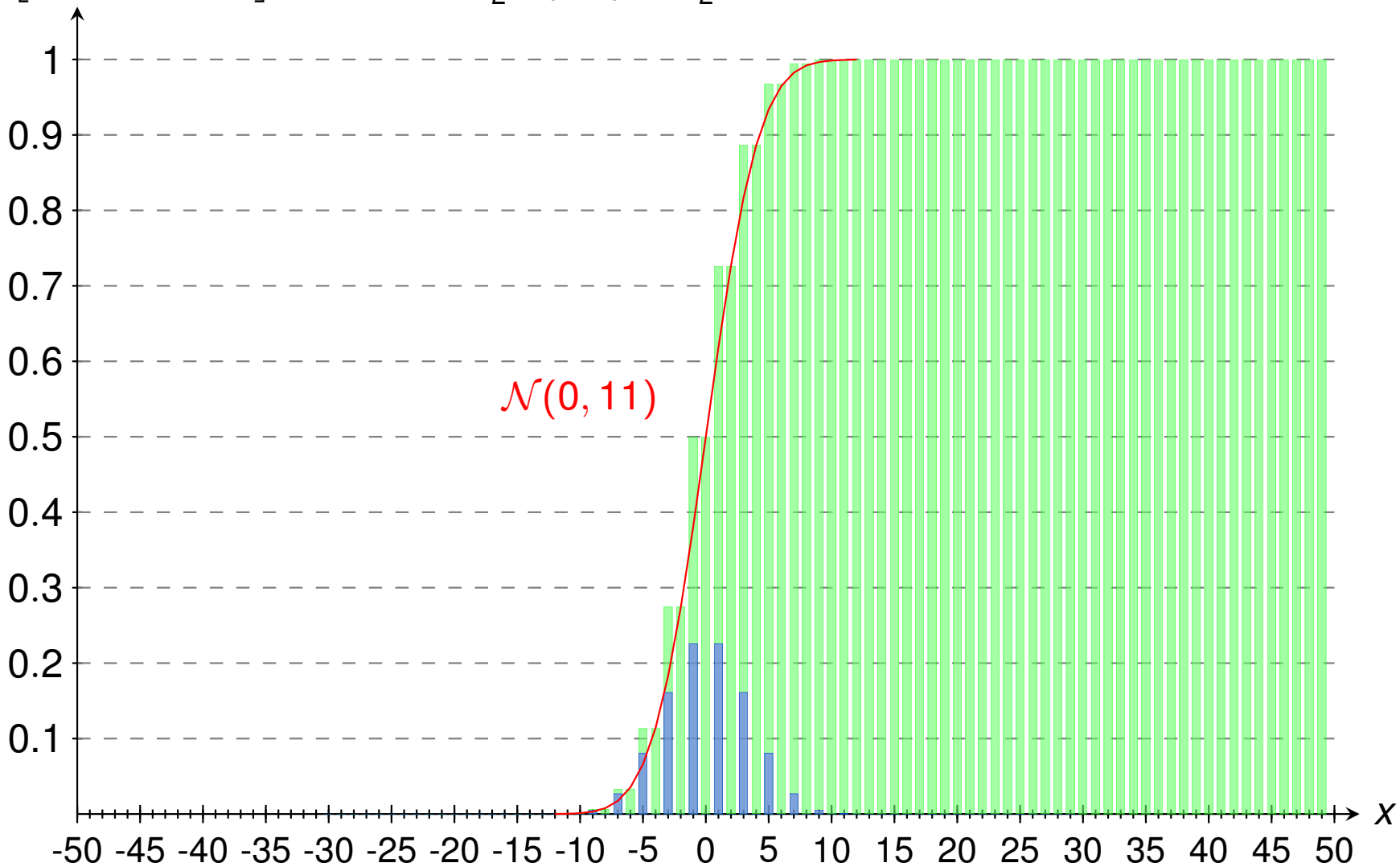

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{12} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

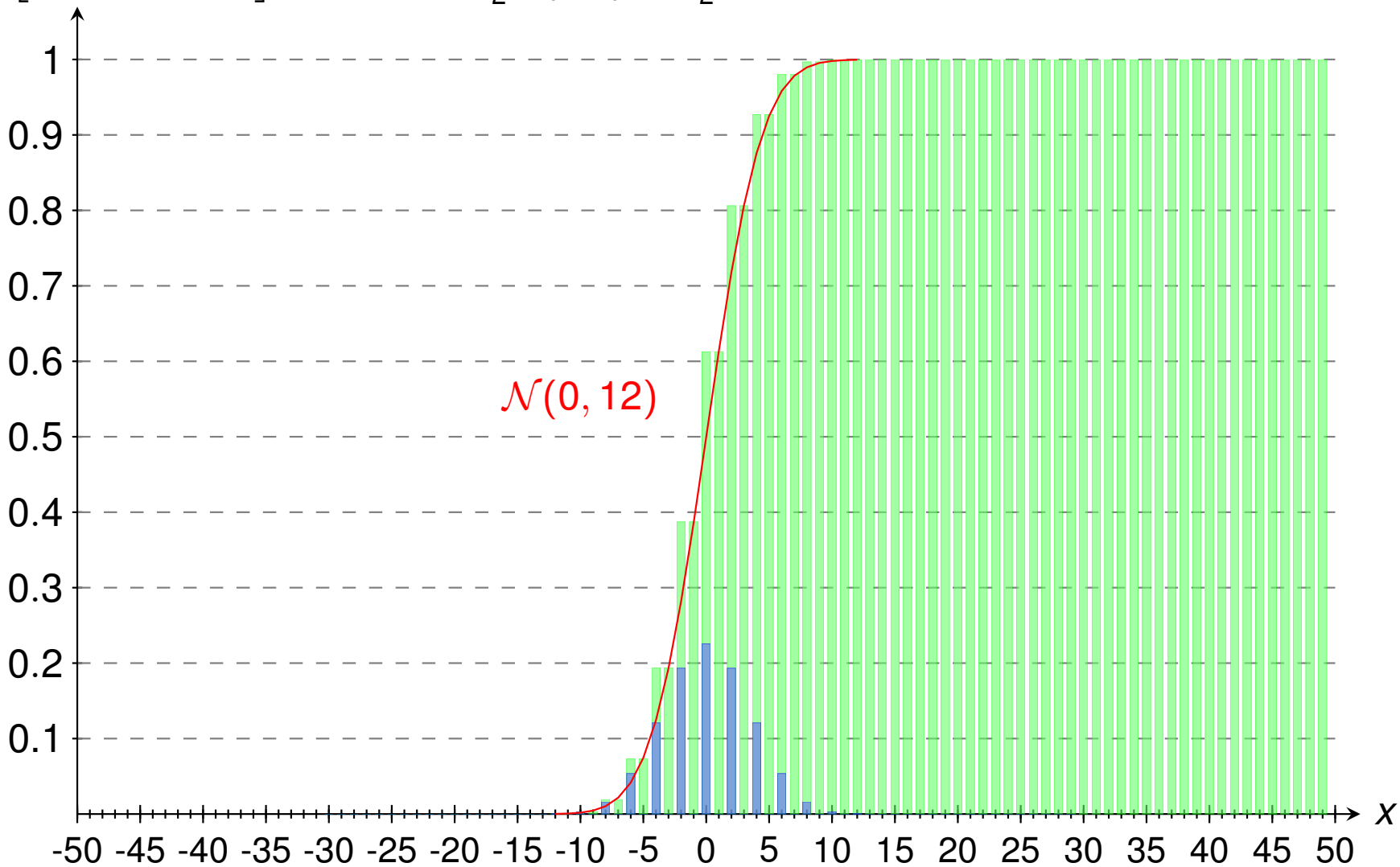

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{13} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

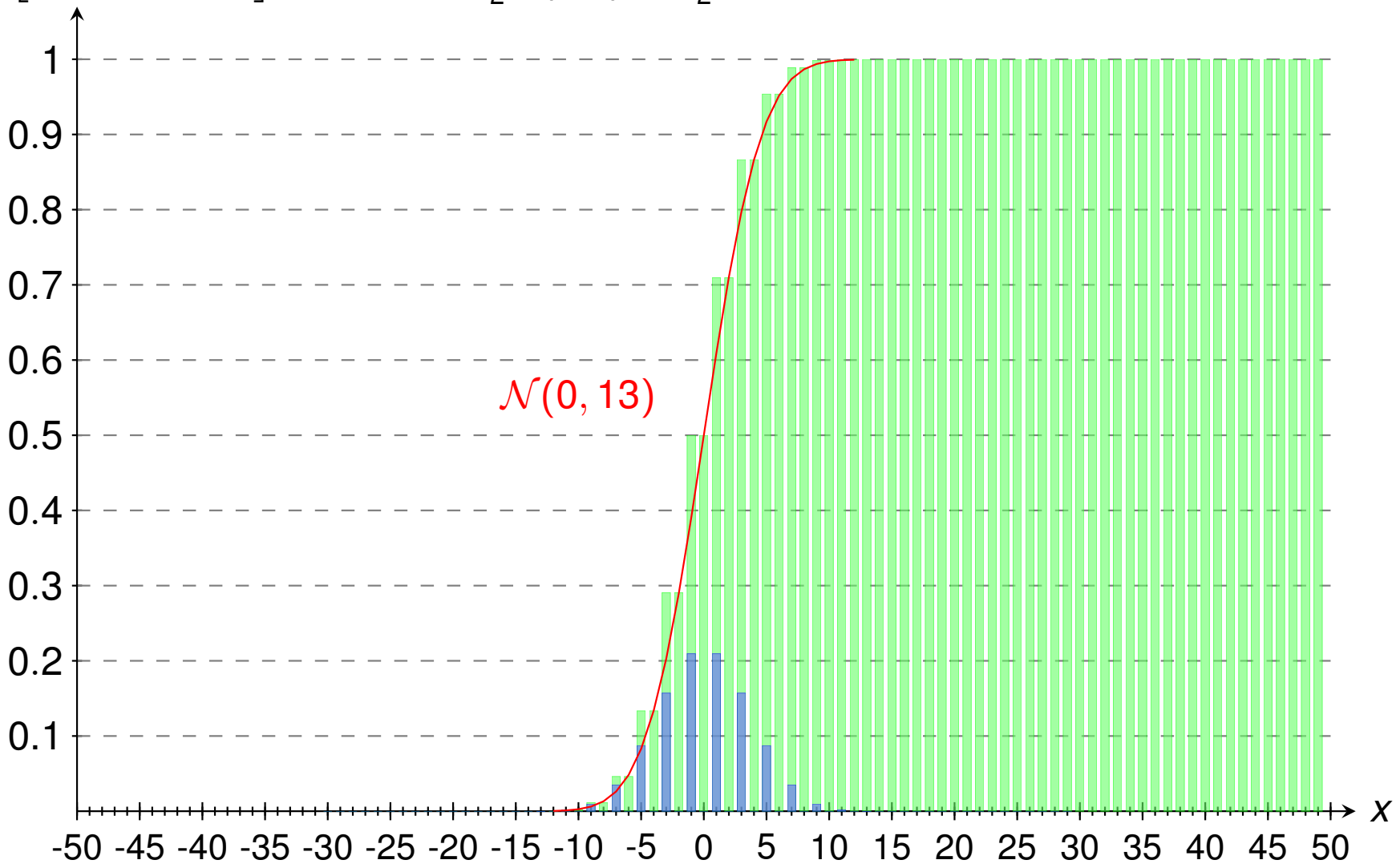

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{14} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

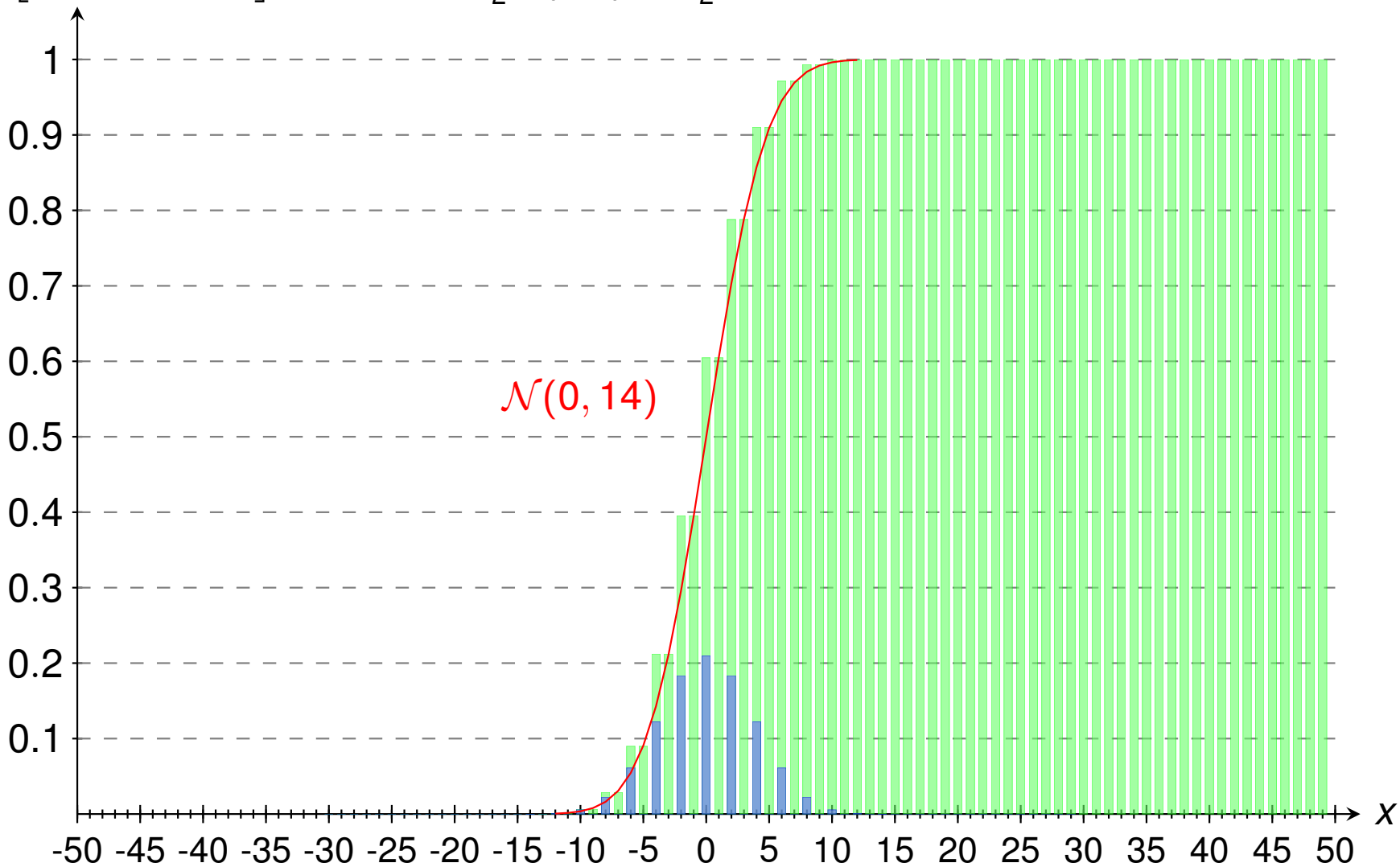

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{15} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

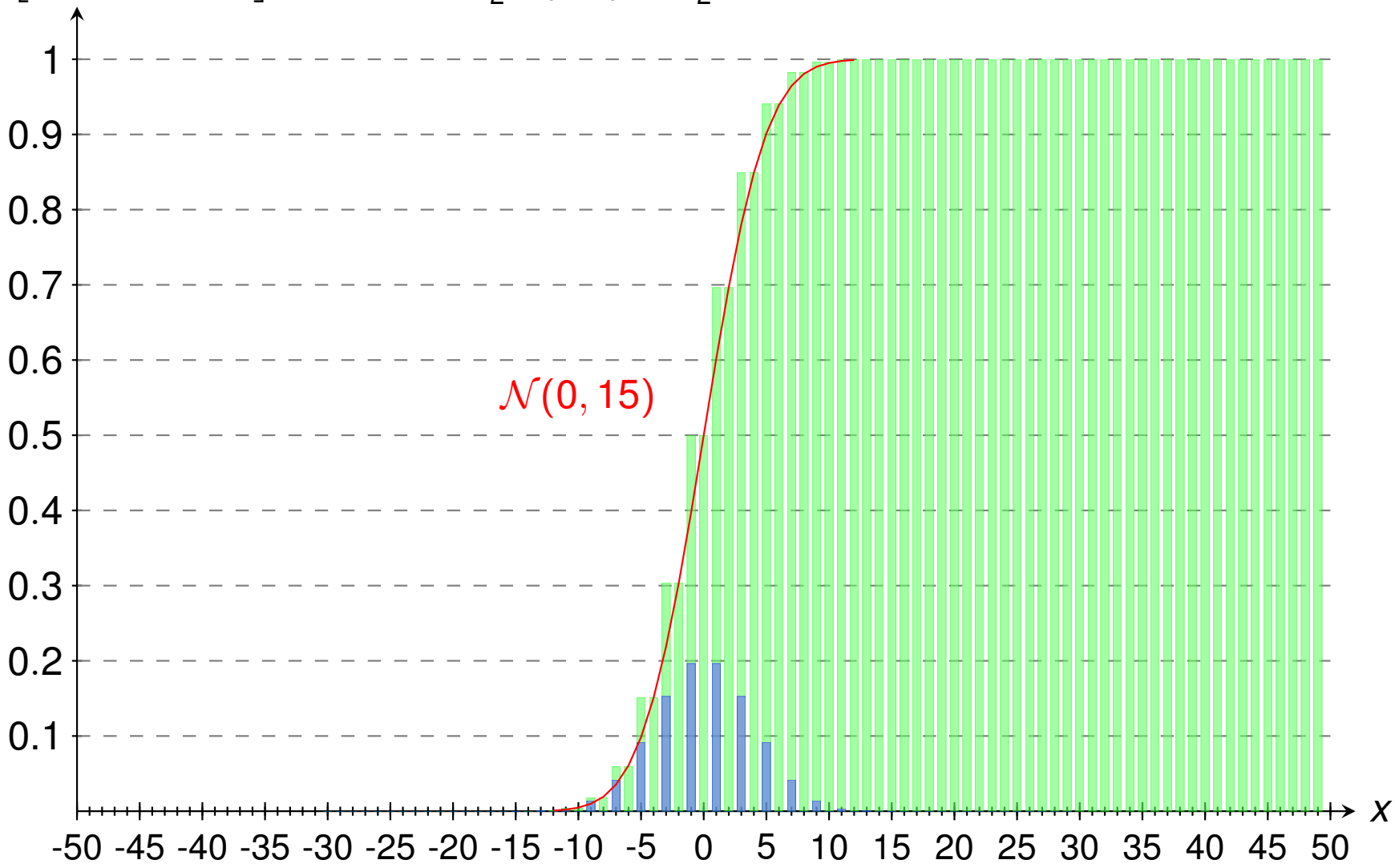

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{16} X_j \leq x\right]$

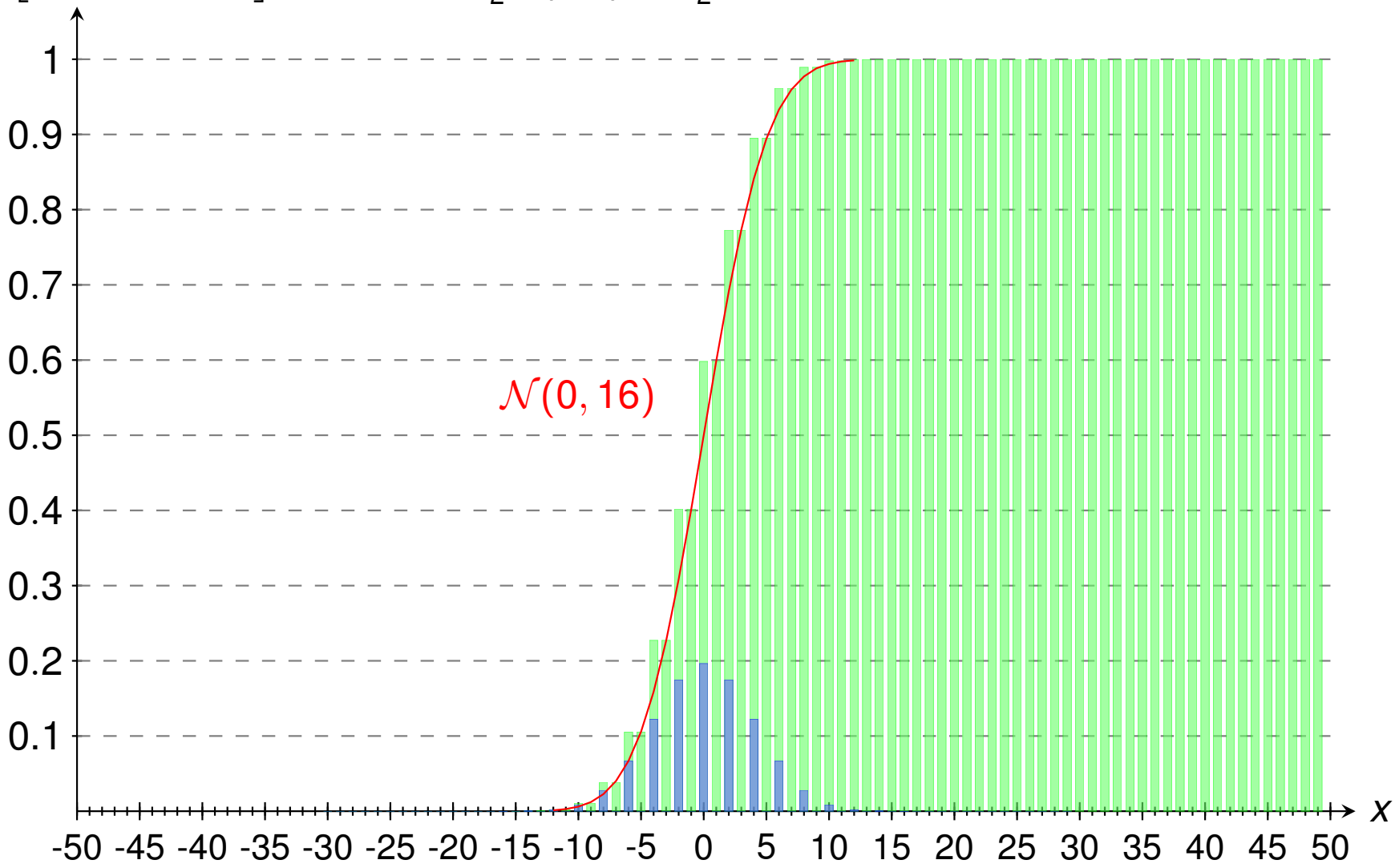

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{17} X_j \leq x\right]$

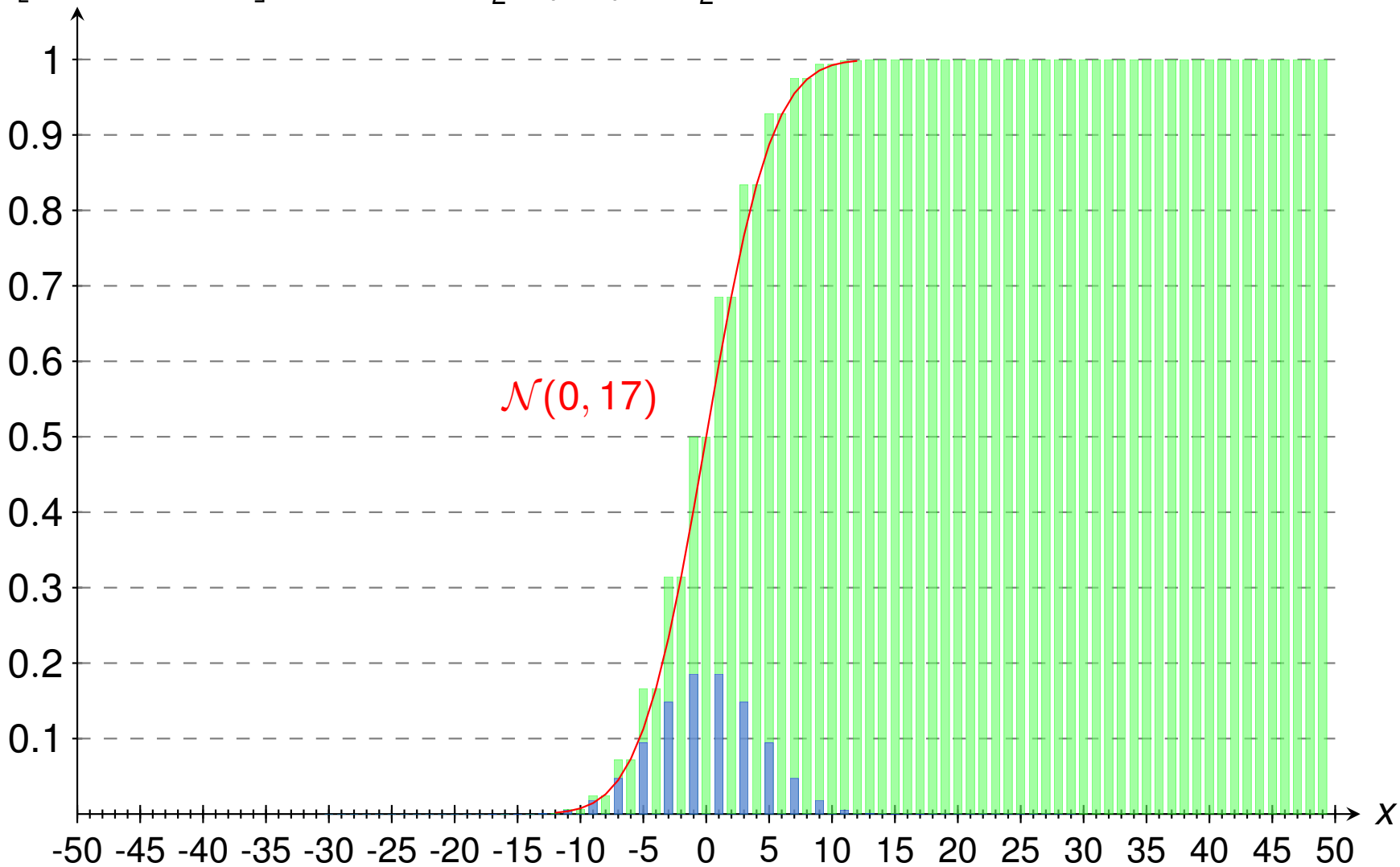

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{18} X_j \leq x\right]$

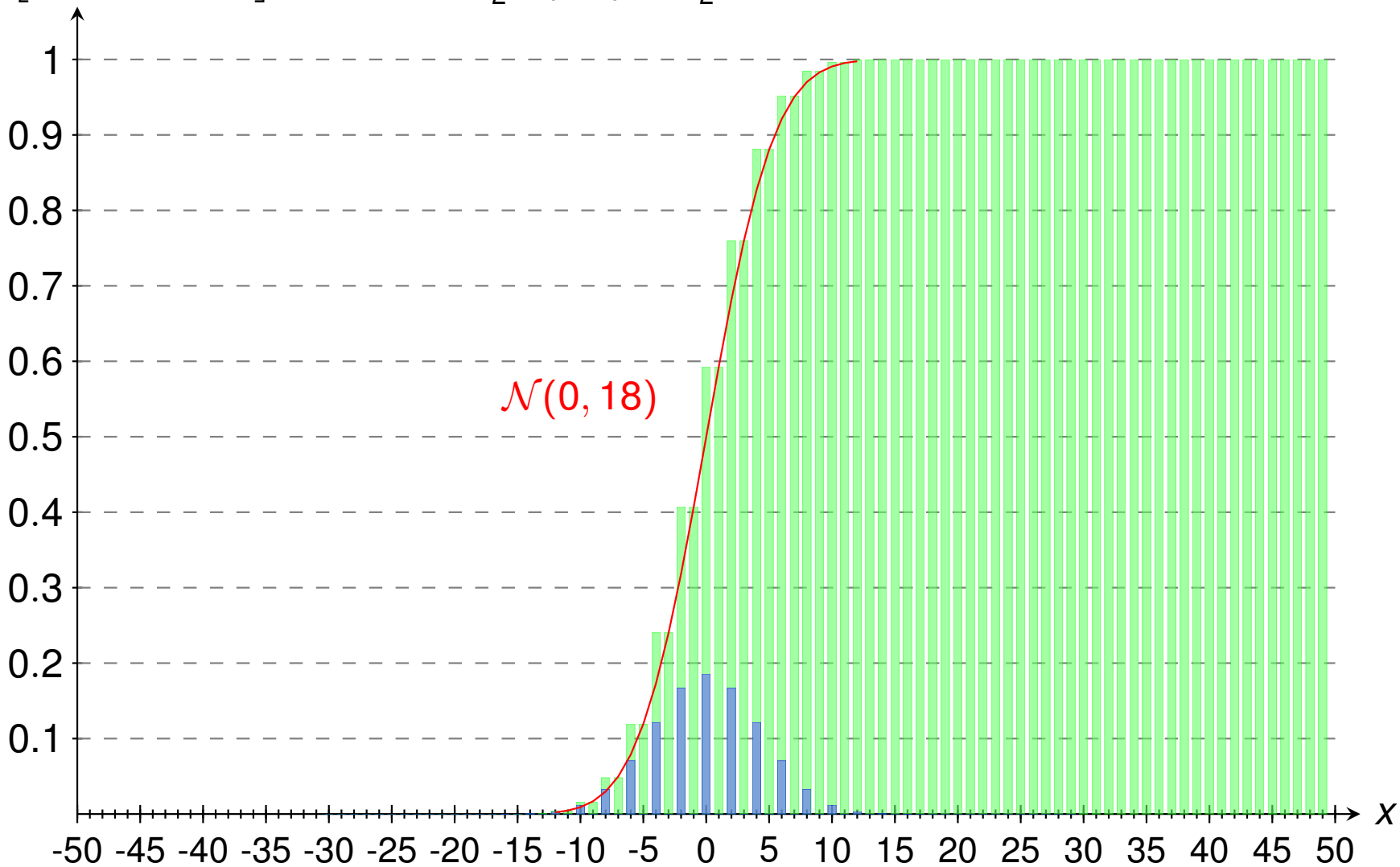

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{19} X_j \leq x\right]$

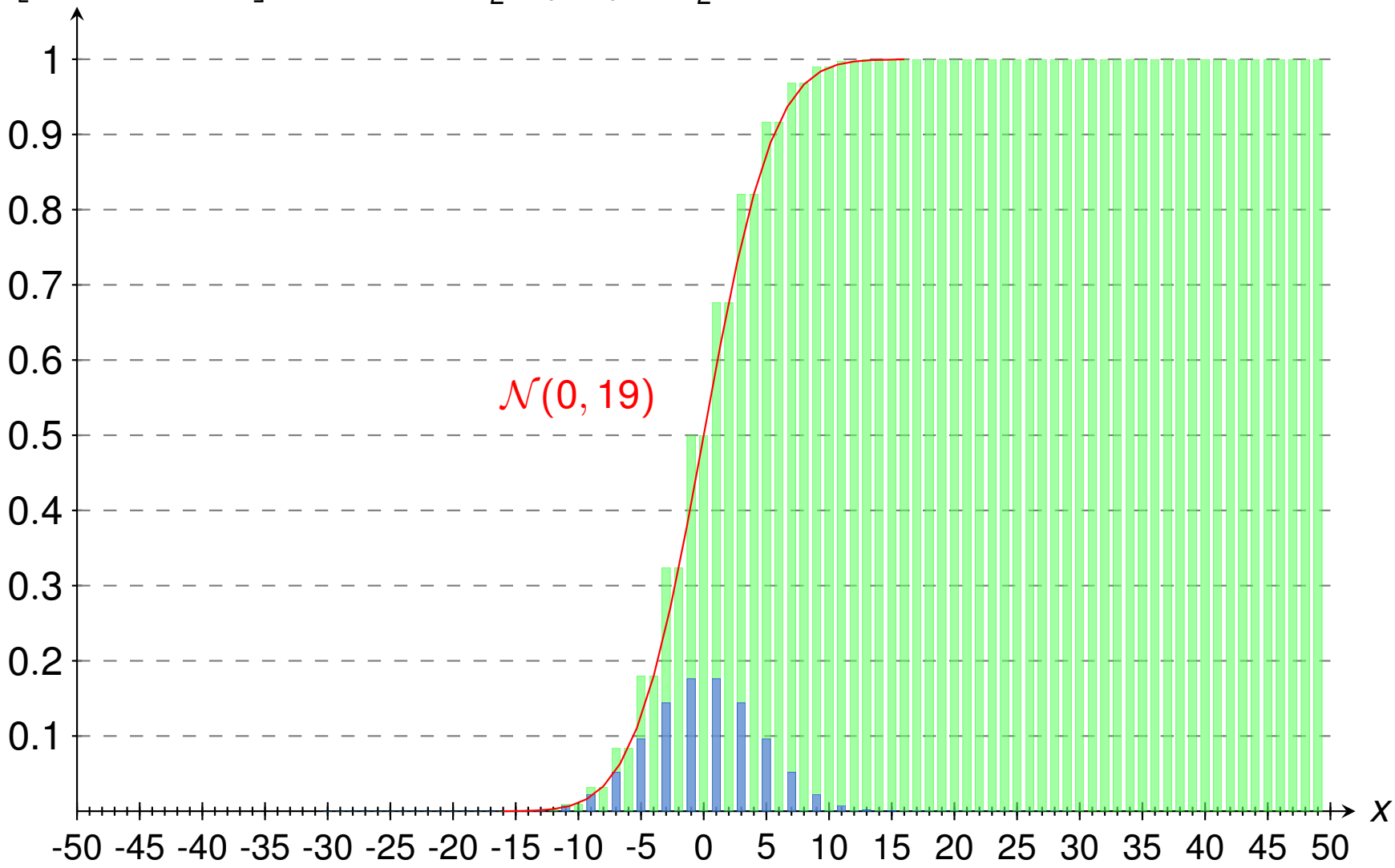

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{20} X_j \leq x\right]$

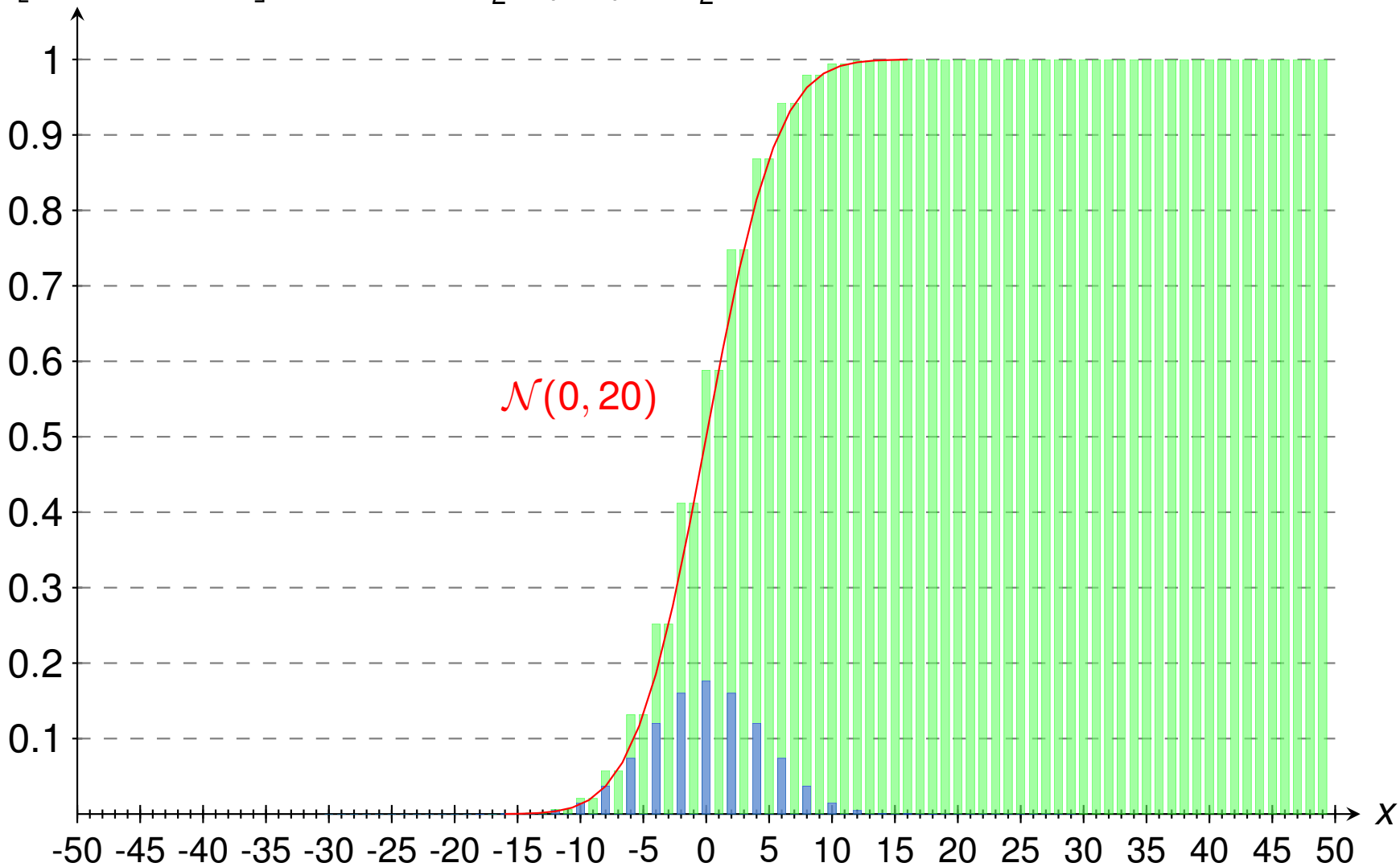

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{21} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

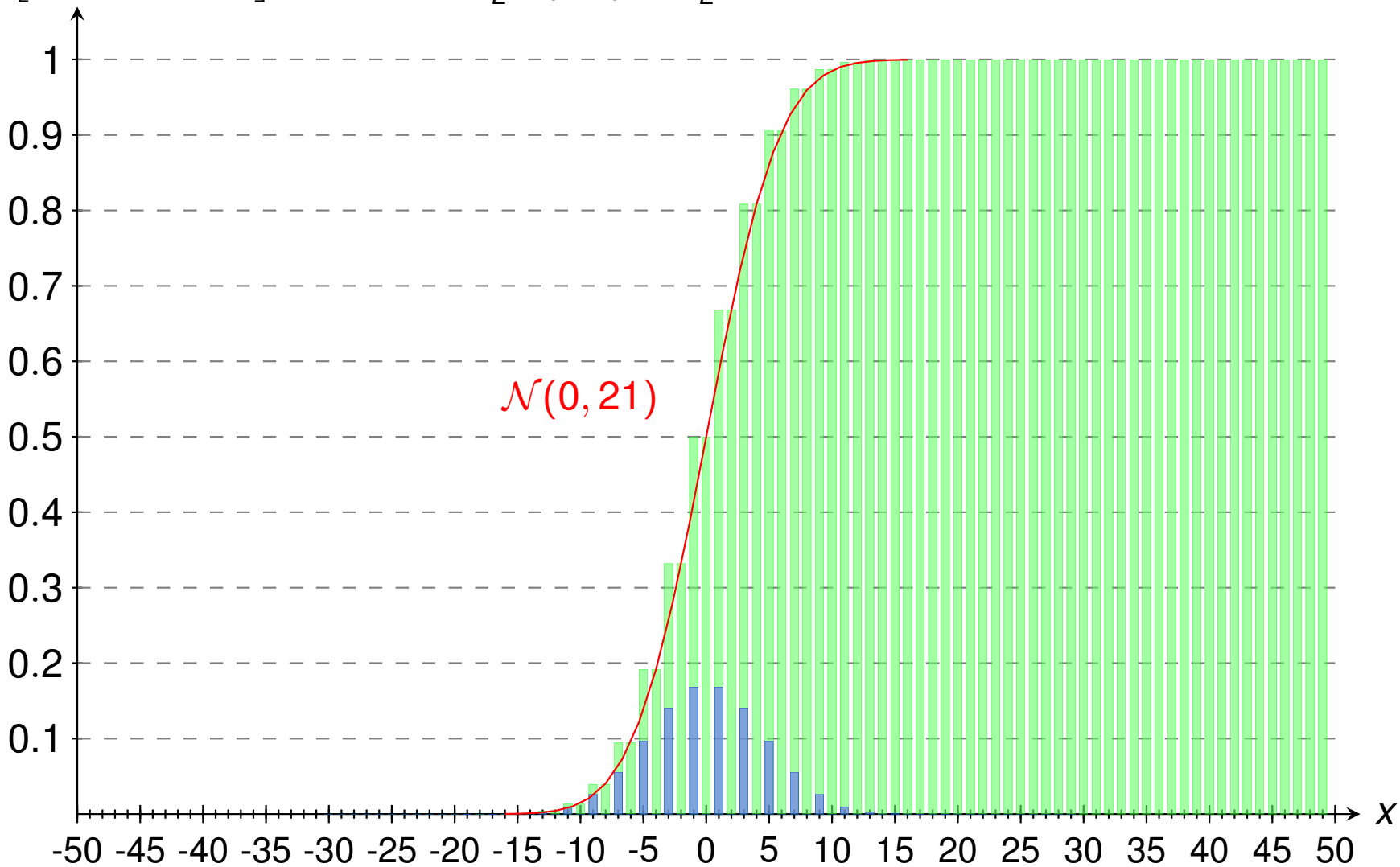

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{22} X_j \leq x\right]$

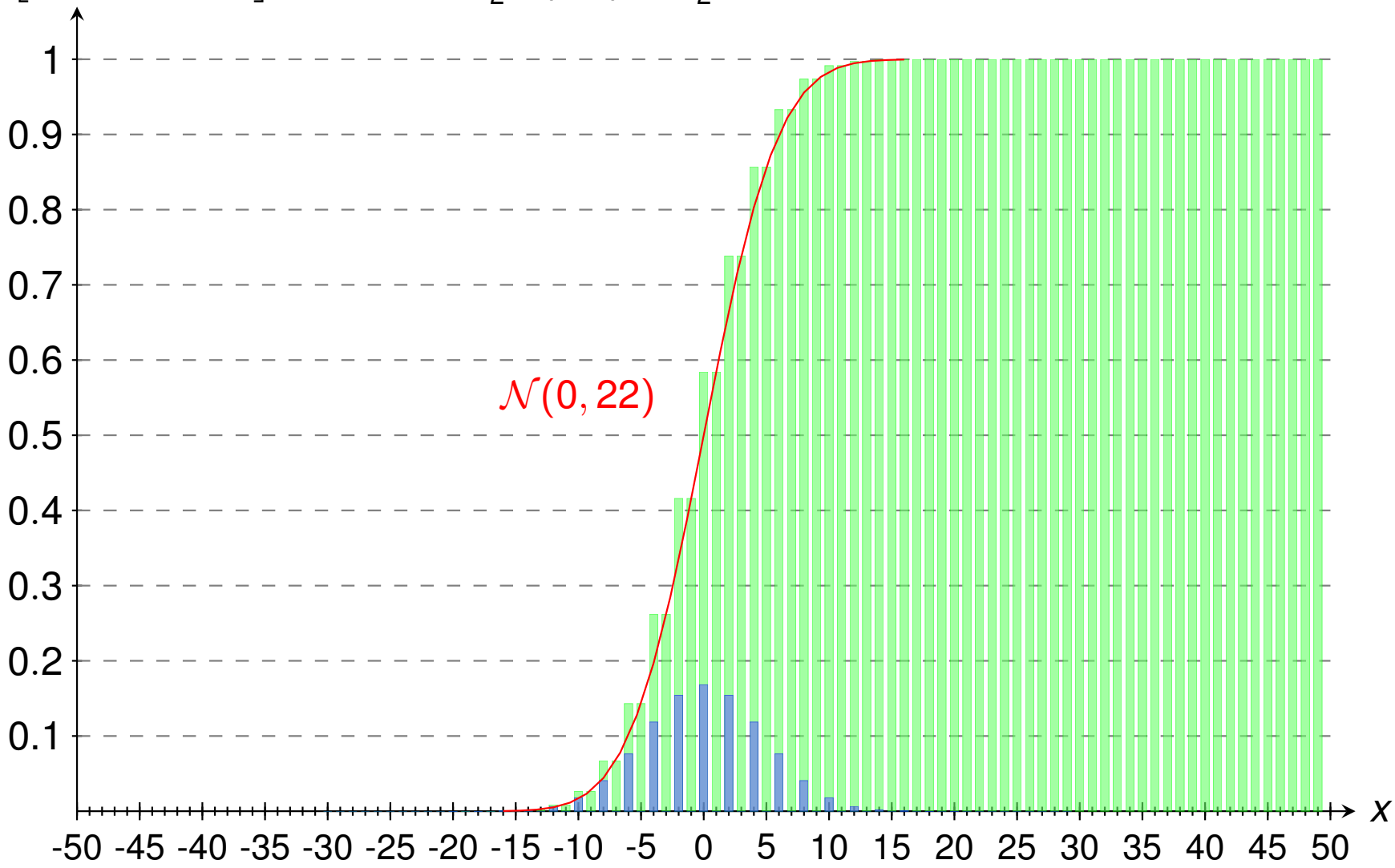

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

$\mathbf{P}\left[\sum_{j=1}^{23} X_j \leq x\right]$

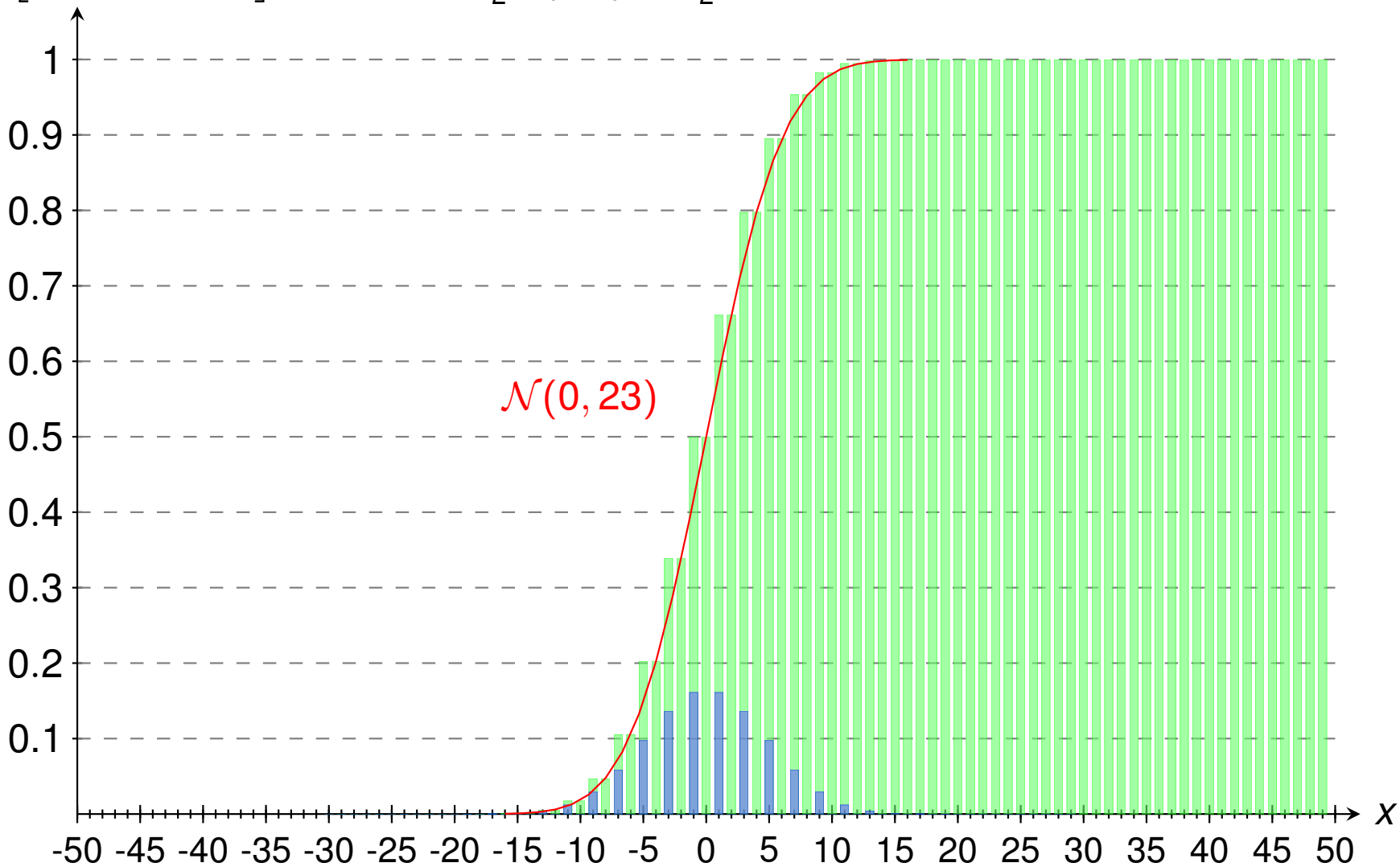

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{24} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$

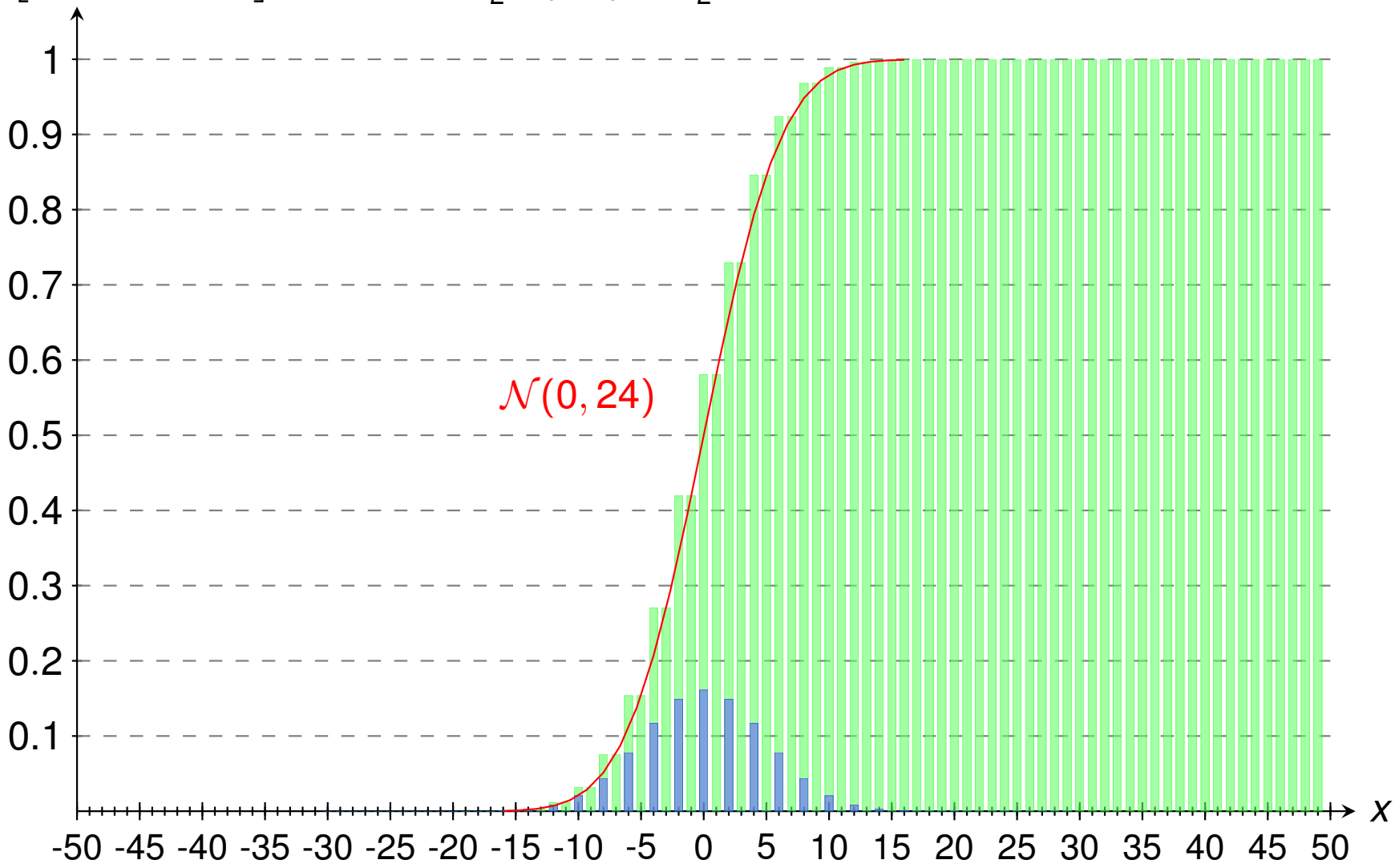

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
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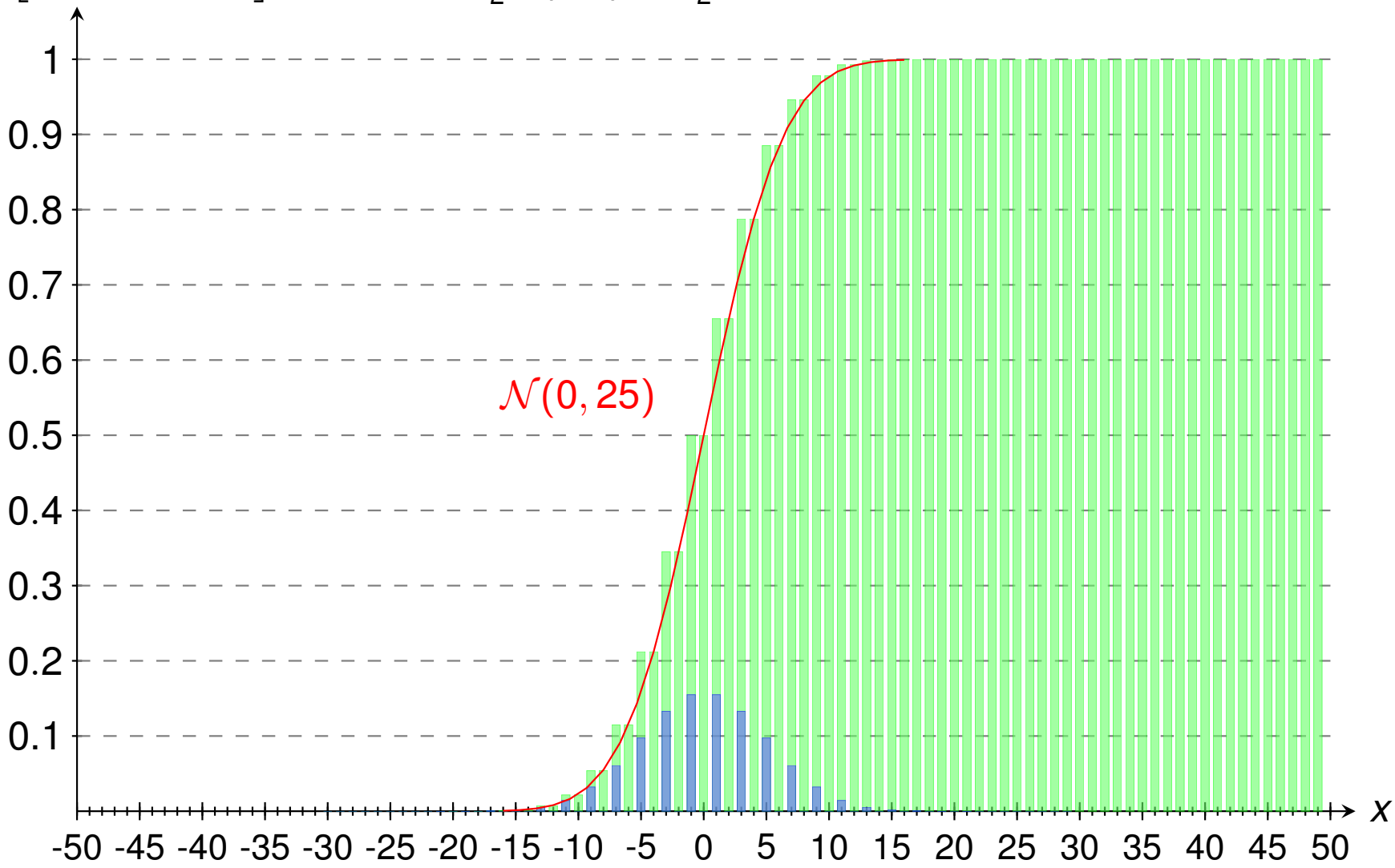
$\mathbf{P}\left[\sum_{j=1}^{25} X_j \leq x\right]$



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$\mathbf{P}\left[\sum_{j=1}^{26} X_j \leq x\right]$

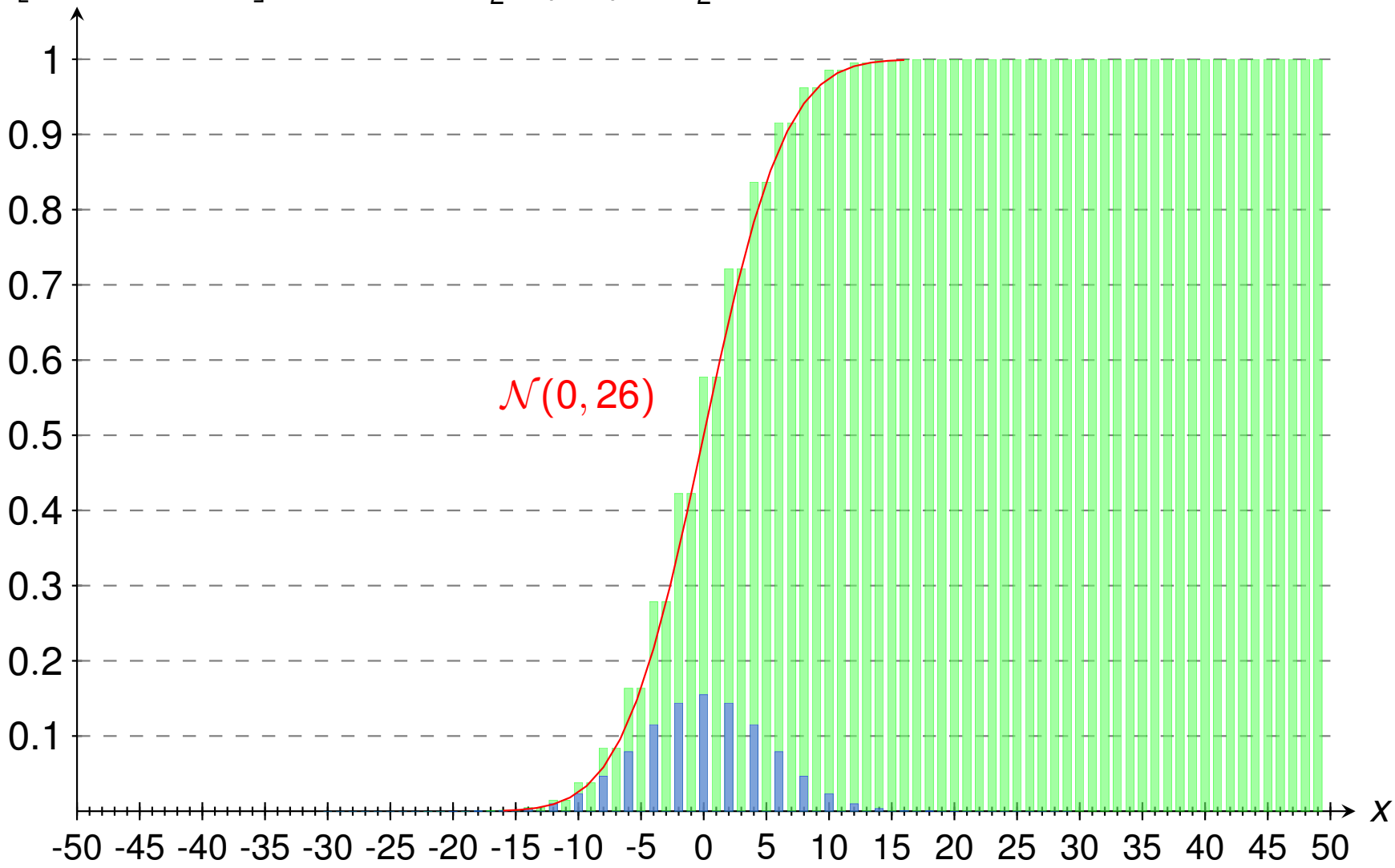

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{27} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
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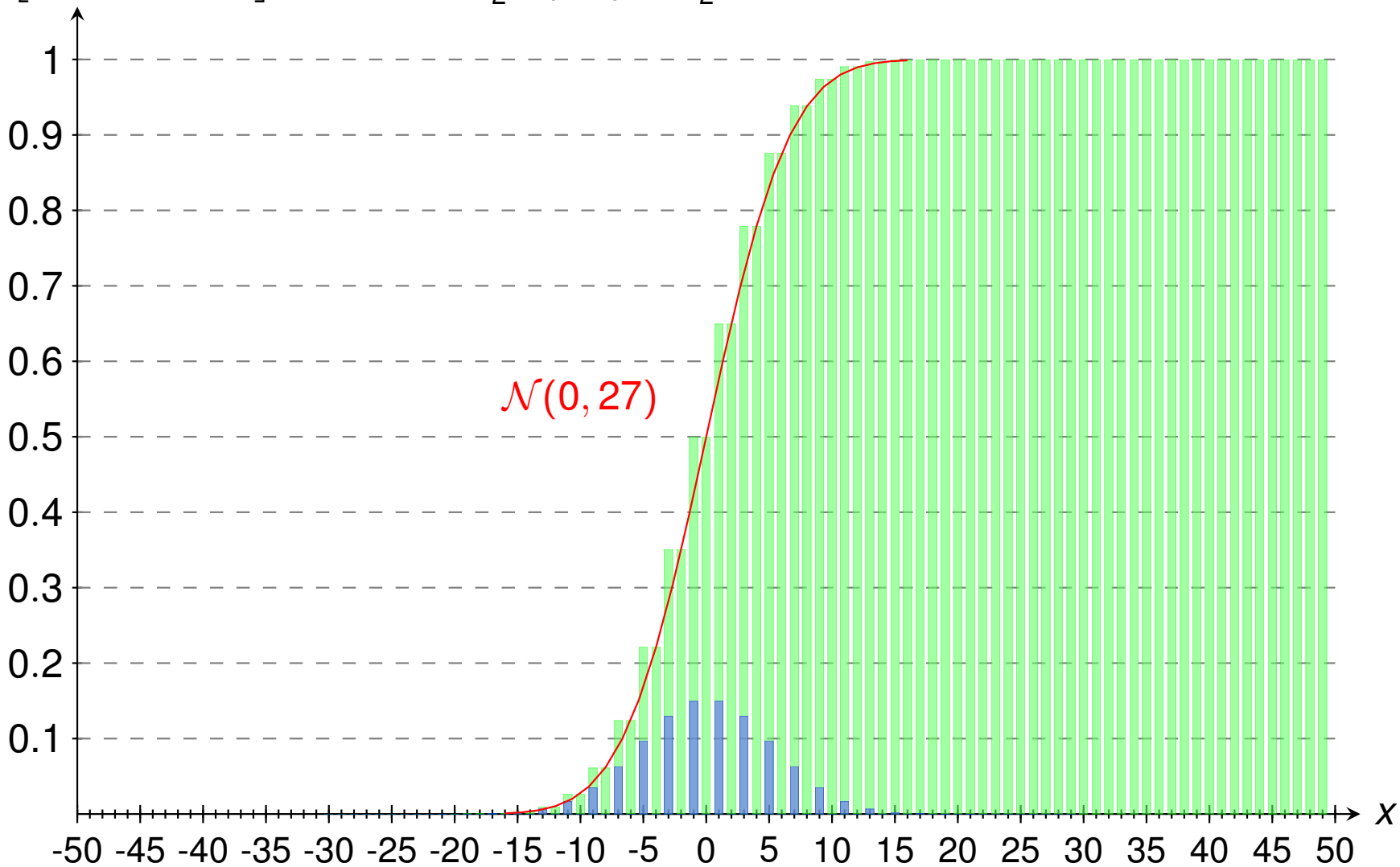

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{28} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
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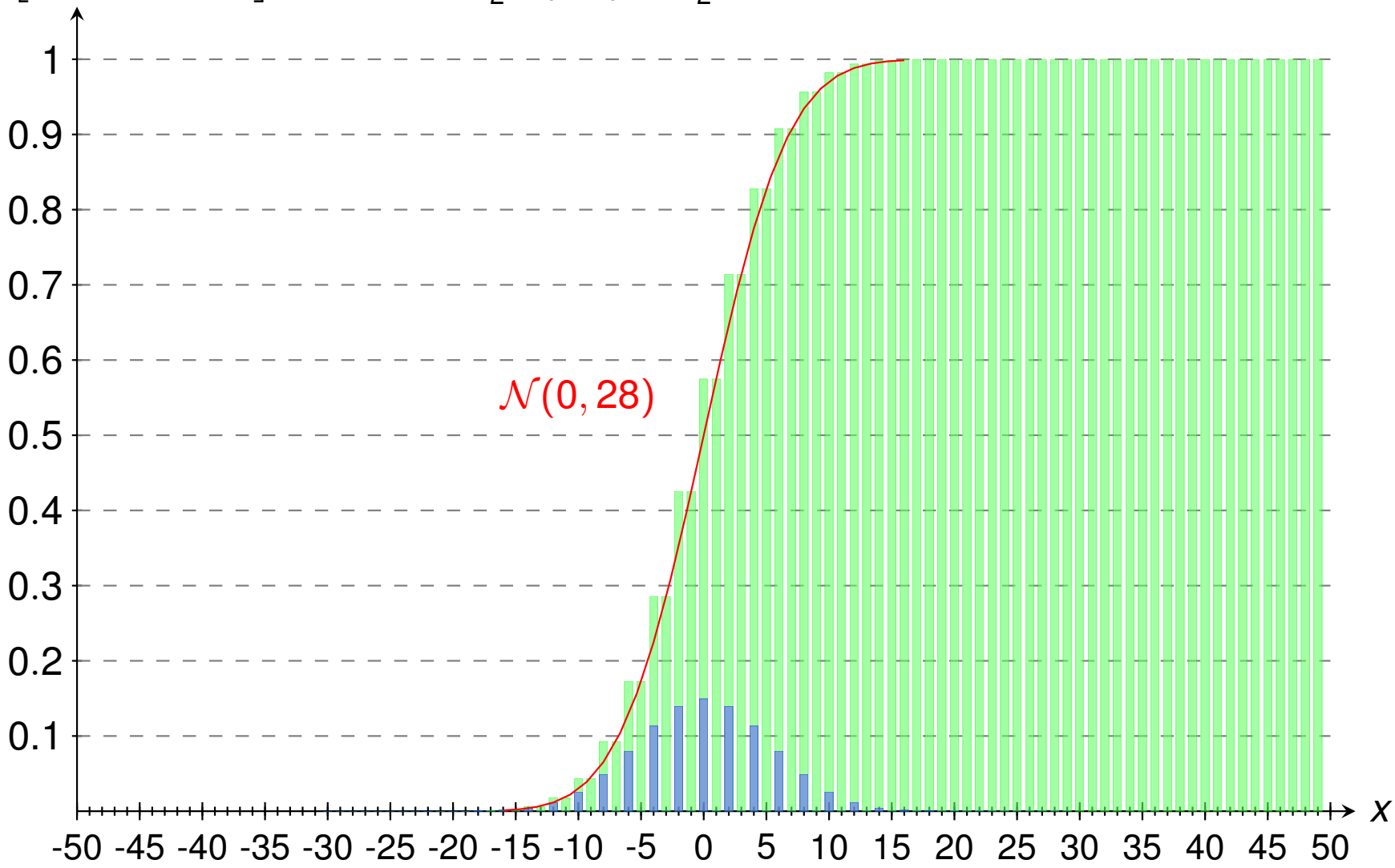

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{29} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
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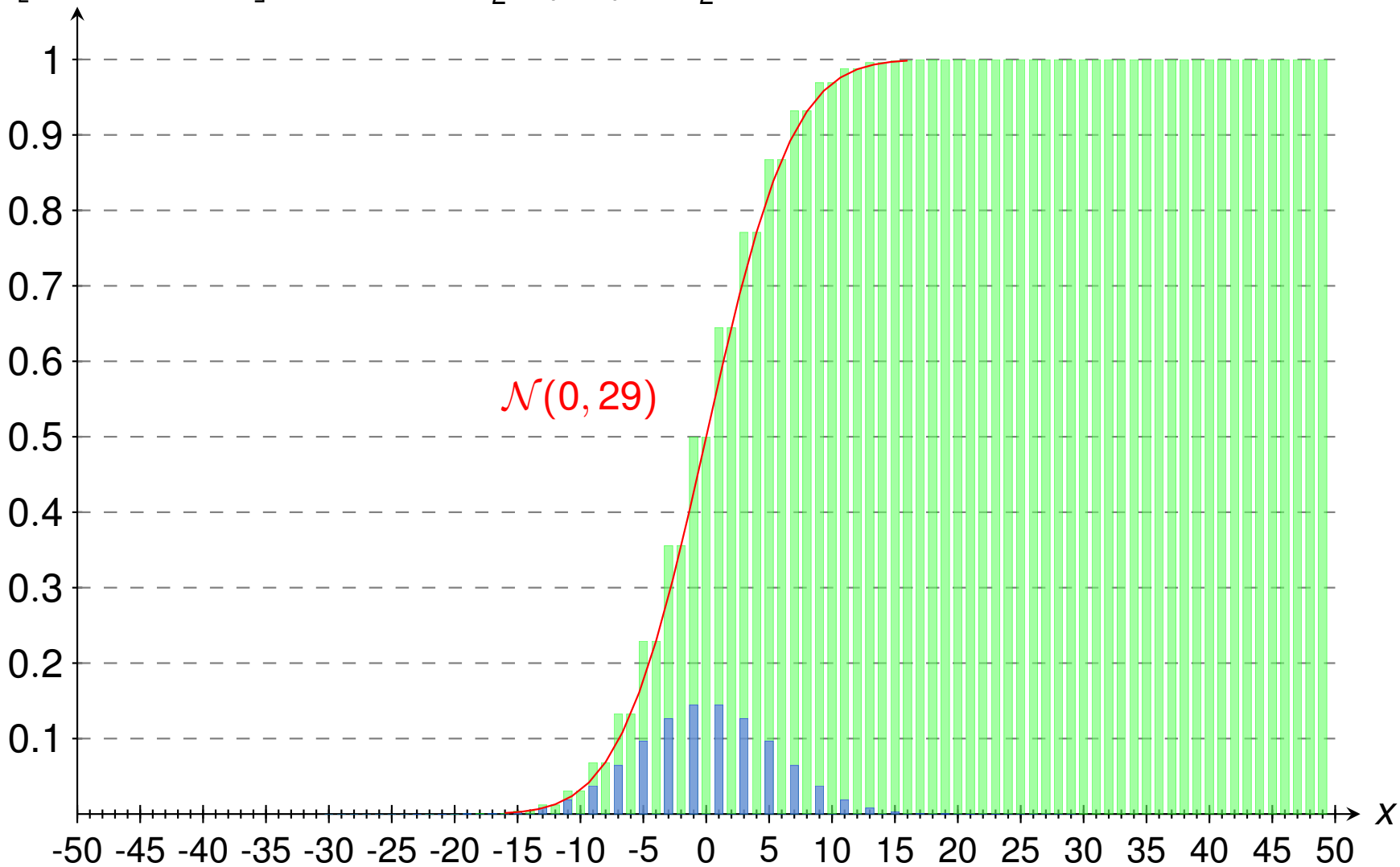

Illustration of CLT (3, Part II) (Distribution from Lecture 8)

$\mathbf{P}\left[\sum_{j=1}^{30} X_j \leq x\right]$

- $\mu = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$
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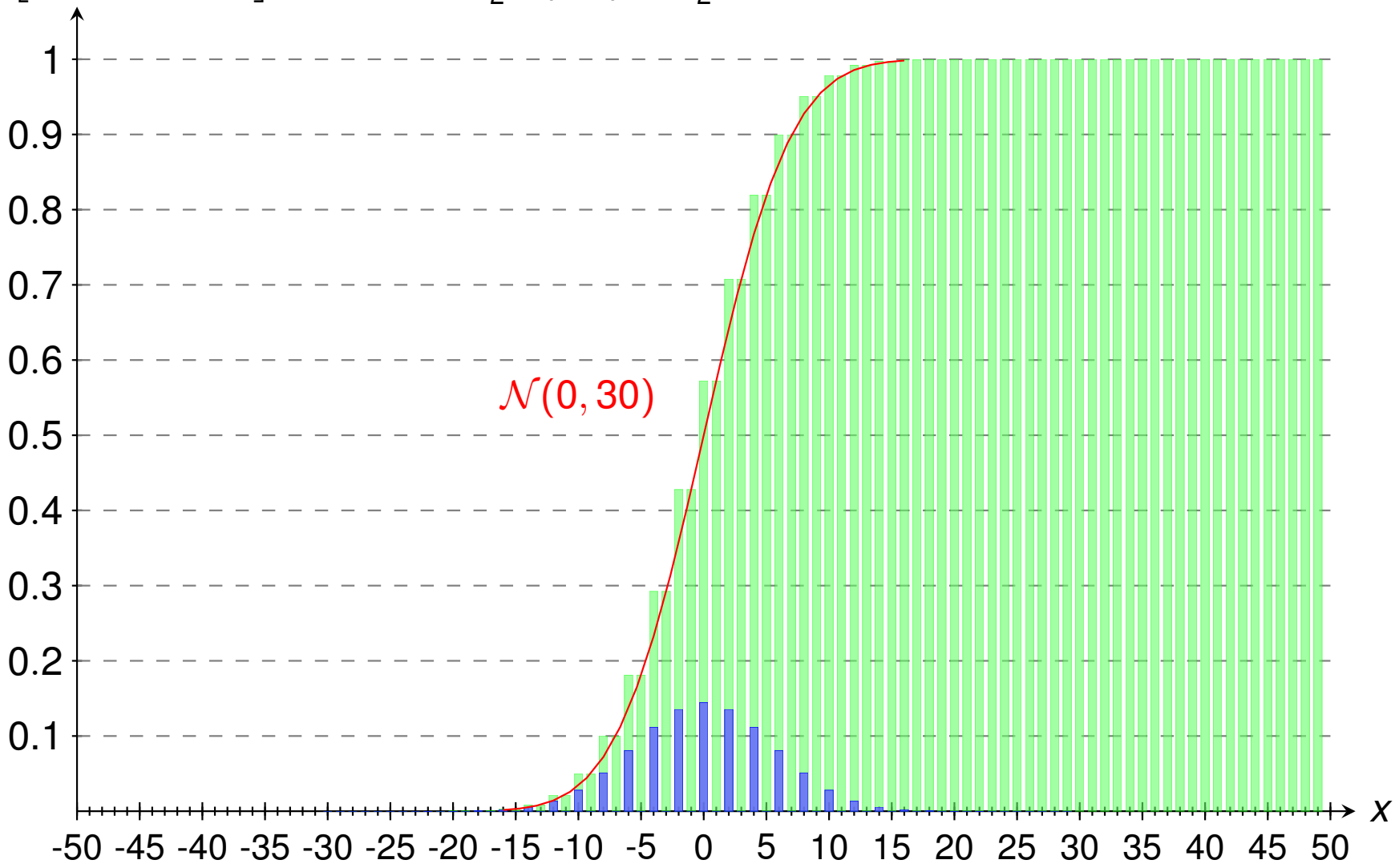

Illustration of CLT (4, Part I) with Standardising

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
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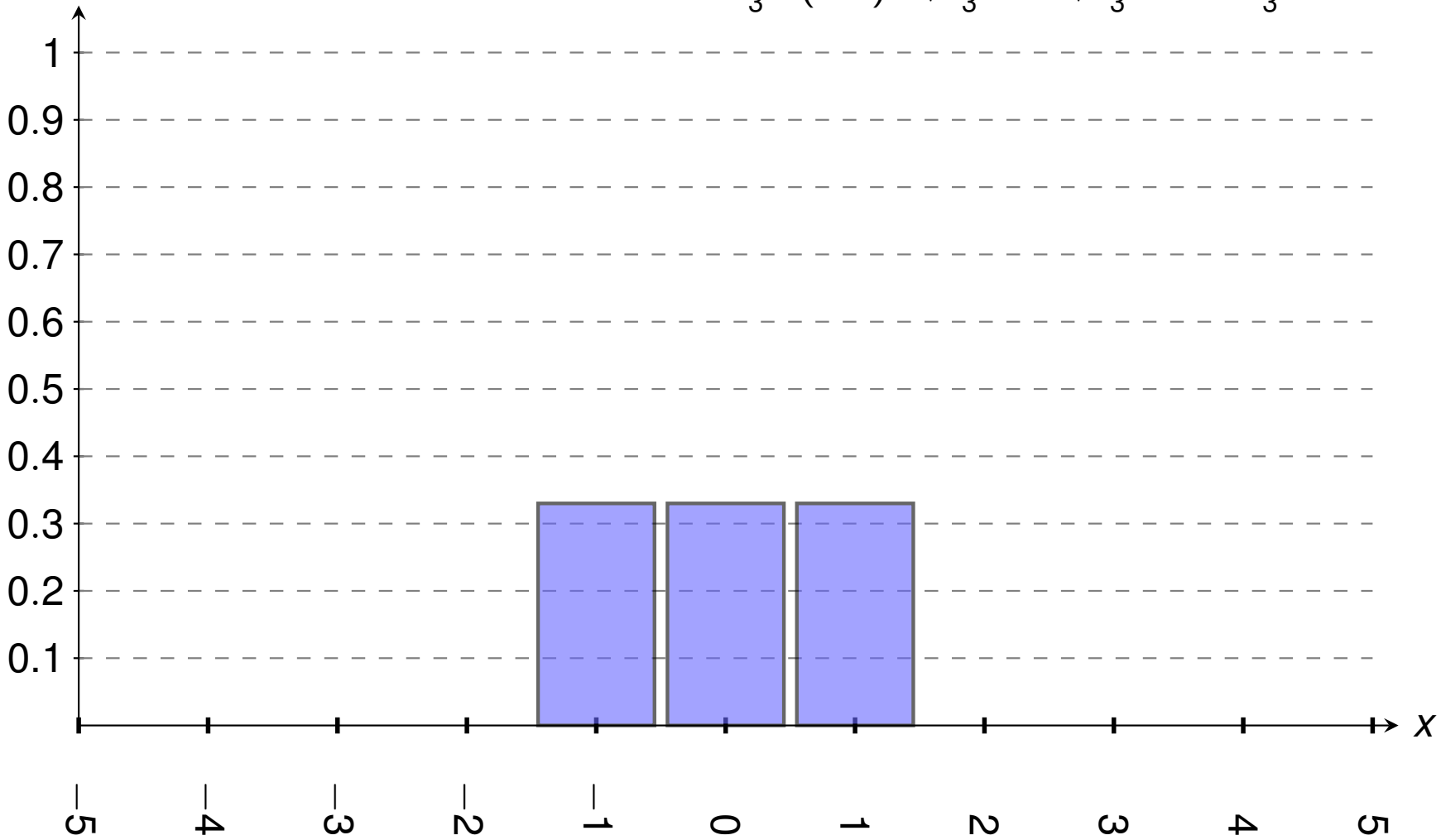

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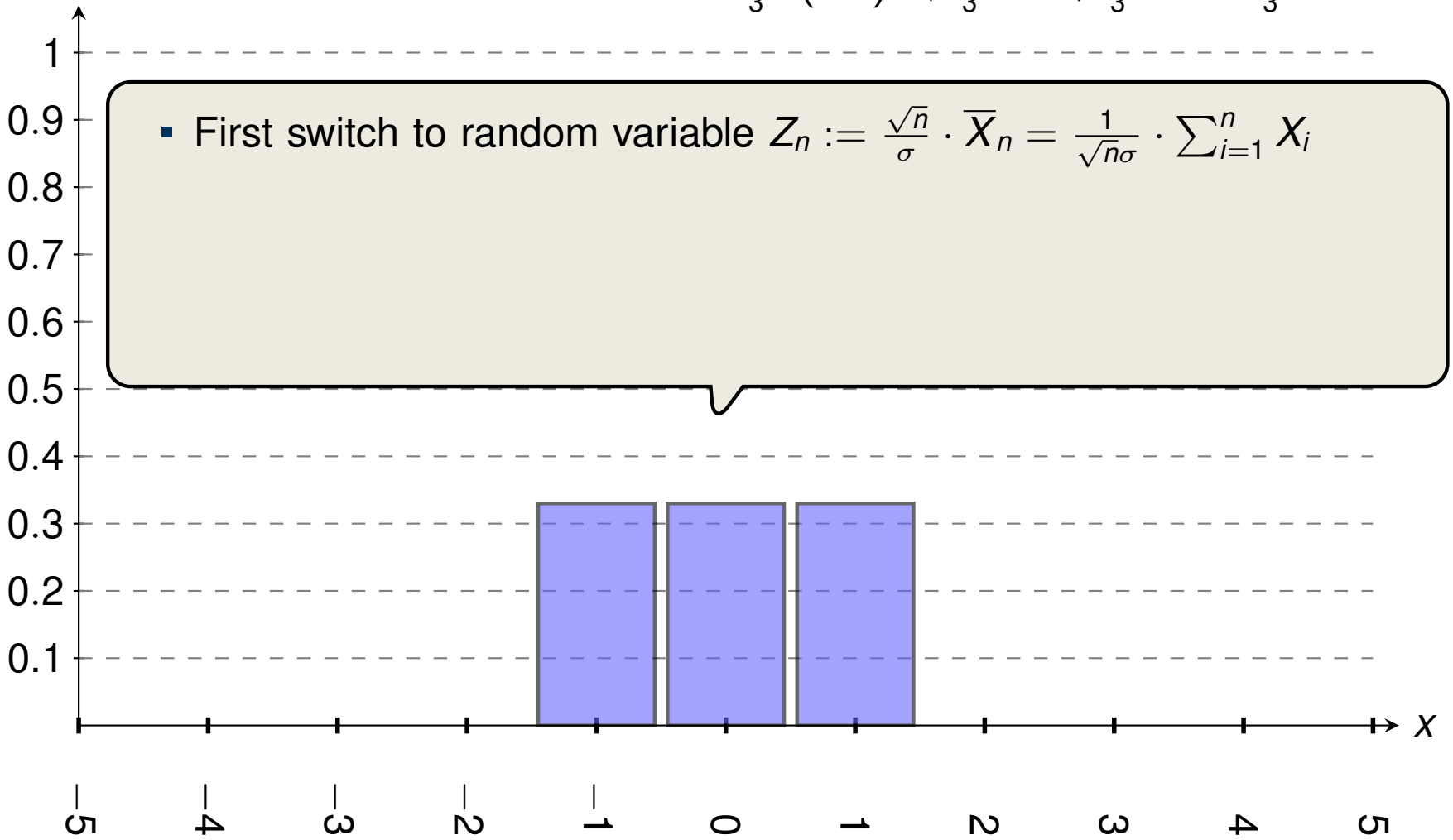

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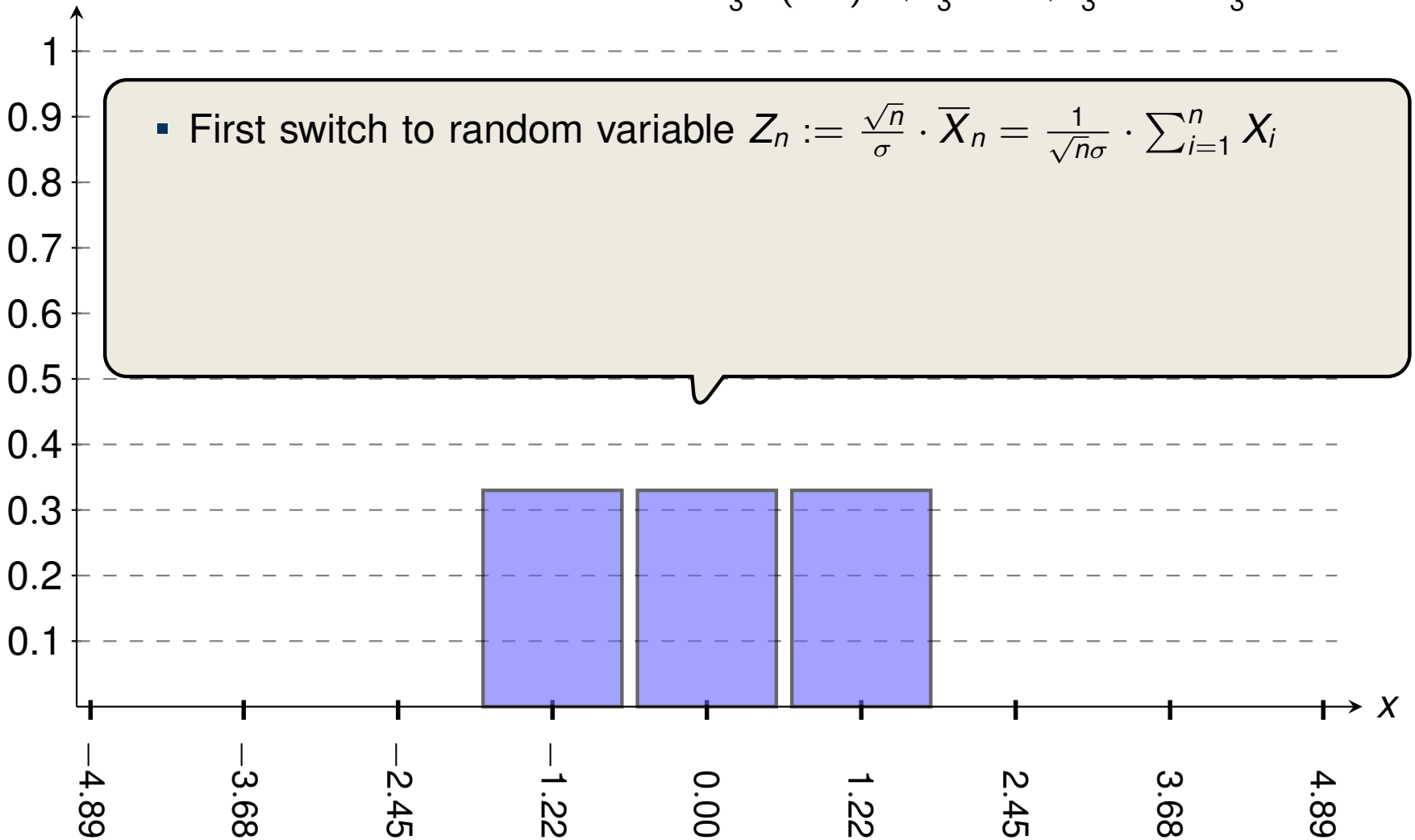

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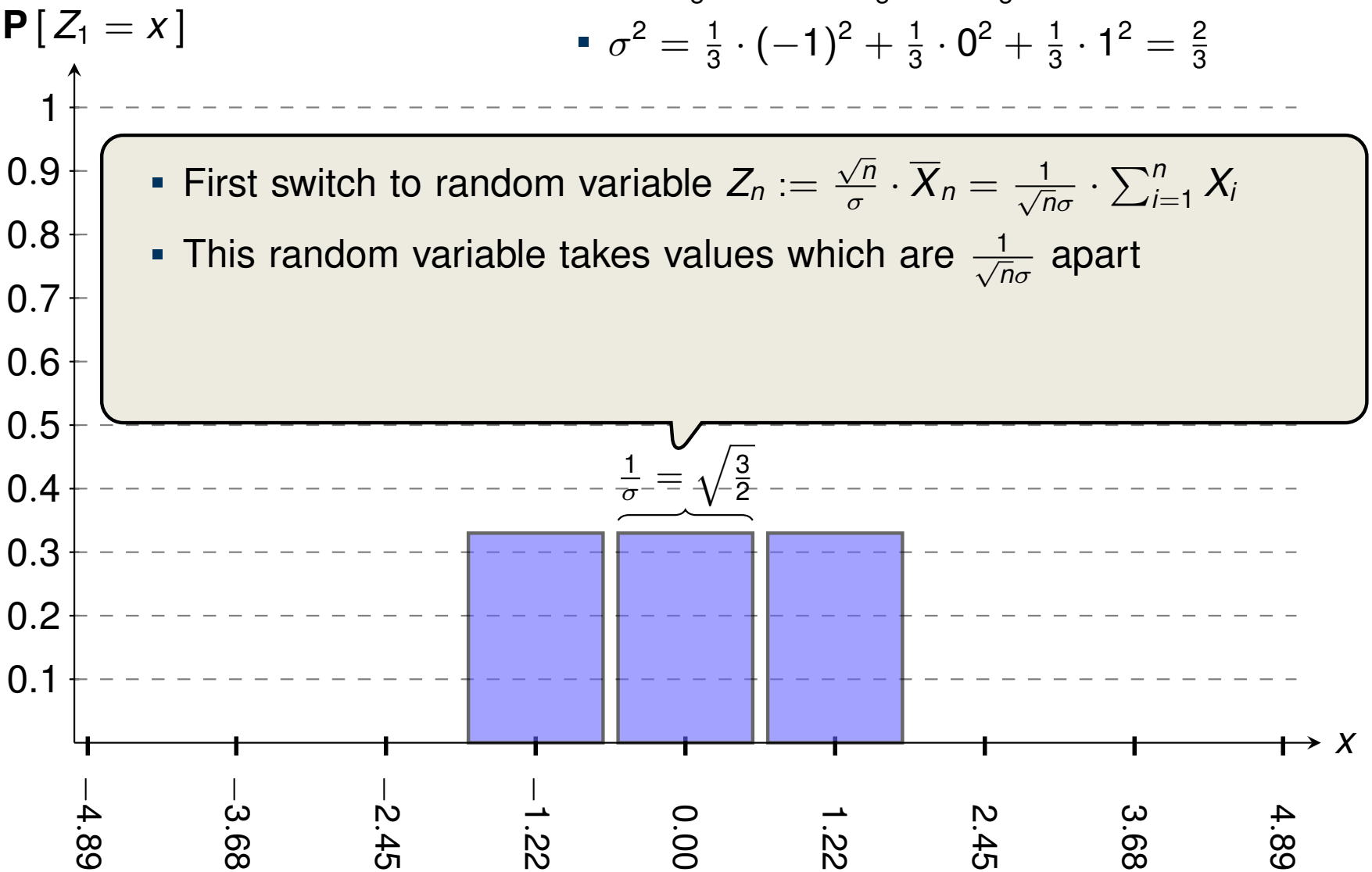

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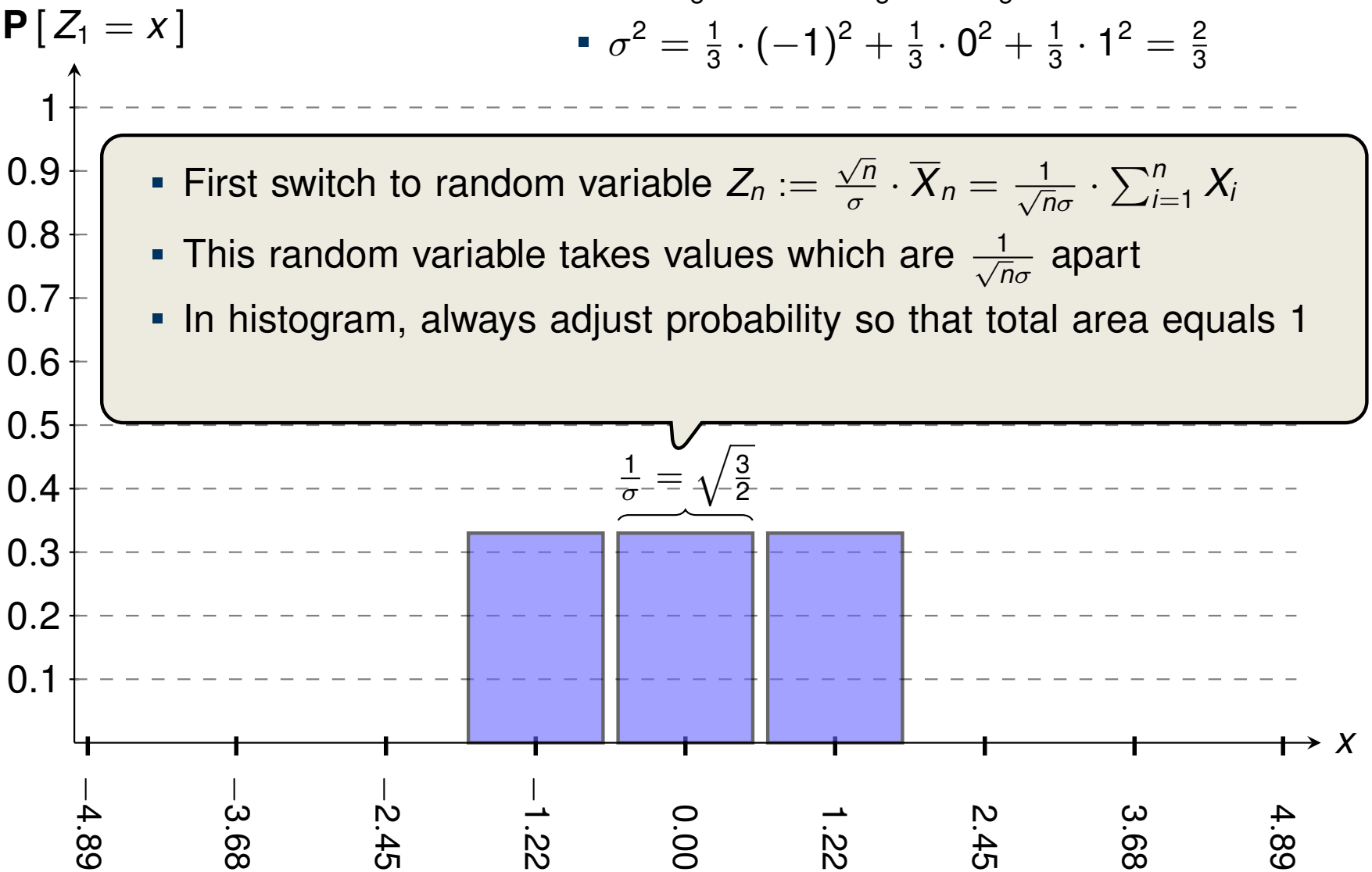

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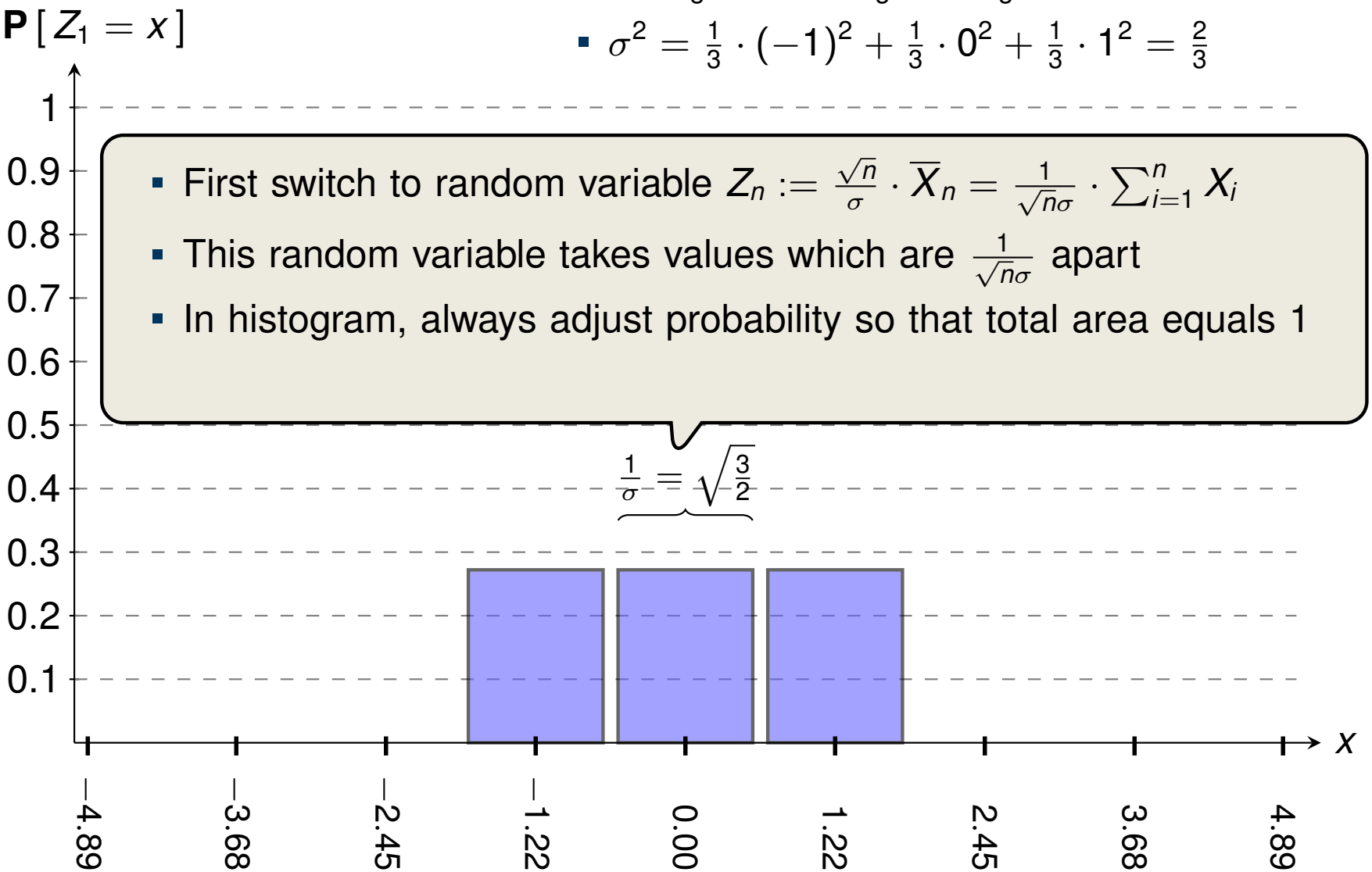

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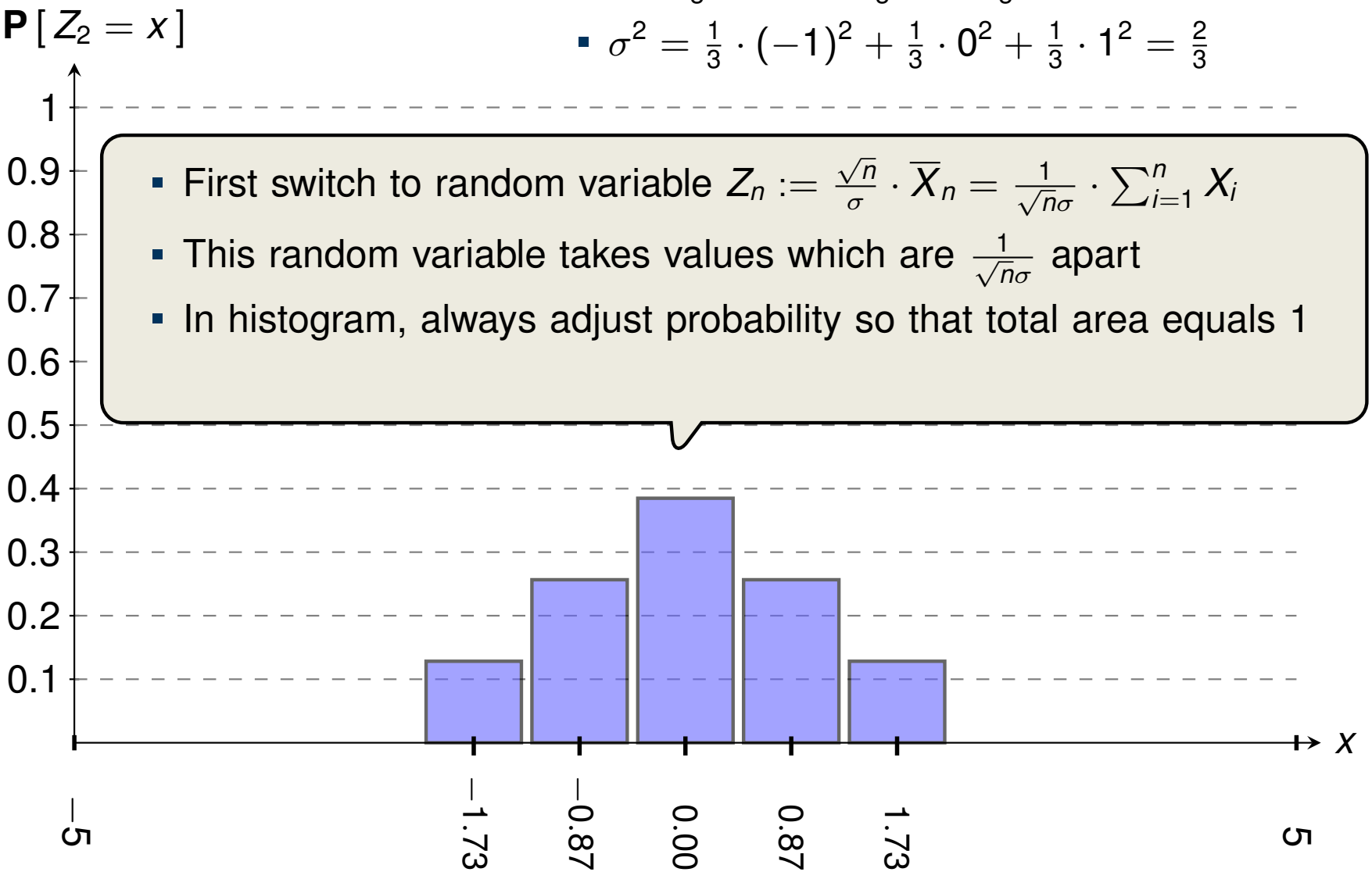

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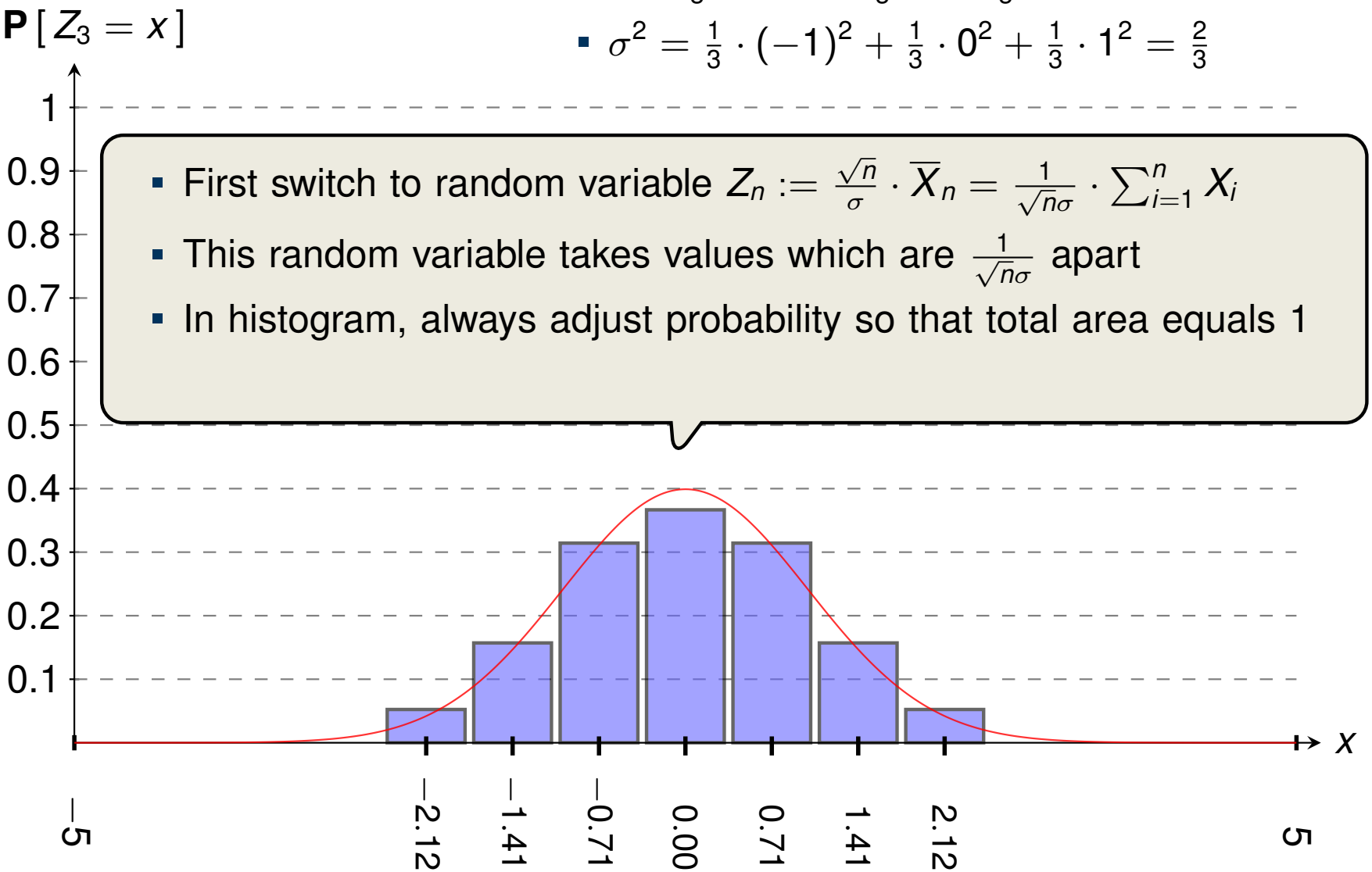

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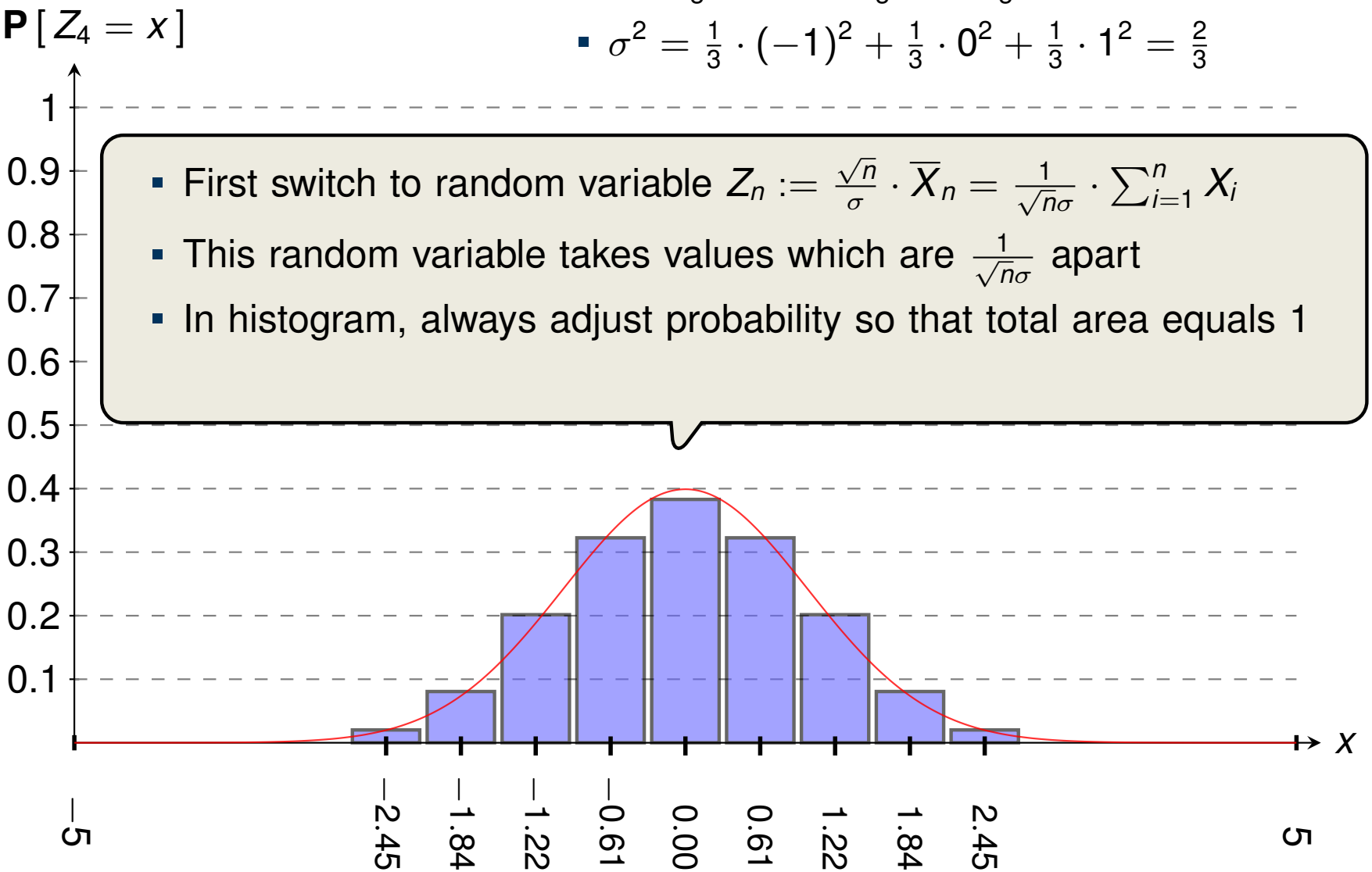

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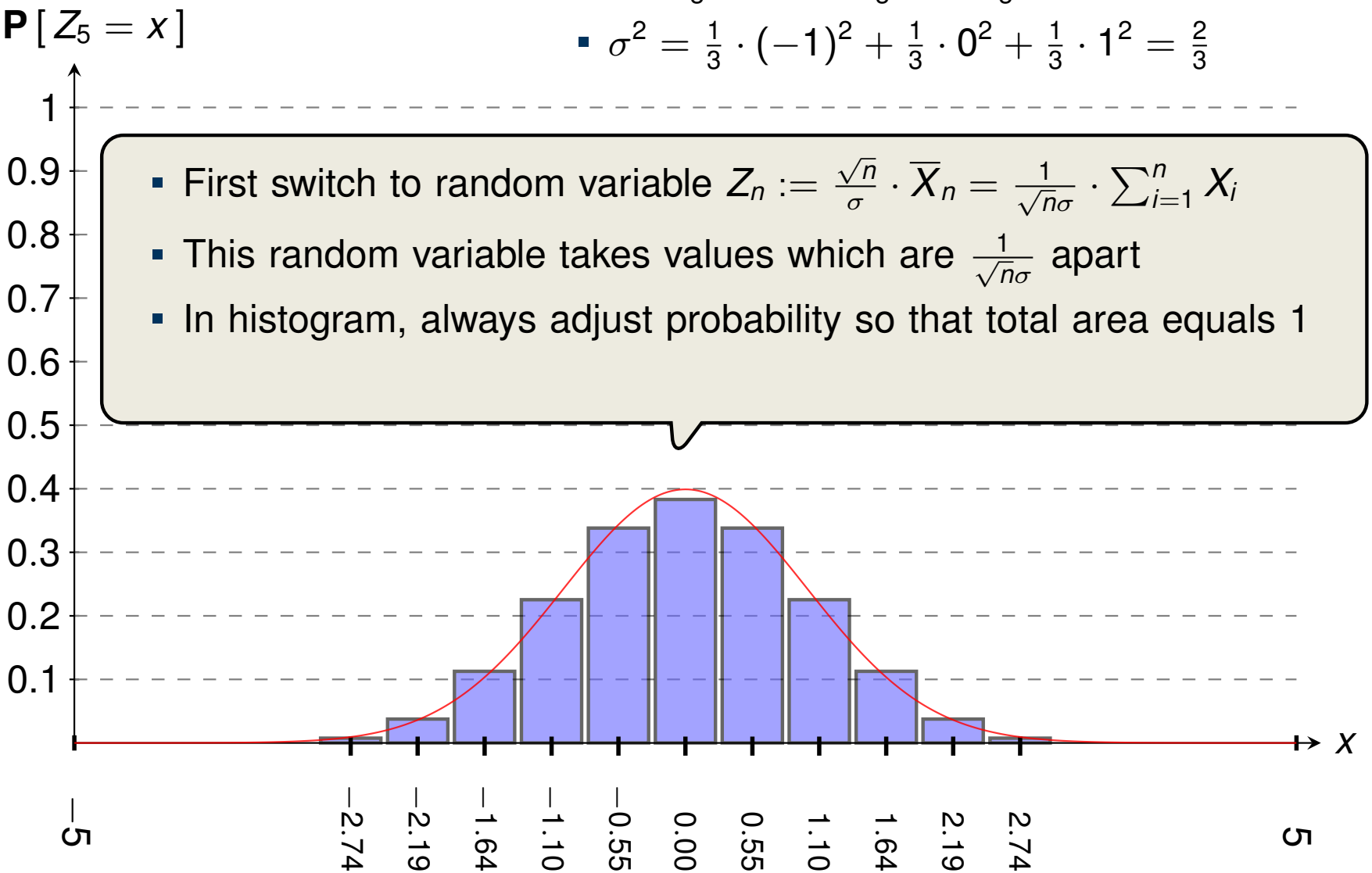

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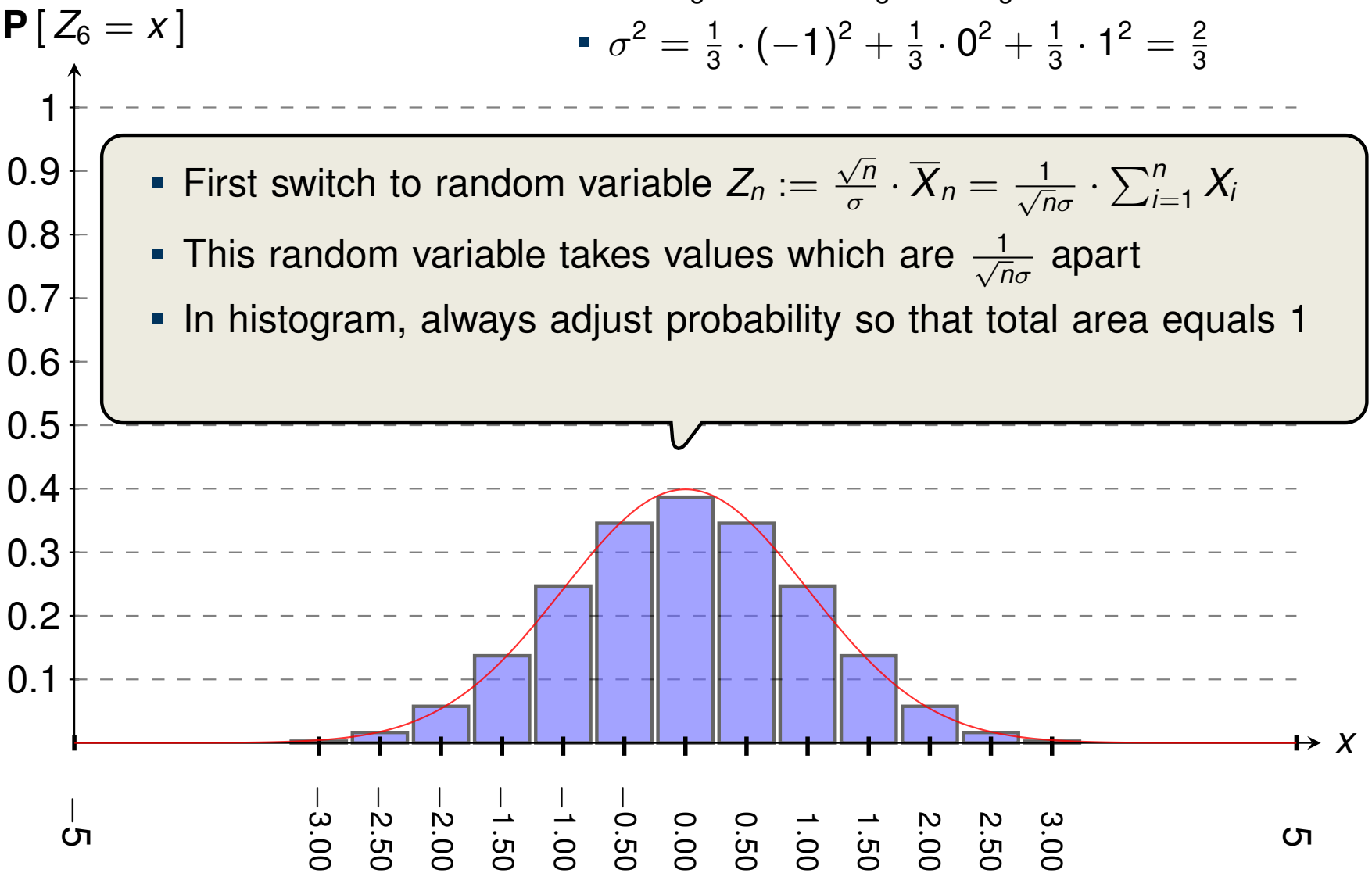

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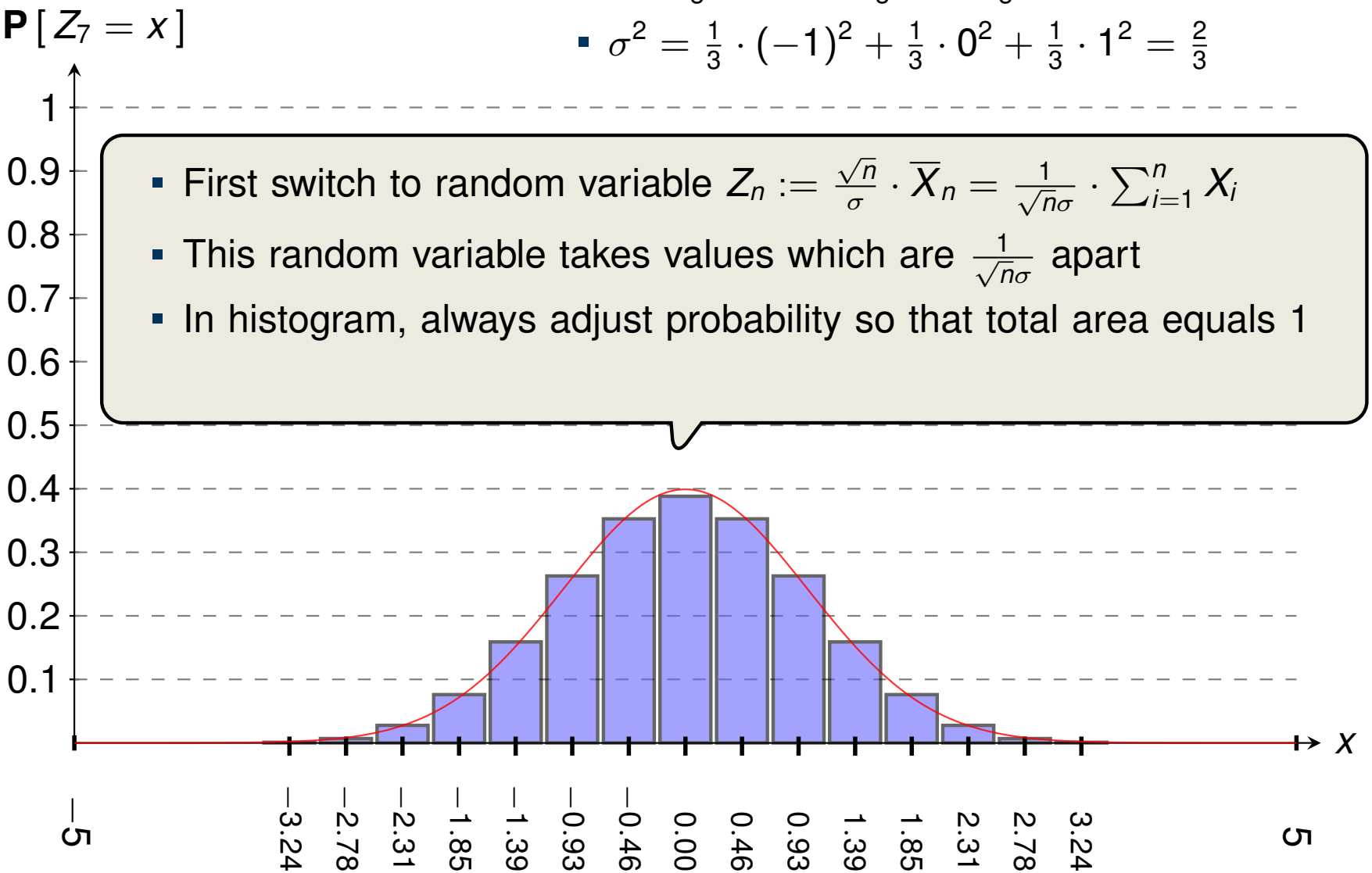

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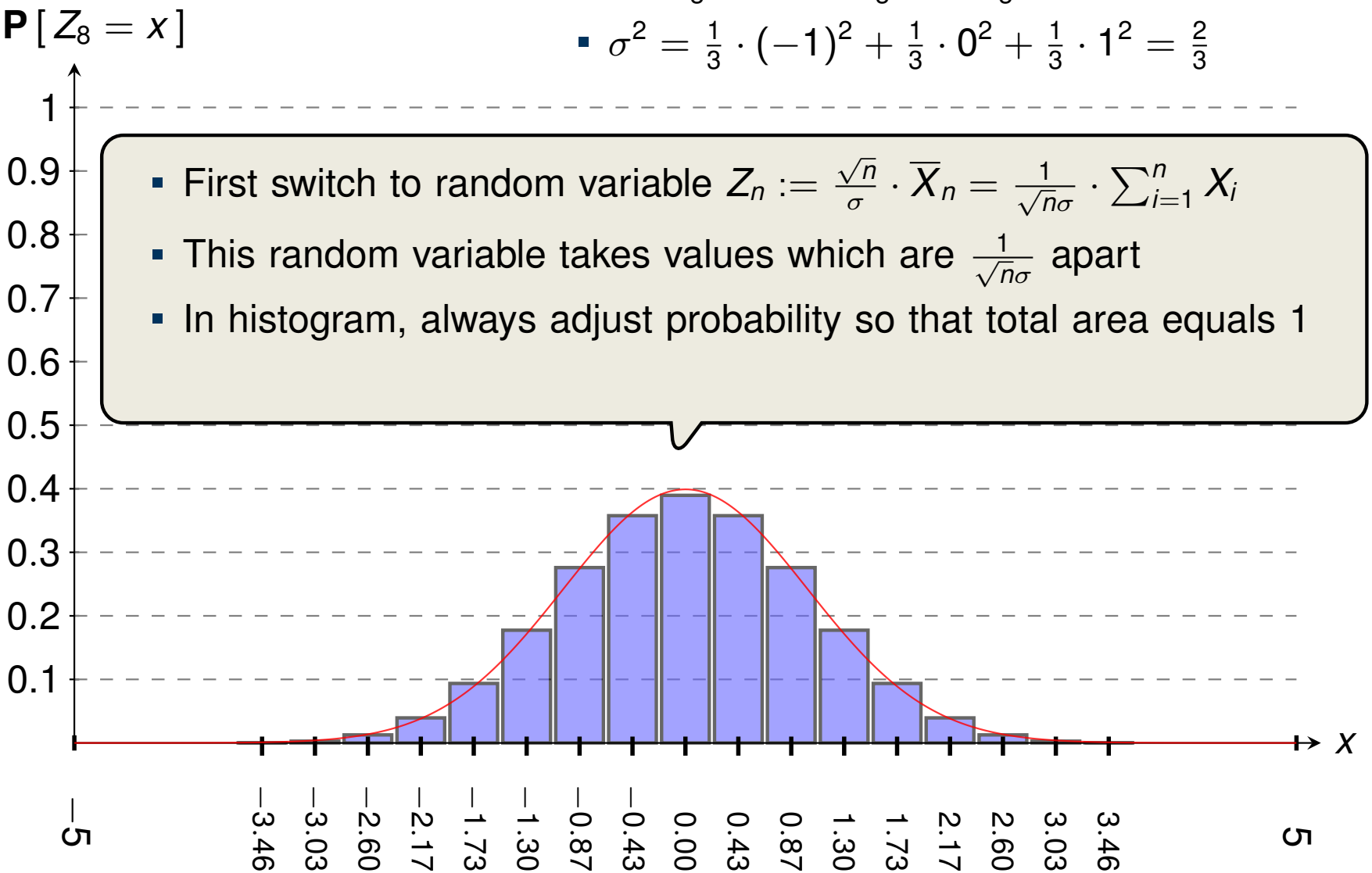

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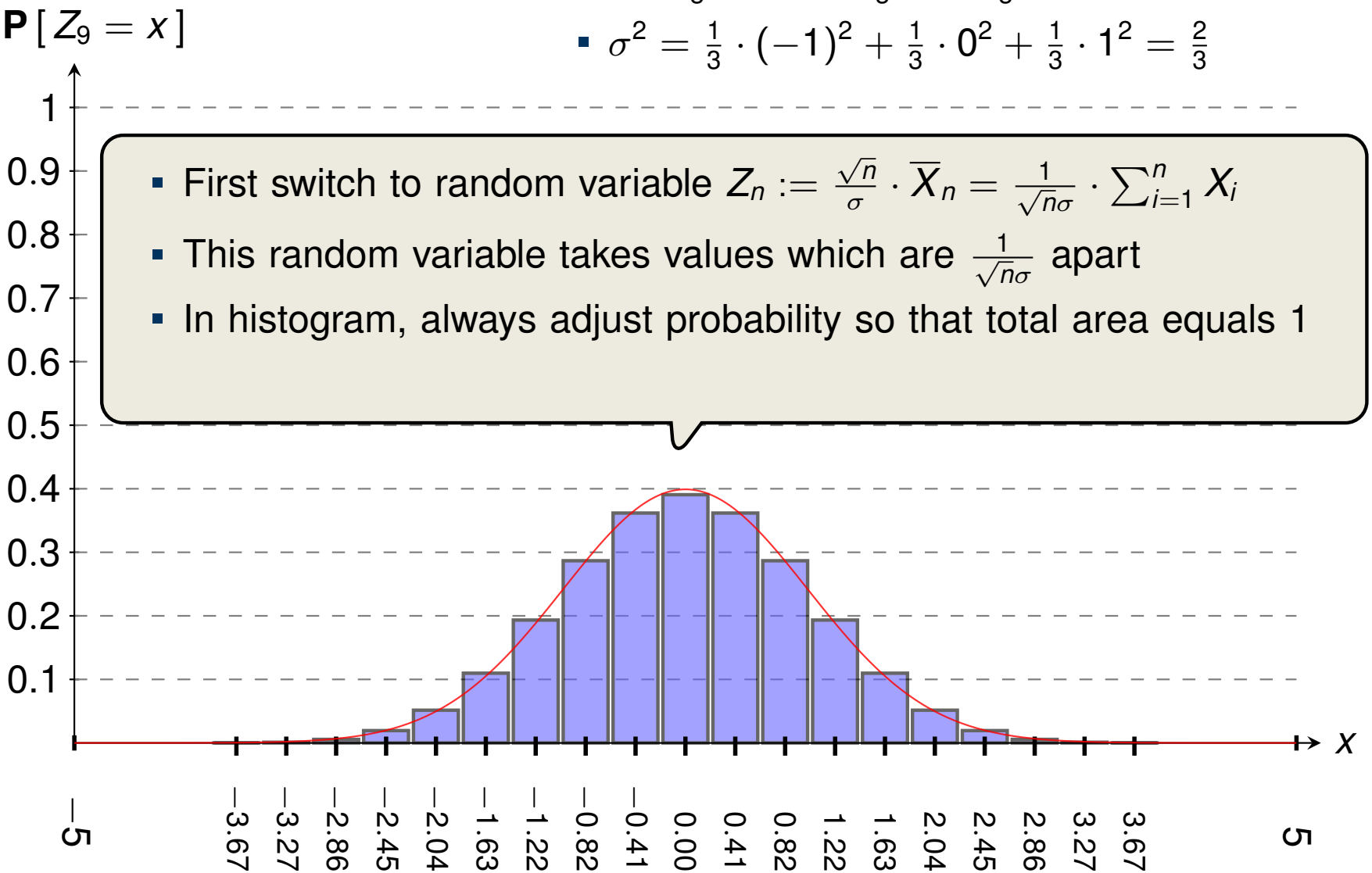

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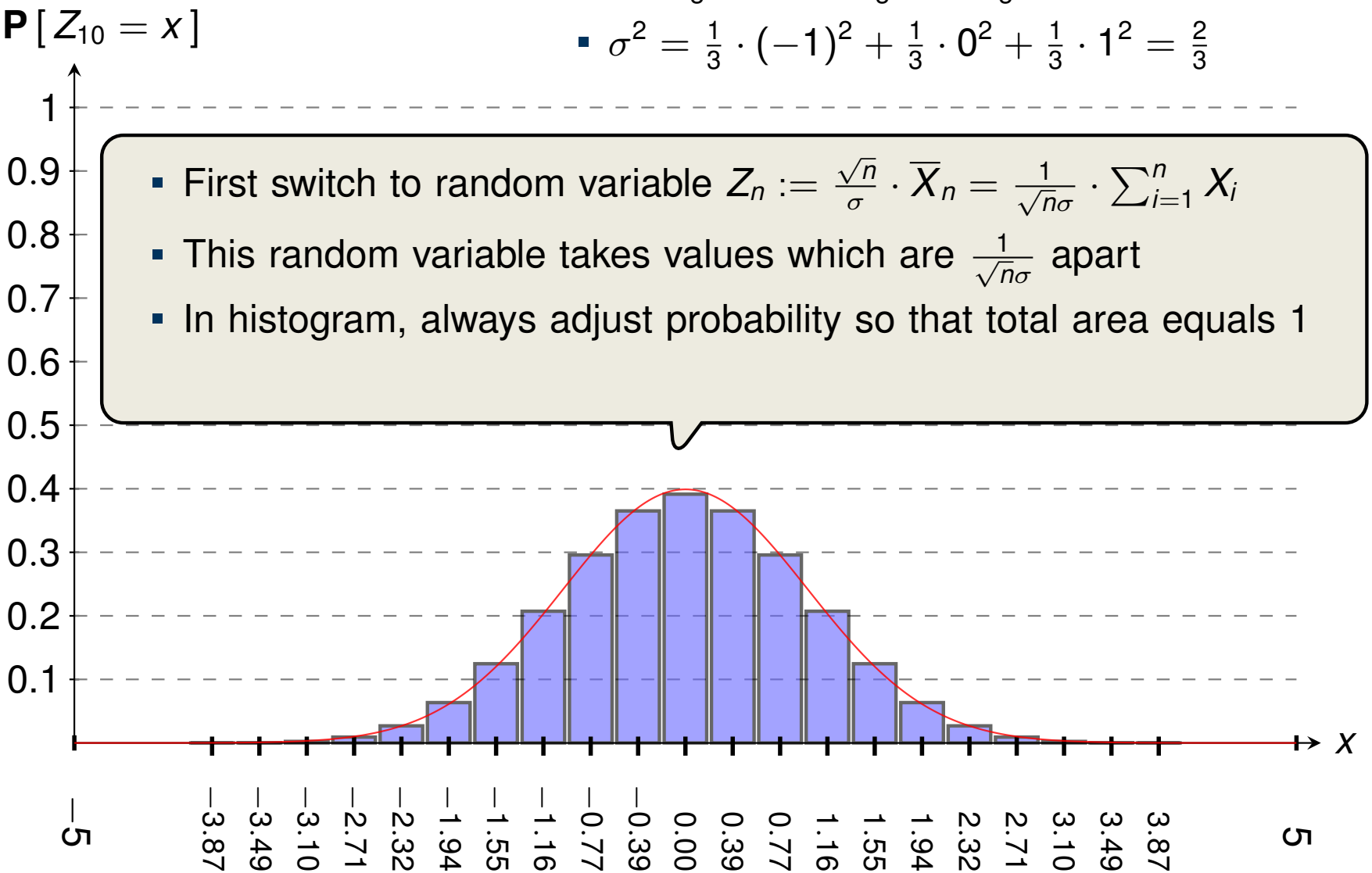

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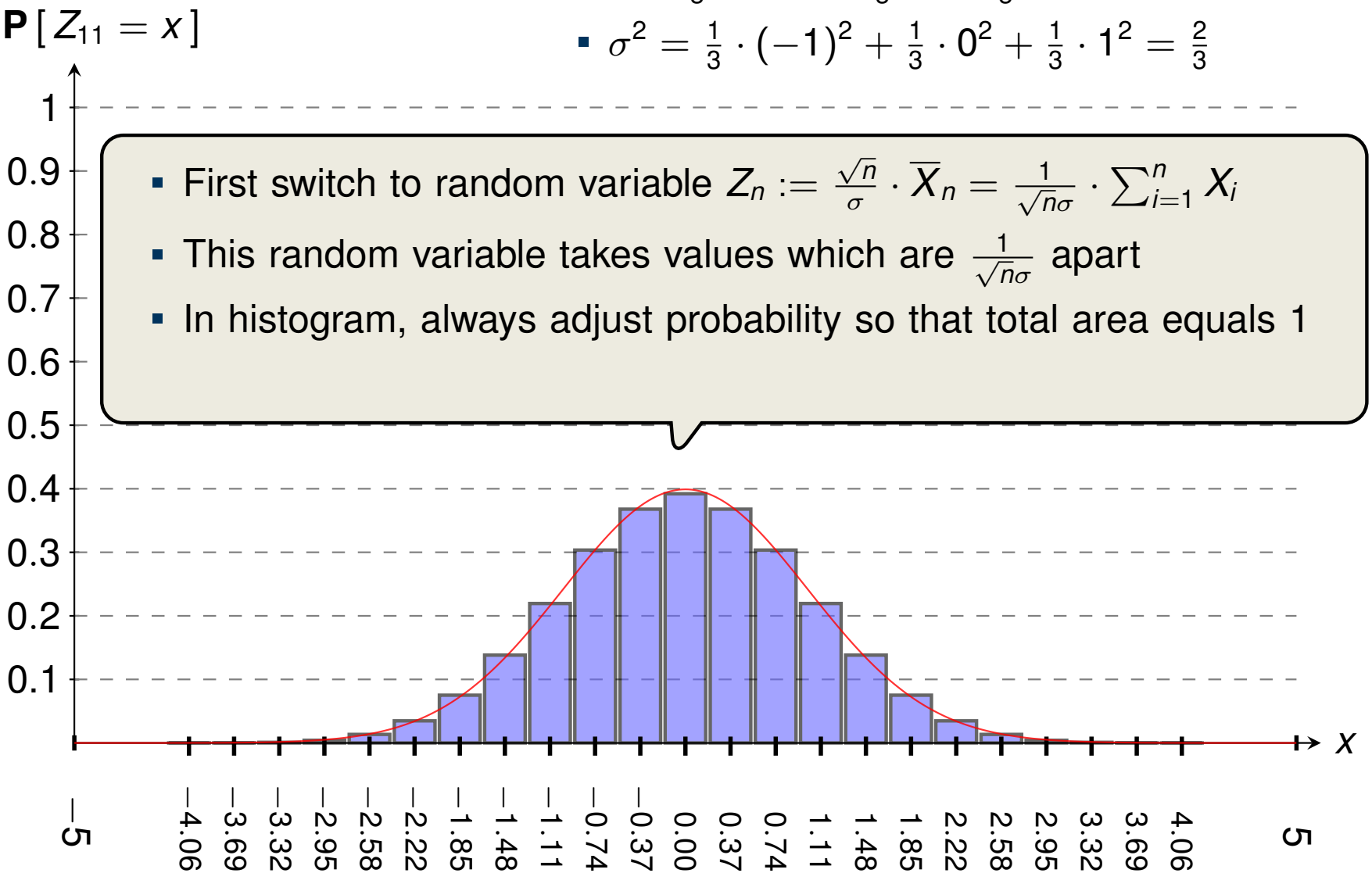

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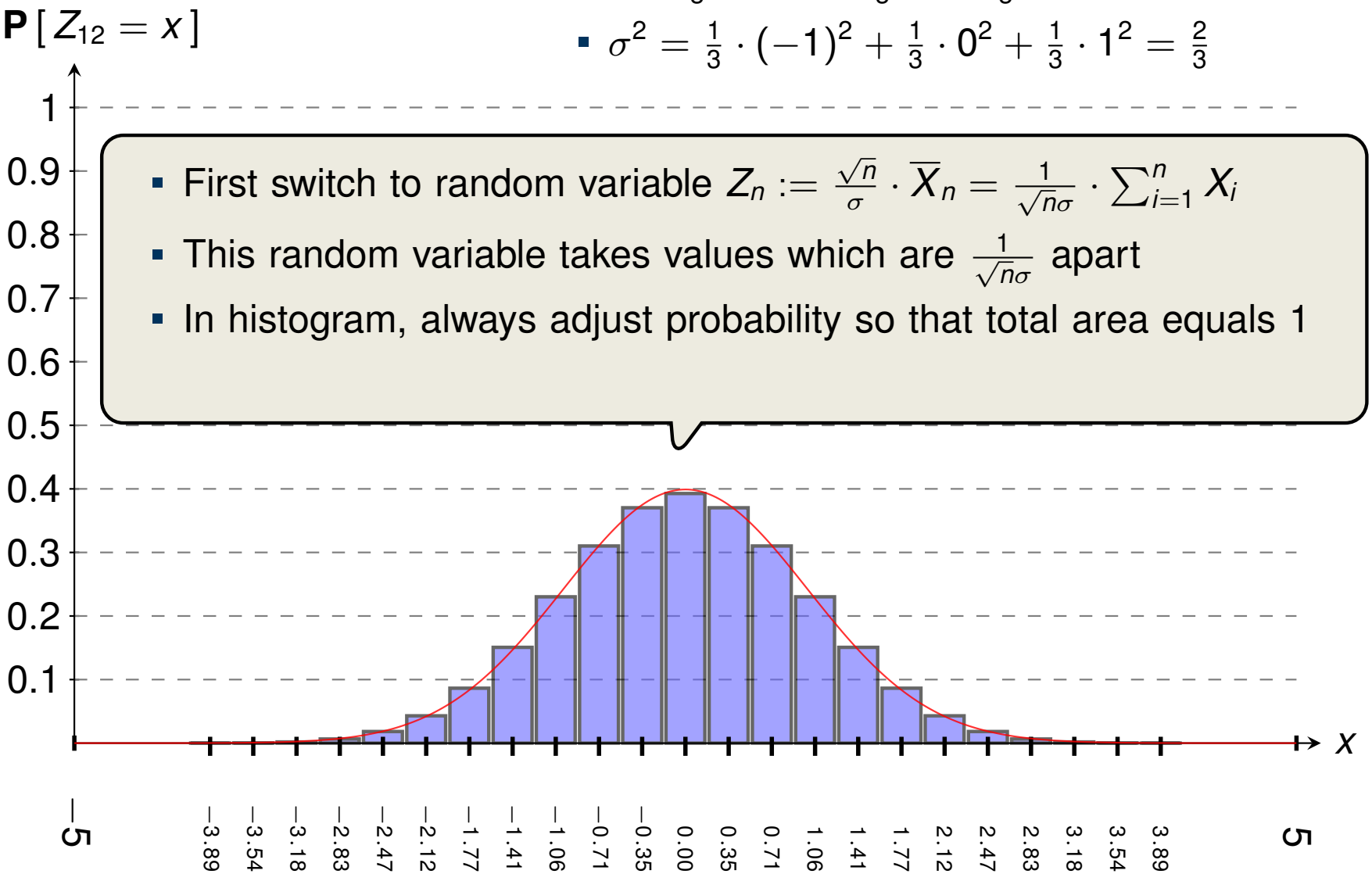

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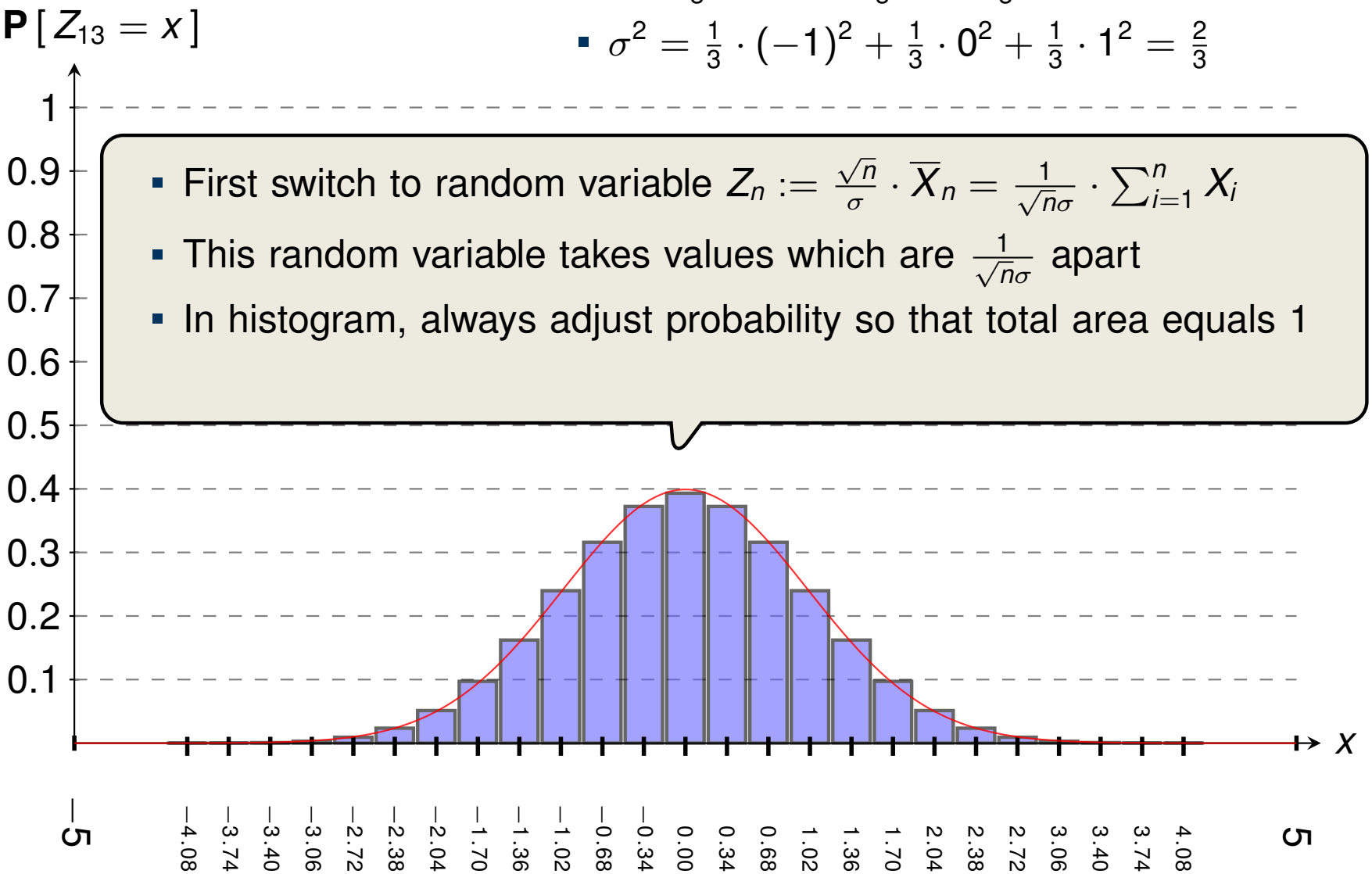

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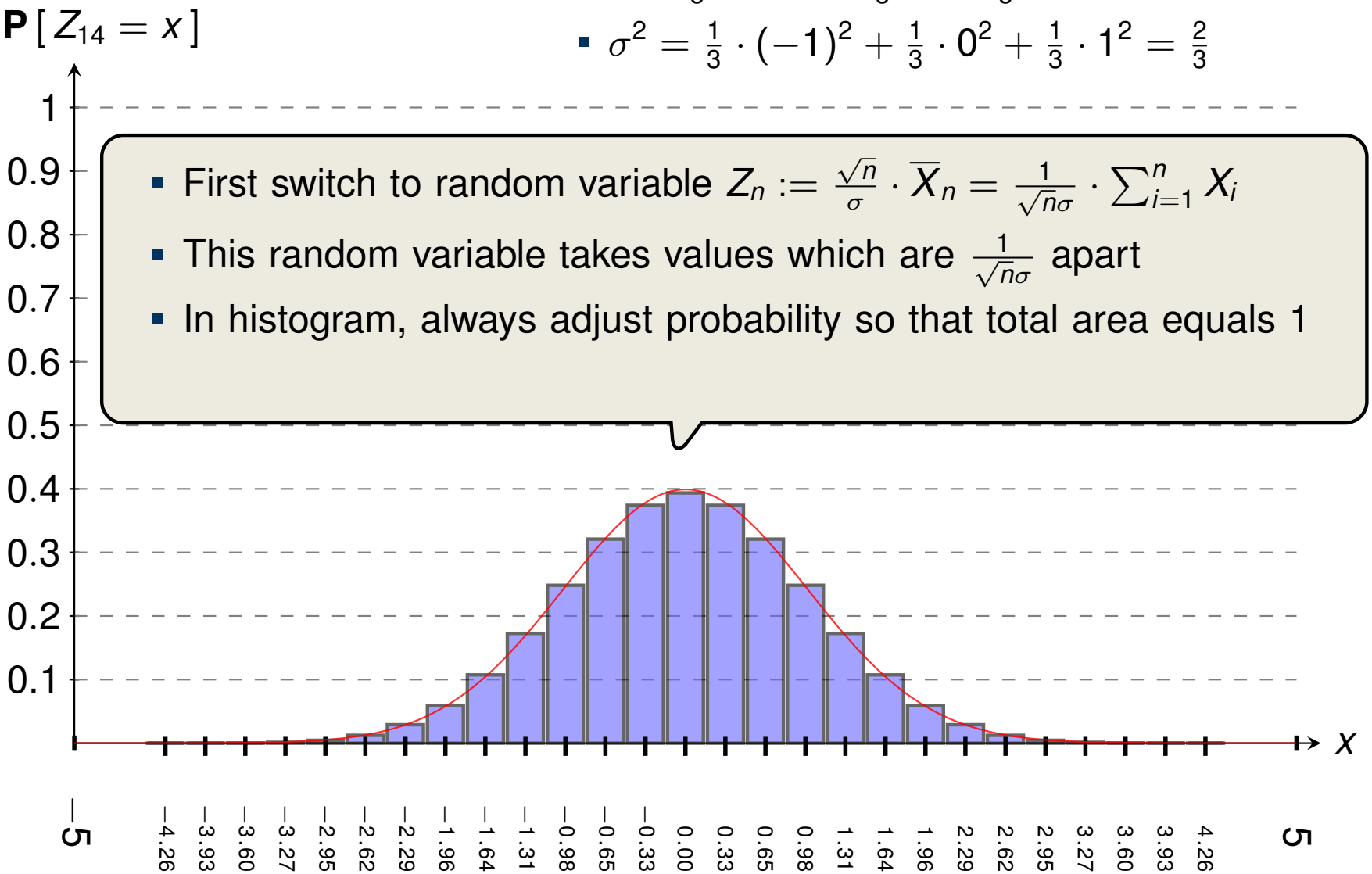

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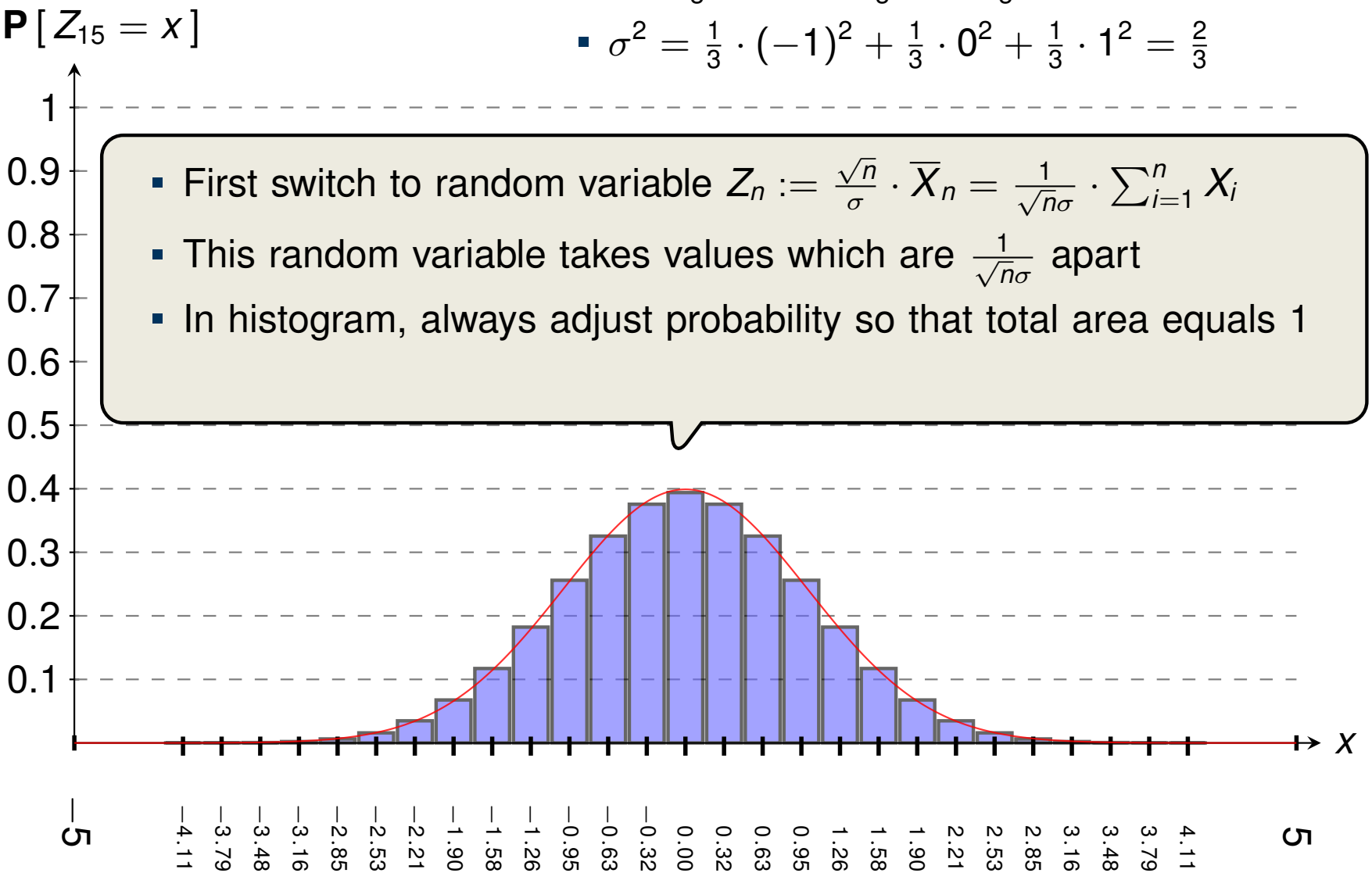

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- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
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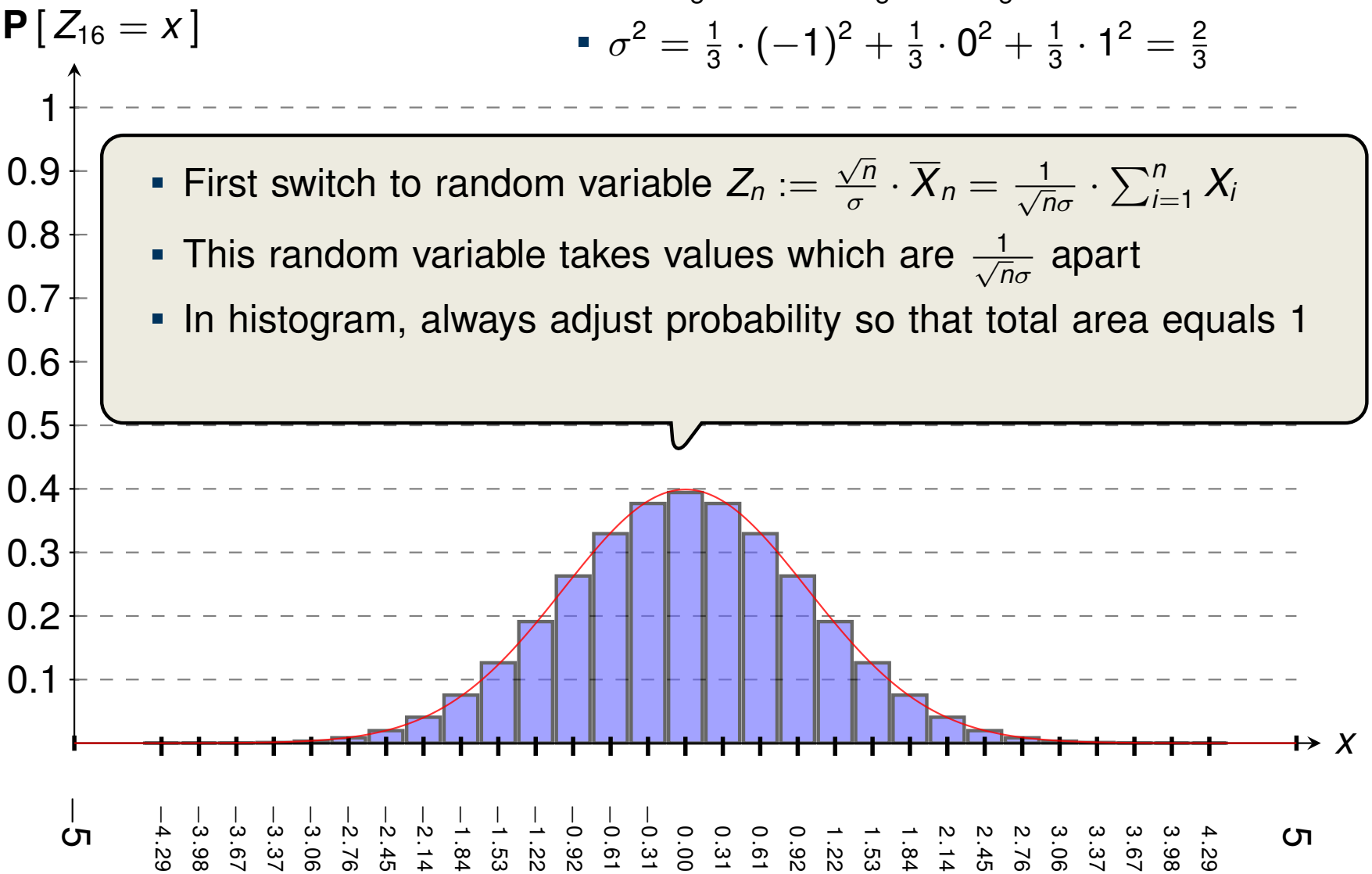

- First switch to random variable $Z_n := \frac{\sqrt{n}}{\sigma} \cdot \overline{X}_n = \frac{1}{\sqrt{n}\sigma} \cdot \sum_{i=1}^{n} X_i$
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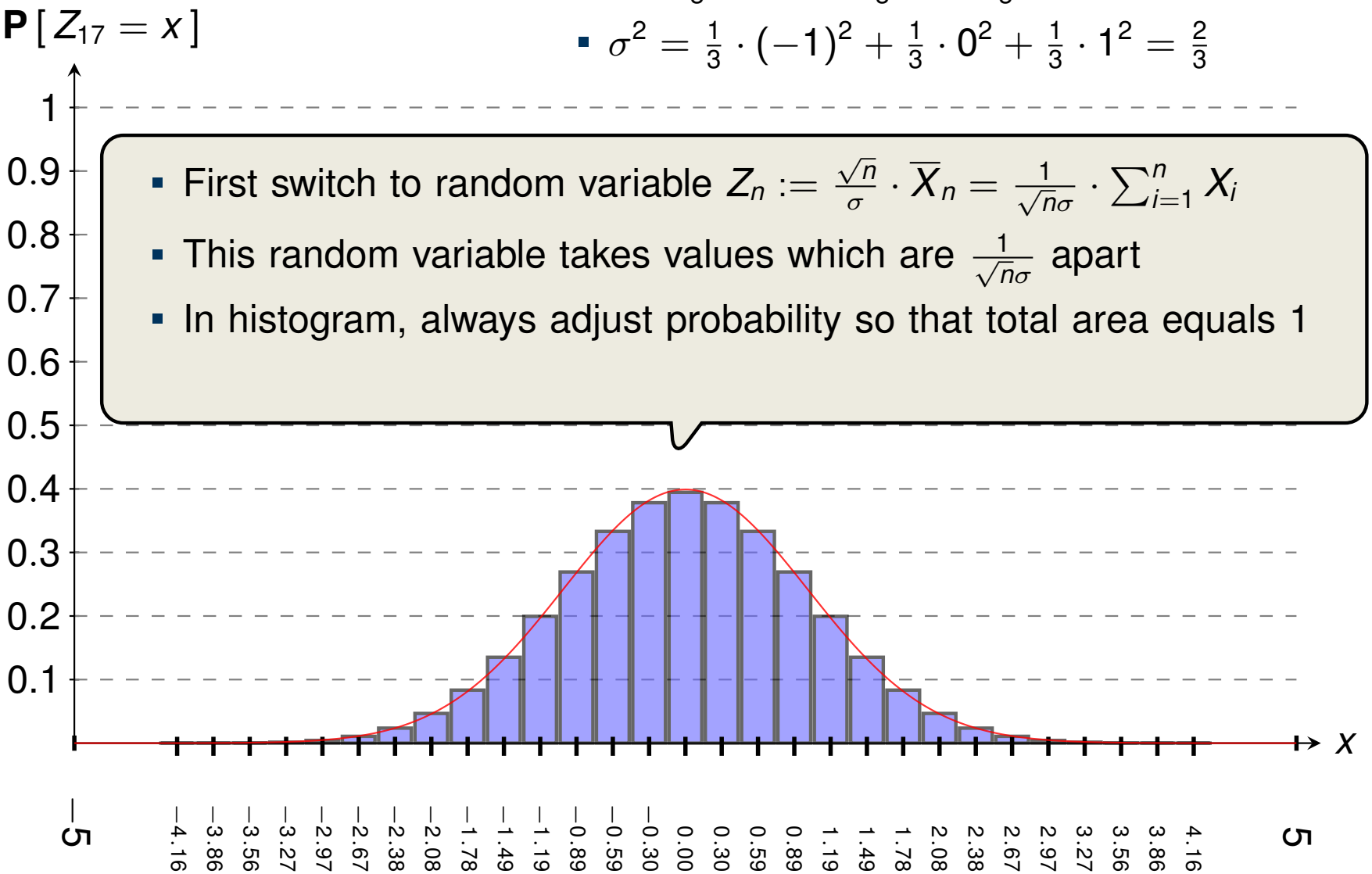

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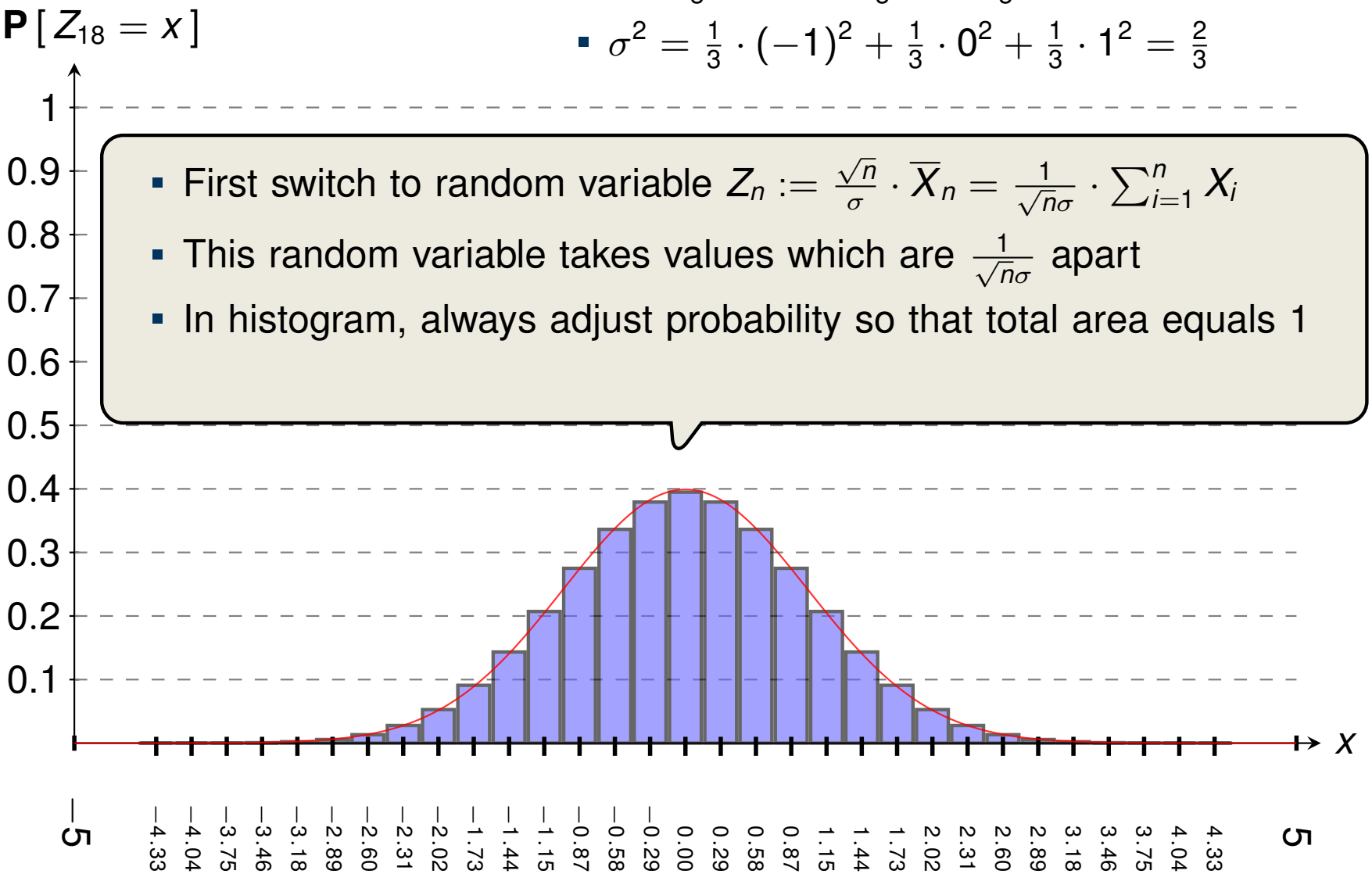

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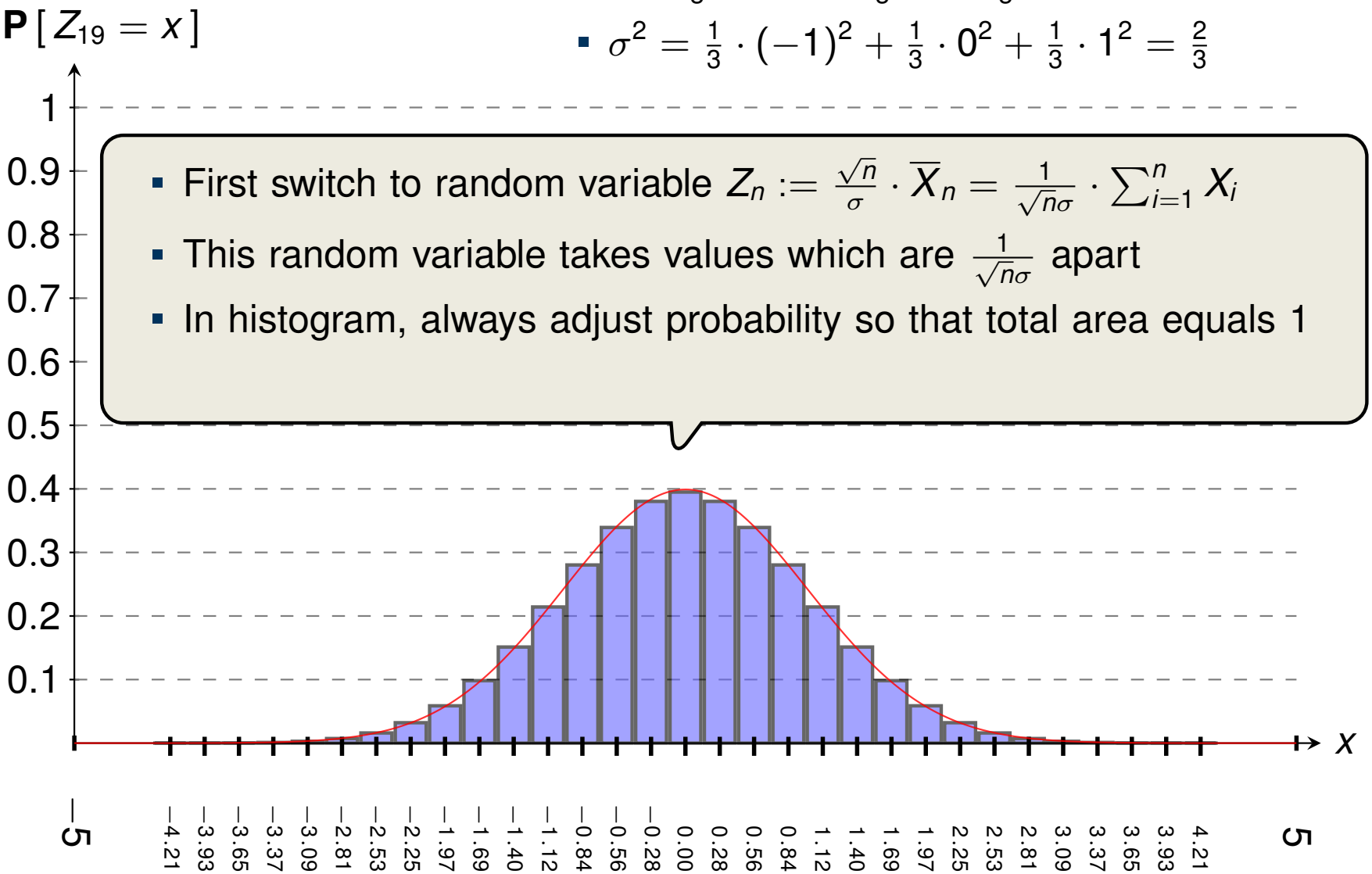

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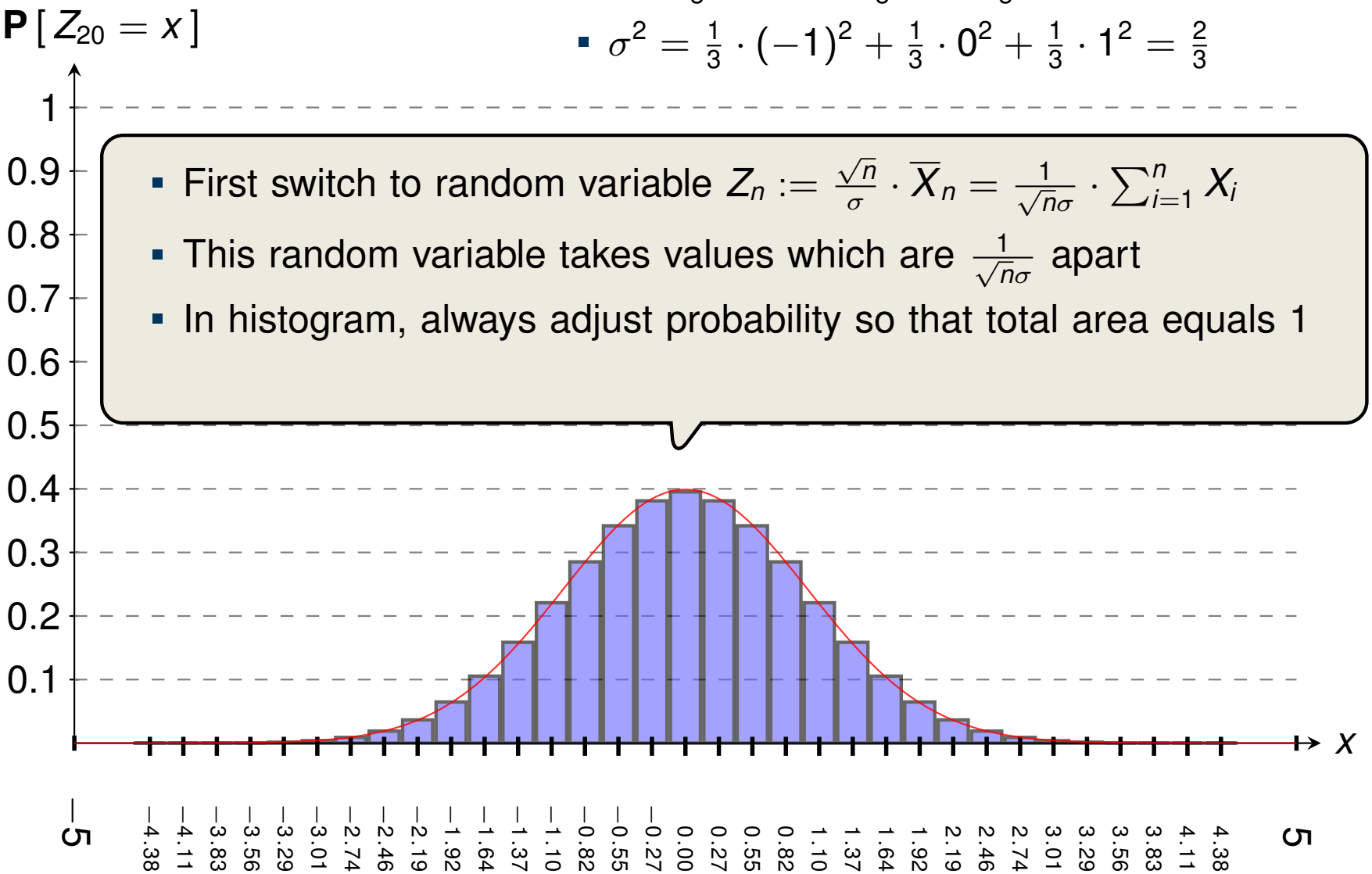

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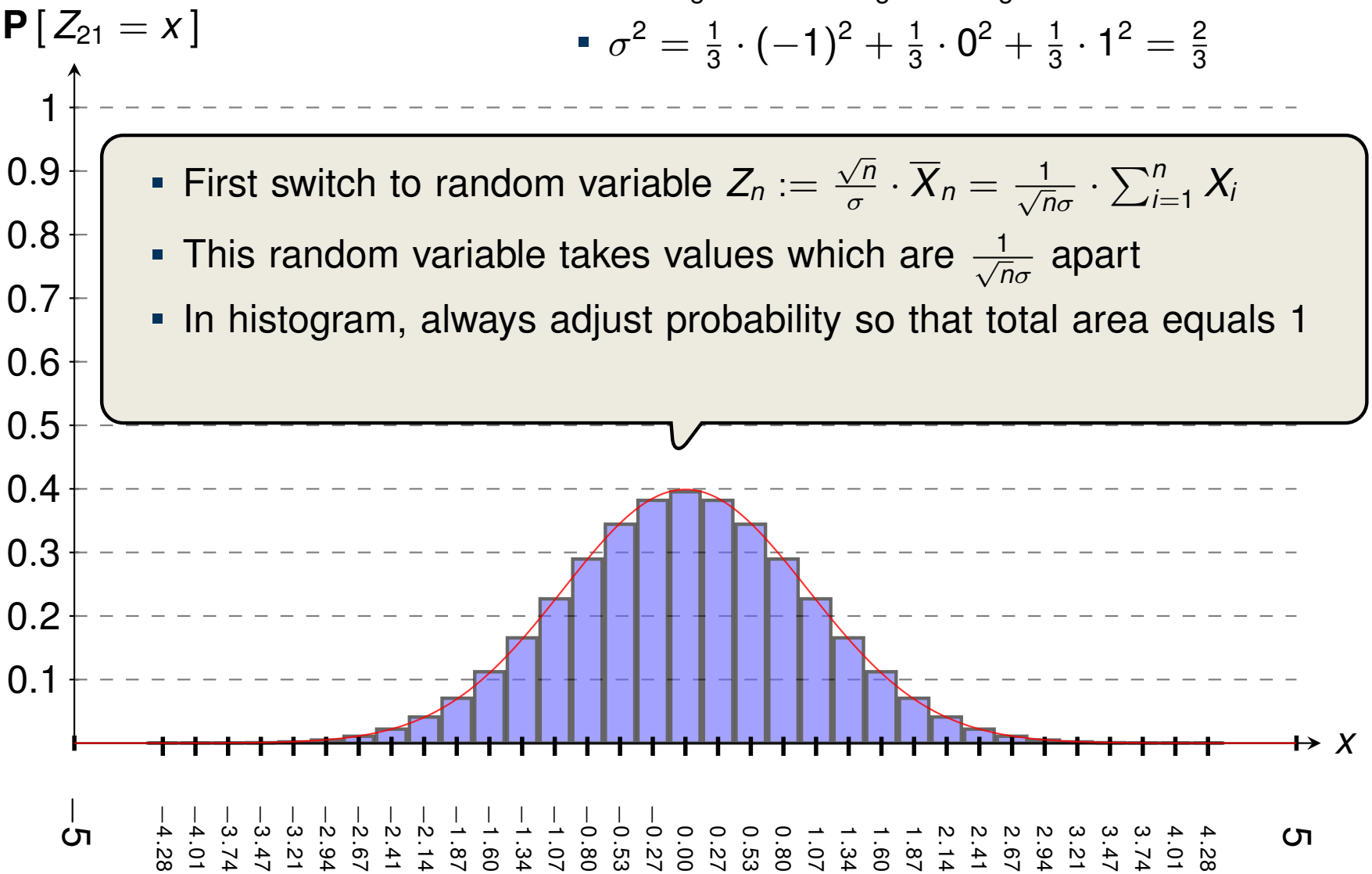

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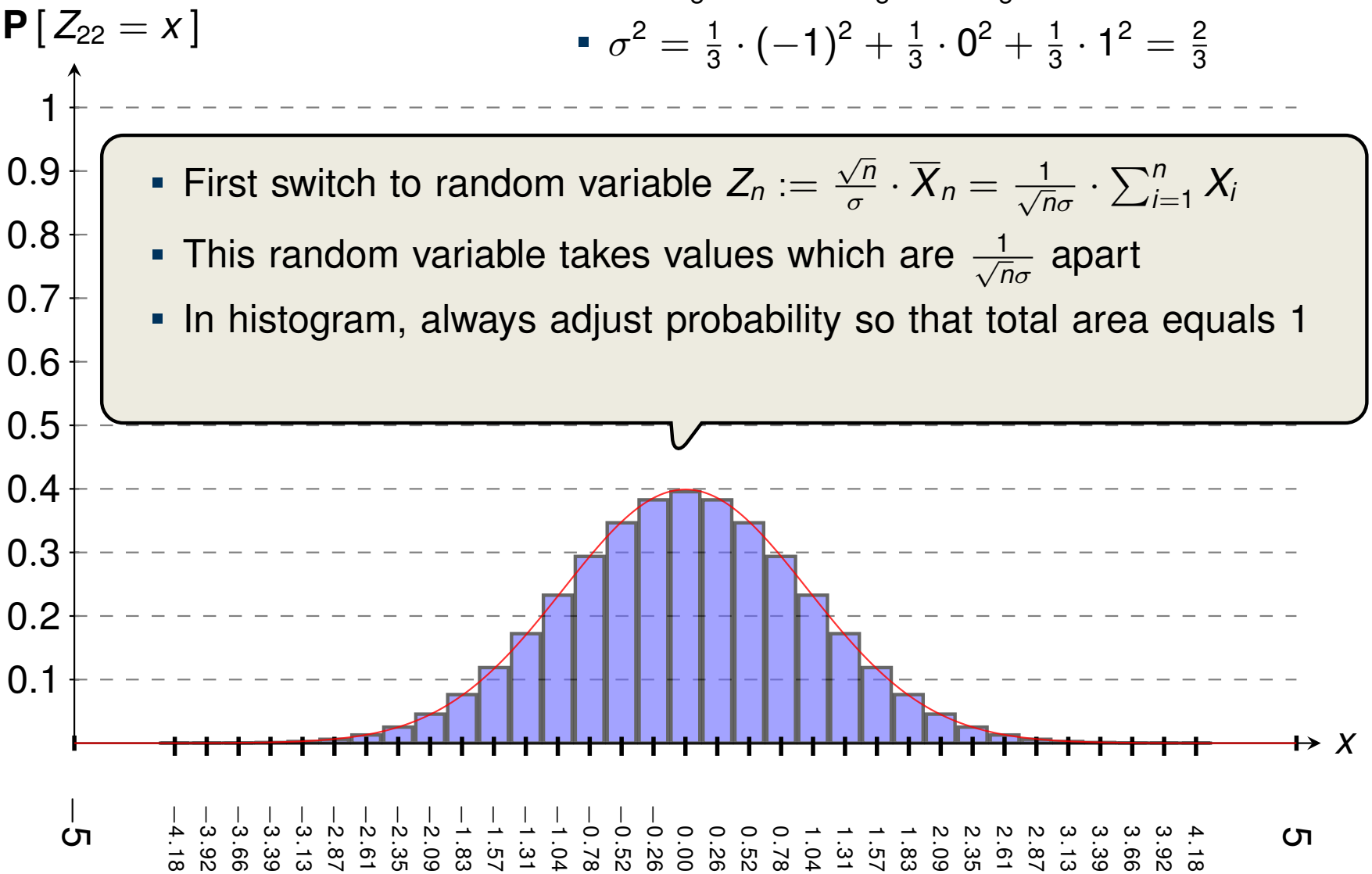

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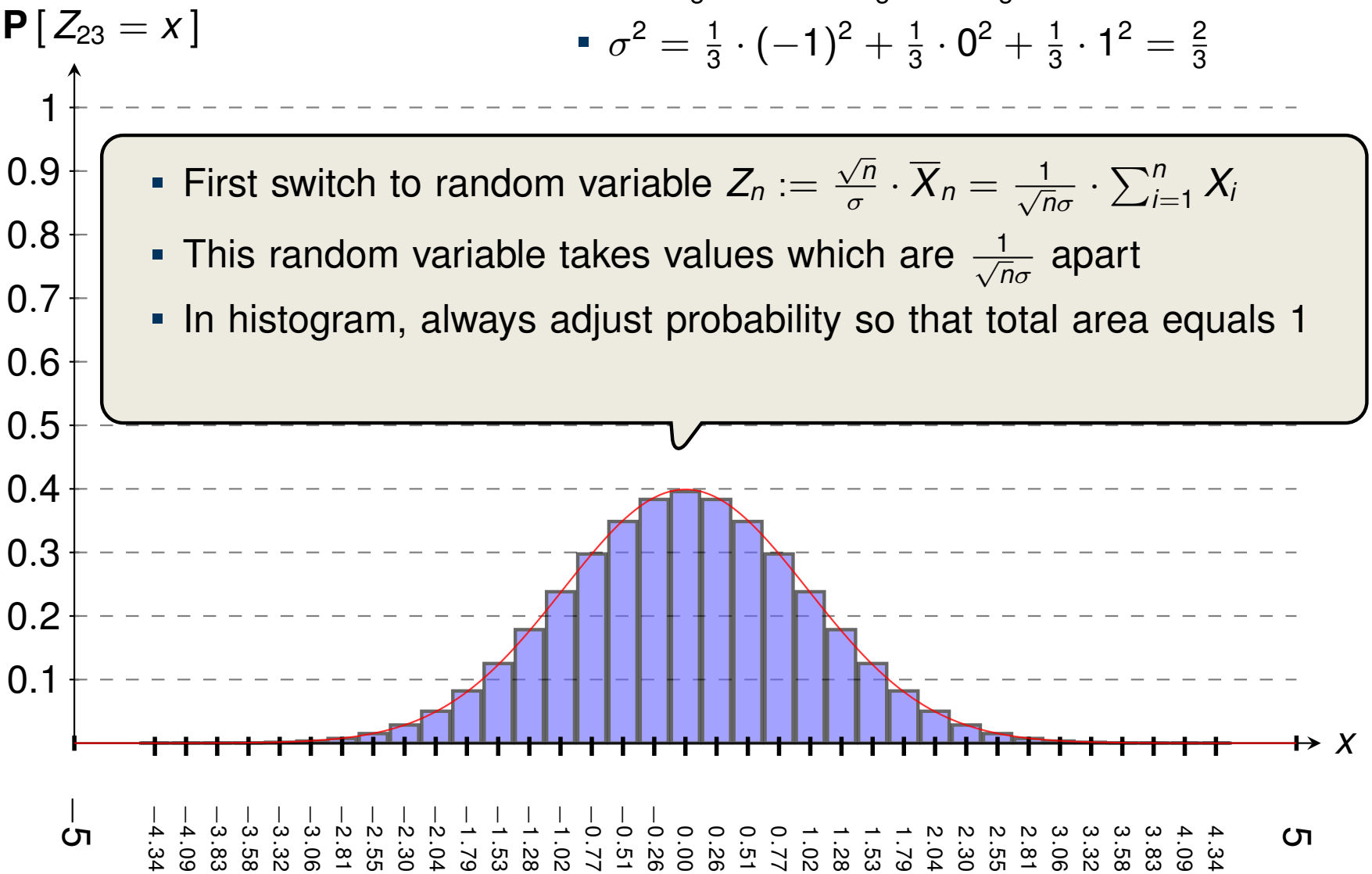

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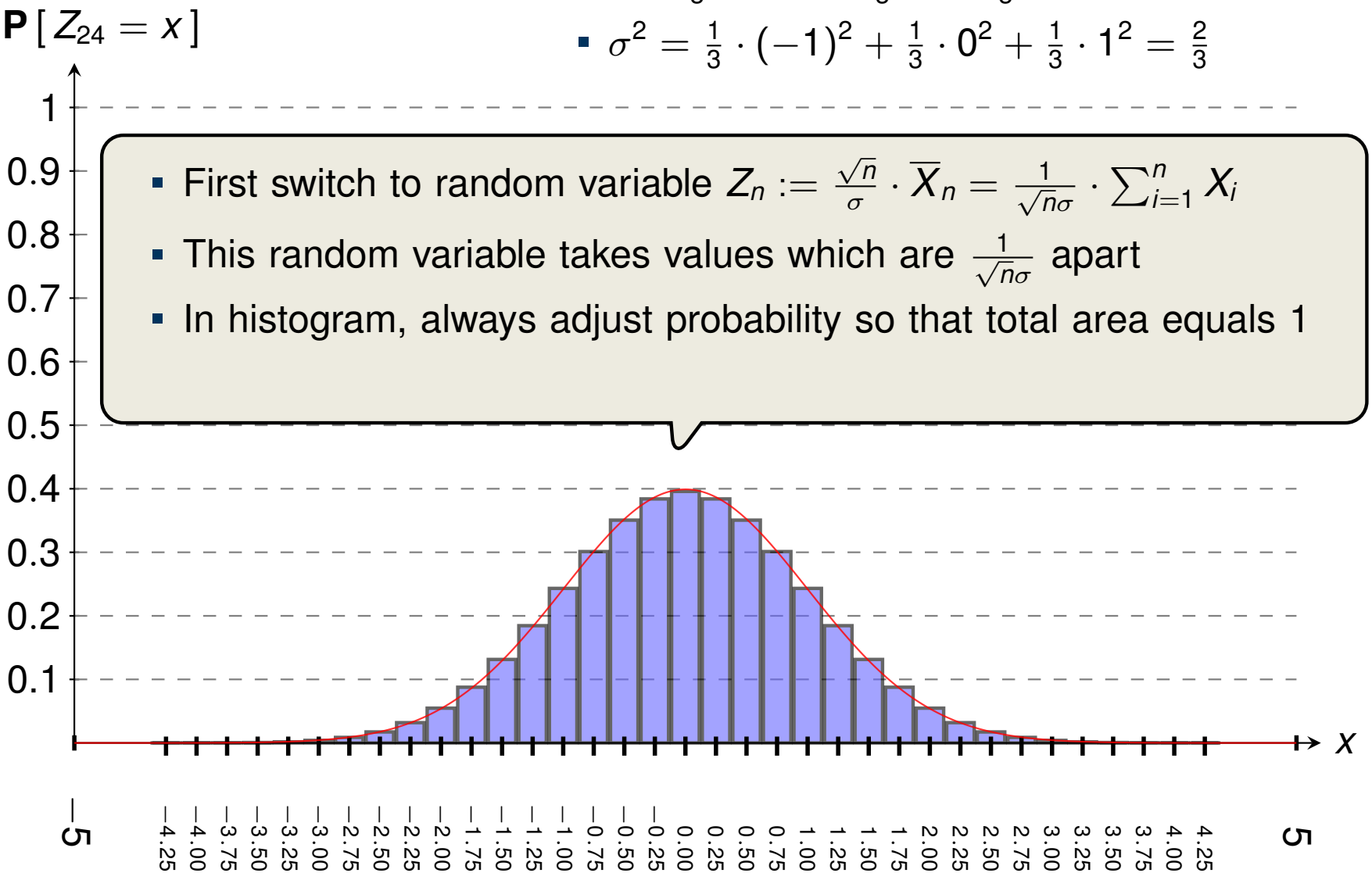

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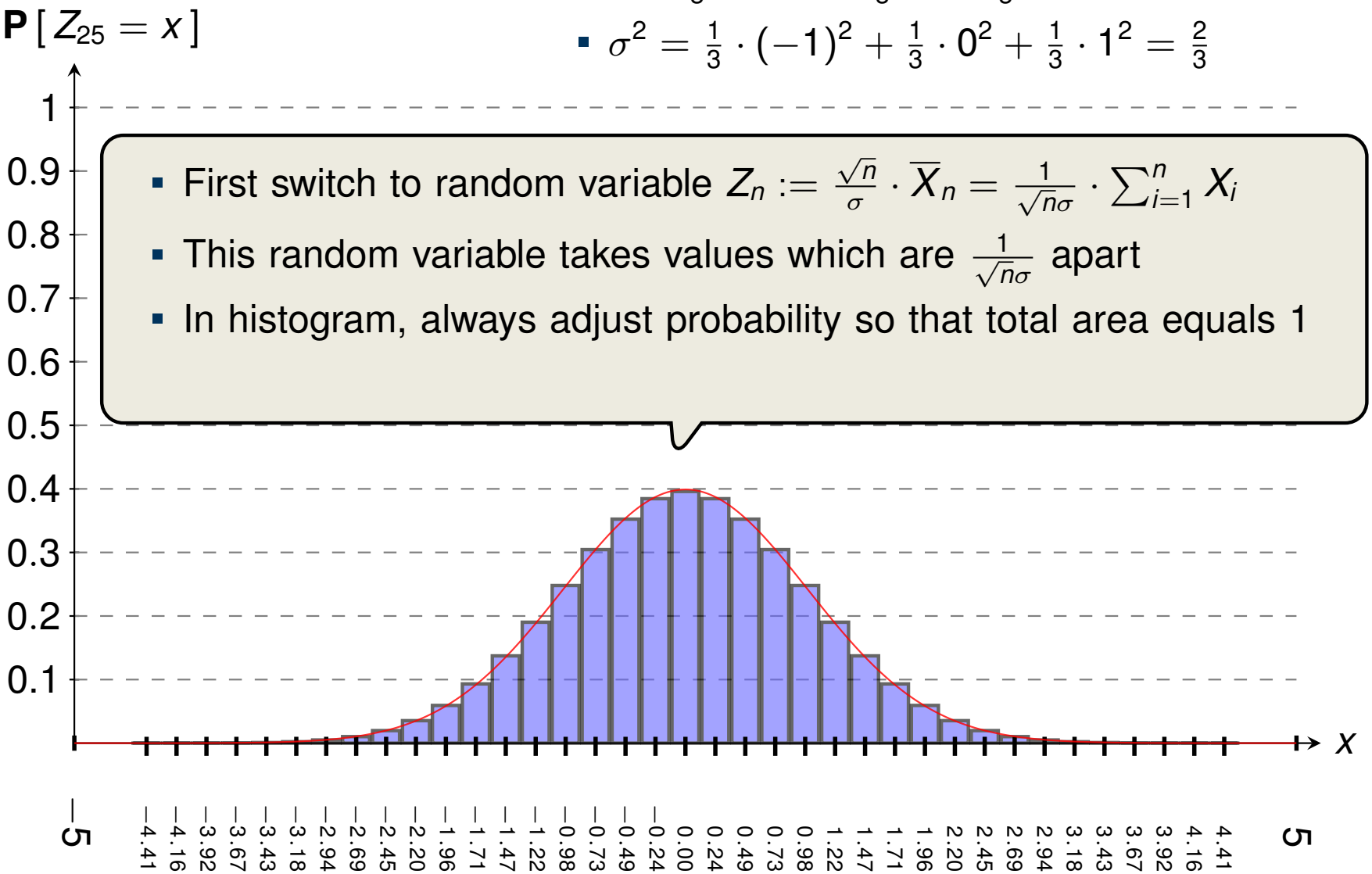

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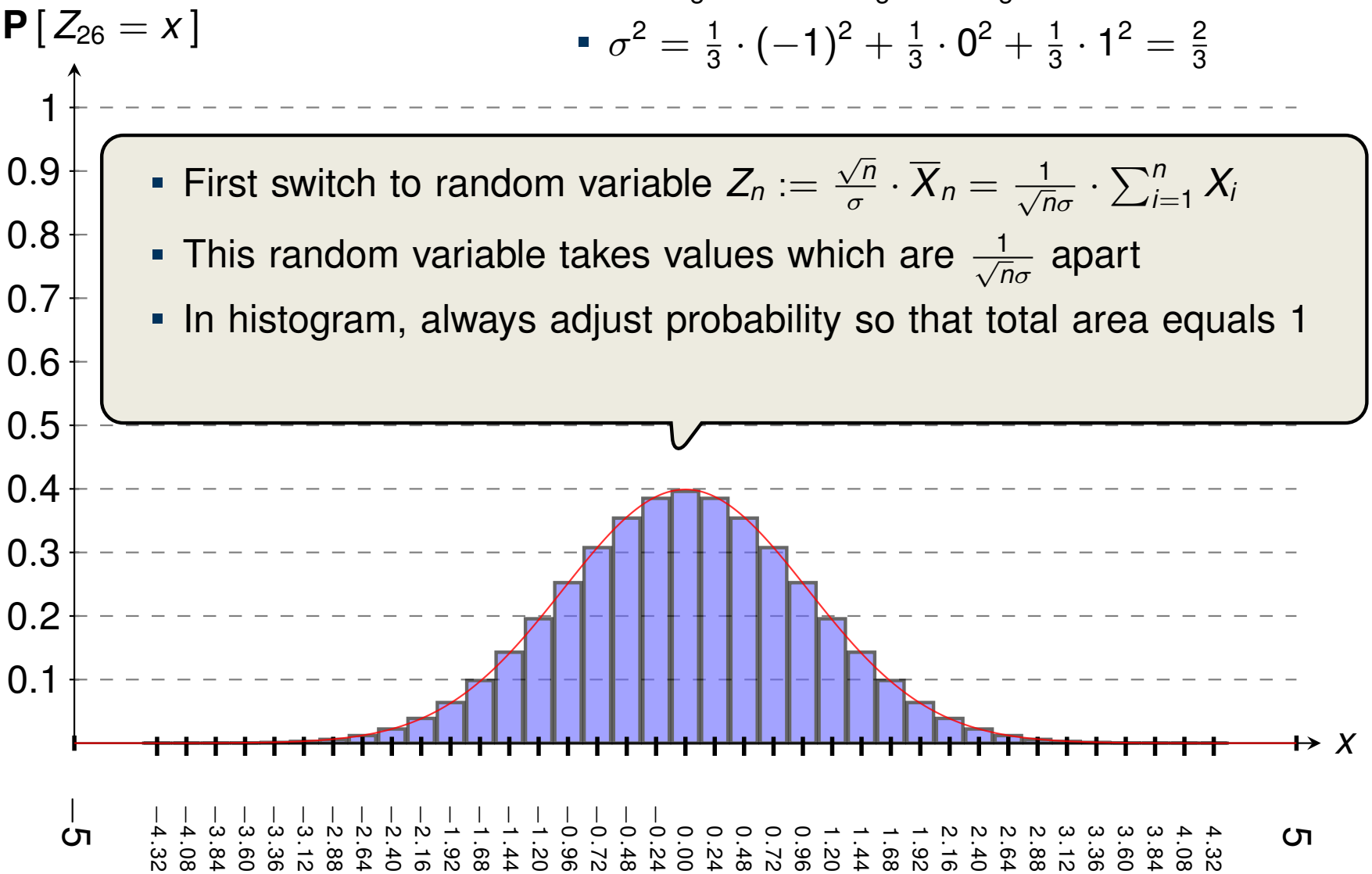

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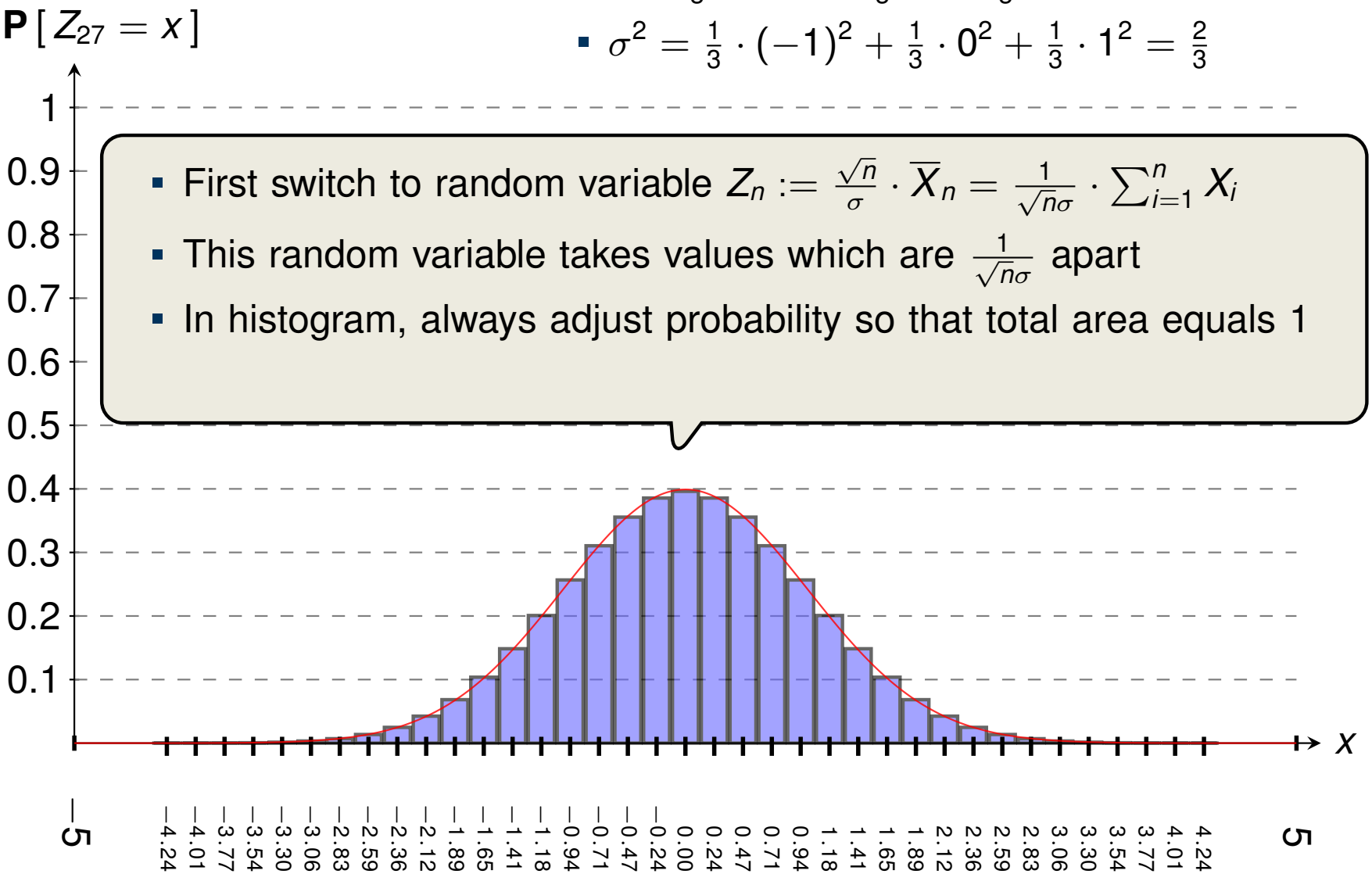

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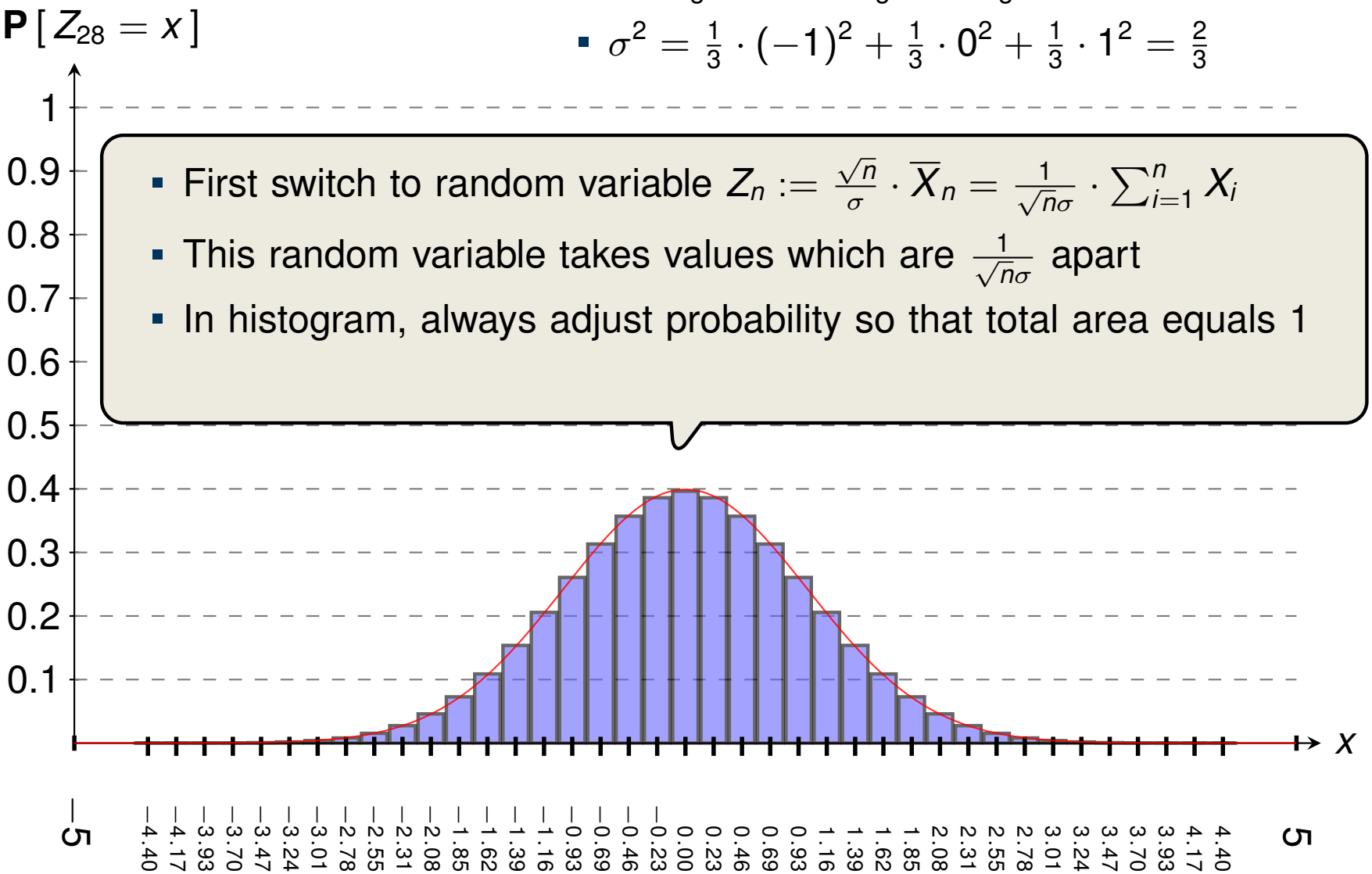

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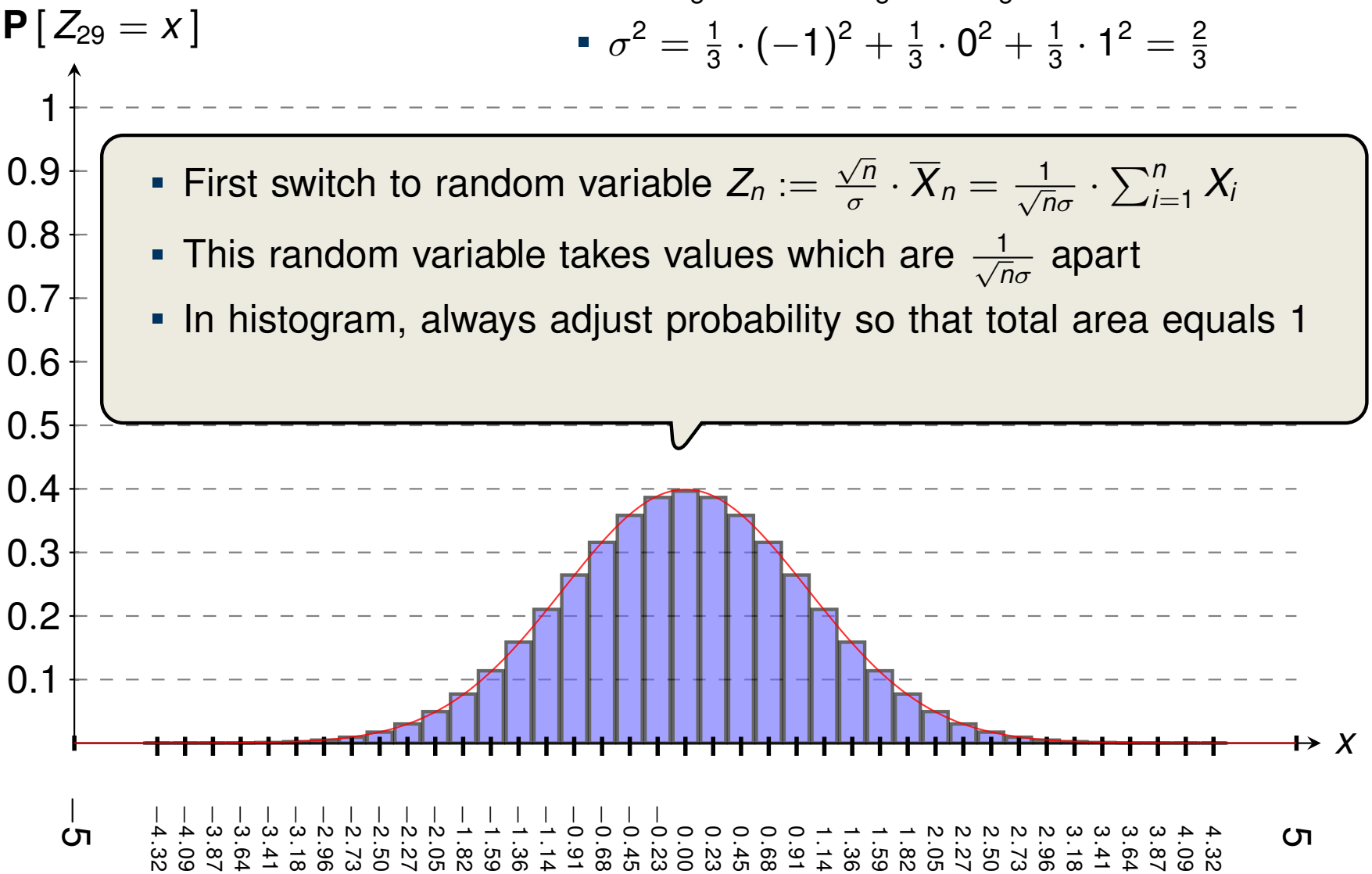

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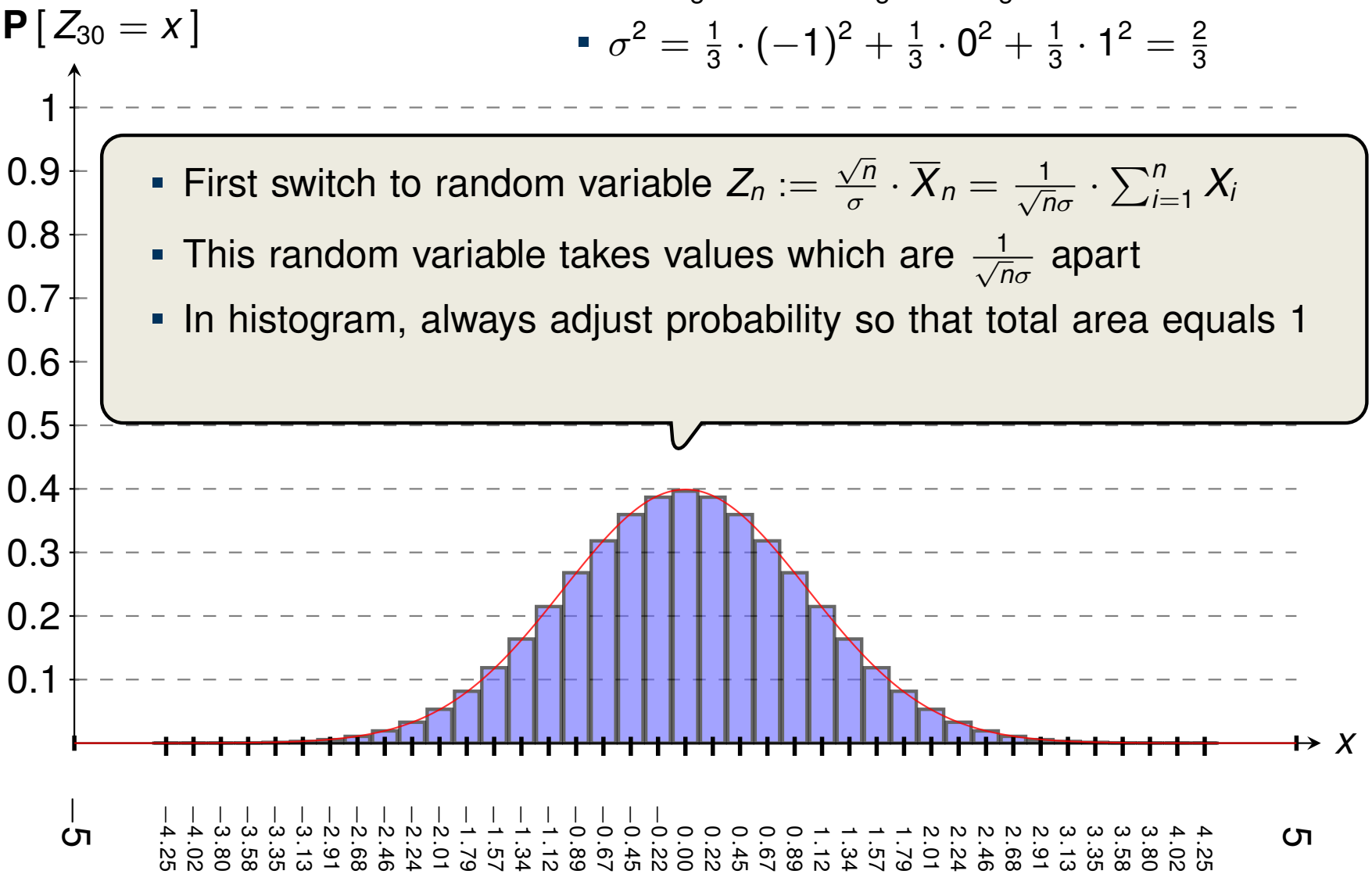

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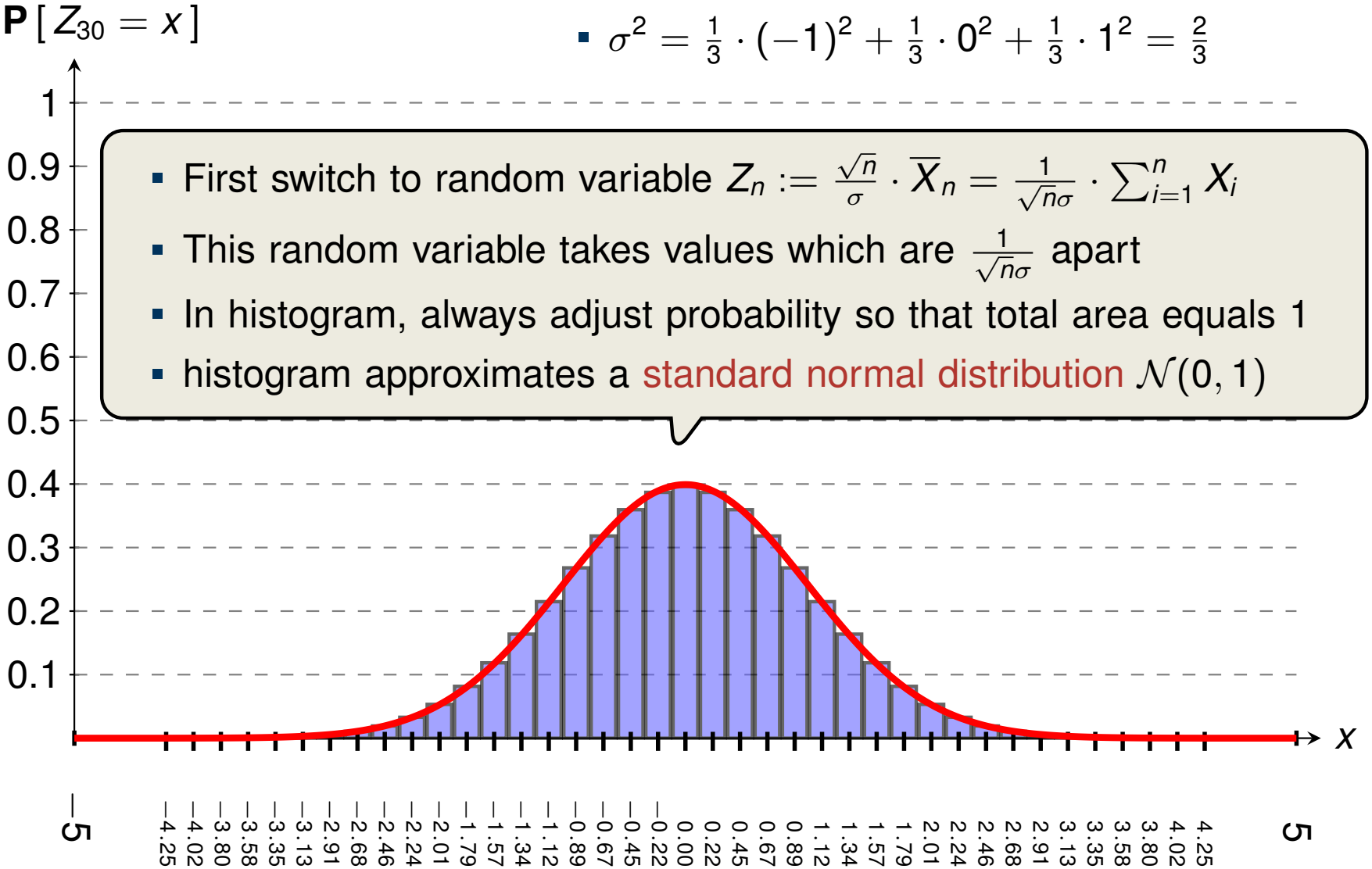

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- This random variable takes values which are $\frac{1}{\sqrt{n}\sigma}$ apart
- In histogram, always adjust probability so that total area equals 1
- histogram approximates a standard normal distribution $\mathcal{N}(0, 1)$

Illustration of CLT (4, Part II) with Standardising

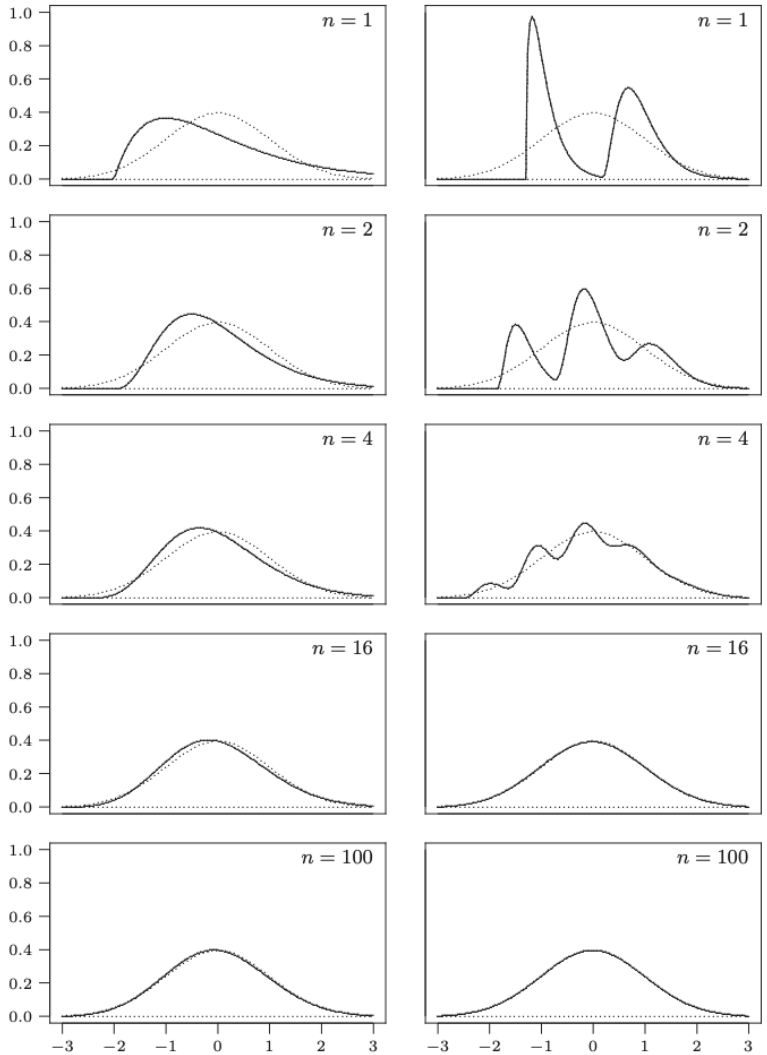

Fig. 14.2. Densities of standardized averages Z_n. Left column: from a gamma density; right column: from a bimodal density. Dotted line: $N(0,1)$ probability density.

Source: Dekking et al., Modern Introduction to Statistics

Outline

Recall: Standard Normal Table

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Source: Ross, Probability 8th ed.

$$Z \sim \mathcal{N}(0,1) \qquad \mathbf{P}[Z \leq x] = \Phi(x)$$

Recall: Standard Normal Table

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.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Source: Ross, Probability 8th ed.

Question: What if we need $\Phi(x)$ for negative x?

$$Z \sim \mathcal{N}(0,1) \qquad \mathbf{P}[Z \leq x] = \Phi(x)$$

Recall: Standard Normal Table

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
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Question: What if we need $\Phi(x)$ for negative x?

$$Z \sim \mathcal{N}(0,1) \qquad \mathbf{P}[Z \leq x] = \Phi(x)$$

Due to symmetry of density we have $\Phi(x) = 1 - \Phi(-x)$.

Normal Approximation of the Binomial Distribution

Example 1

Suppose you are attending a multiple-choice exam of 10 questions and you are completely unprepared. Each question has 4 choices, and you are going to pass the exam if you guess at least 6 correct answers. Use the normal approximation to estimate the probability of passing.

Answer

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True value is 0.0197. Error lies in the discretisation!

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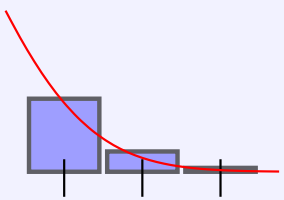

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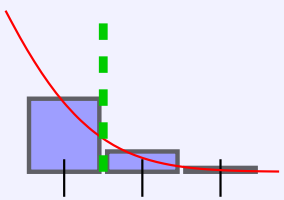
continuity correction: a better approximation is obtained by $\mathbf{P}\left[\sum_{i=1}^{n} X_i \geq 5.5\right] \rightsquigarrow \approx 0.0143$



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A “Reverse” Application of the CLT

Example 2

Suppose we are sequentially loading one container with packets, whose weights are i.i.d. exponential variables with parameter $\lambda = 1/2$. The container has a capacity of 100 weight units. How many packets can we load so that we meet the capacity threshold with at least .95 probability?

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- Using a normal table (looking for value 0.95) yields: $\frac{100-2n}{2\sqrt{n}} = 1.645$.

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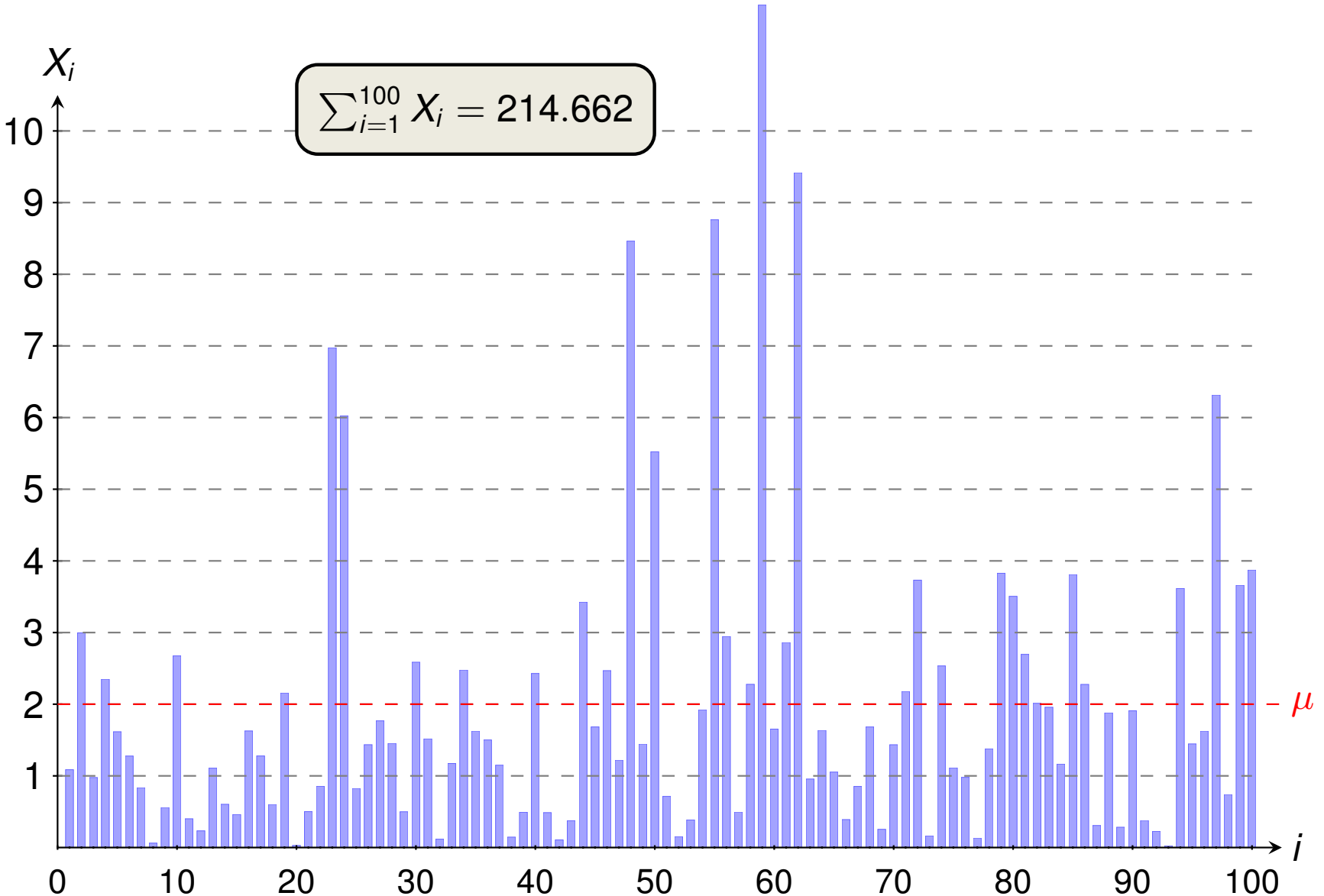
$\Rightarrow$ Solving the quadratic gives $n \leq 39.6$ (so $n \leq 39$)

A Sample of 100 Exponential Random Variables $Exp(1/2)$



Comparison between Markov, Chebyshev and CLT

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- Chebyshev: $\mathbf{V}[X] = \sum_{i=1}^{100} \mathbf{V}[X_i] = 100 \cdot (1/2)^2 = 25$.

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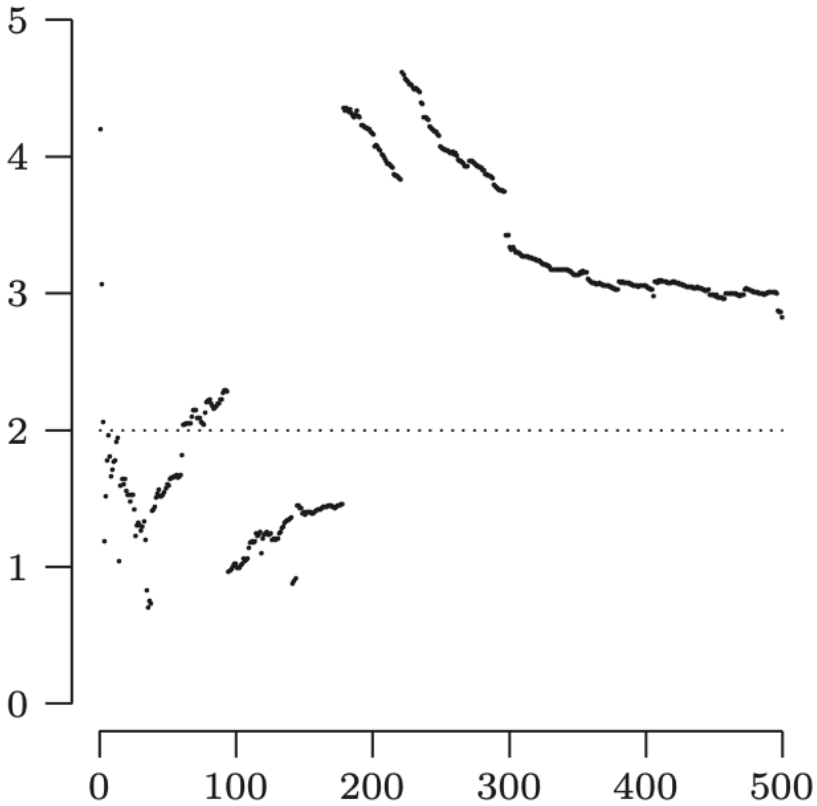
  - In this region, 75 gives a better approximation than 74.5, but for smaller values (e.g., ≤ 63) the continuity corrections gives significantly better results.

A Distribution whose Average does not converge



Cau(2, 1) distribution, Source: Dekking et al., Modern Introduction to Statistics

The Cauchy distribution has "too heavy" tails (no expectation), in particular the average does not converge.

Introduction to Probability

Lecture 10: Estimators (Part I)

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

Easter 2026

Compiled: May 19, 2026 at 18:35

Outline

Introduction

Setting: We can take random samples in the form of i.i.d. random variables $X_1, X_2, \ldots, X_n$ from an unknown distribution.

- Taking enough samples allows us to estimate the mean (WLLN, CLT)
- Using indicator variables, we can estimate $\mathbf{P}[X \leq a]$ for any $a \in \mathbb{R}$
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Definition of Empirical Distribution Function (Empirical CDF)

Let $X_1, X_2, \ldots, X_n$ be i.i.d. samples, and F be the corresponding distribution function. For any $a \in \mathbb{R}$, define

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Thus by taking enough samples, we can estimate the entire distribution (including its expectation and variance).

Empirical Distribution Functions (Example 1/2)

Example 1

Consider throwing an unbiased dice 8 times, and let the realisation be:

$$(x_1, x_2, \ldots, x_8) = (4, 1, 4, 3, 1, 6, 4, 1).$$

What is the Empirical Distribution Function $F_8(a)$?

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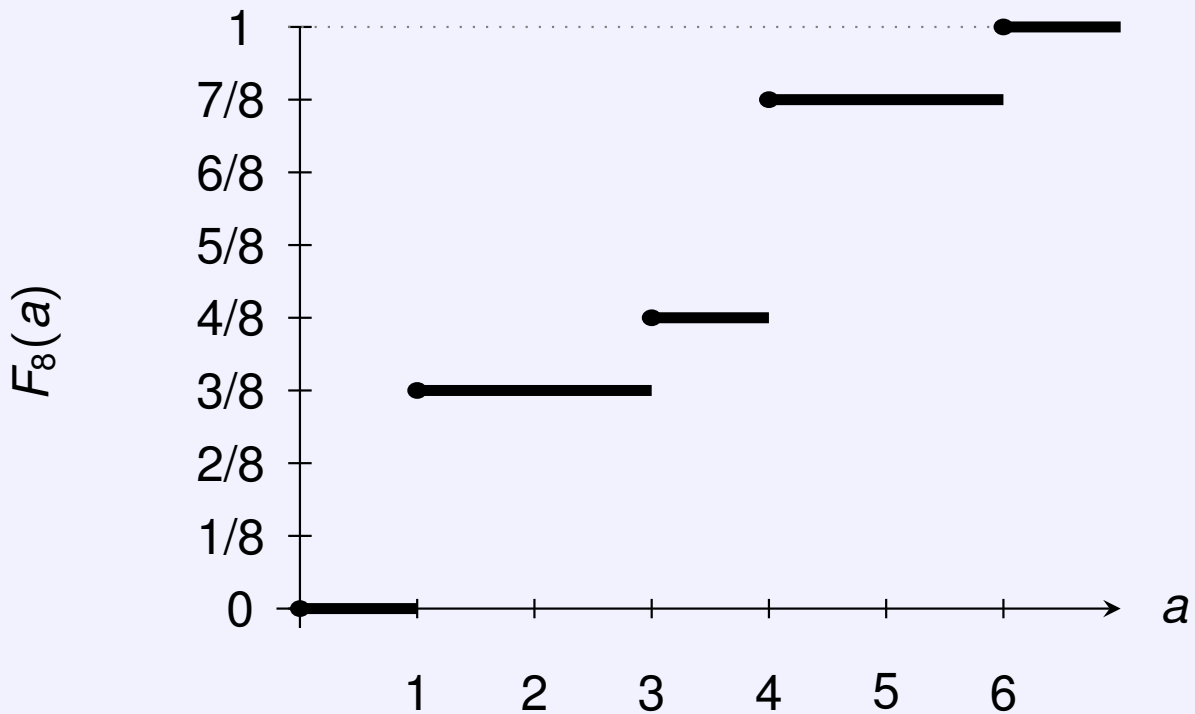

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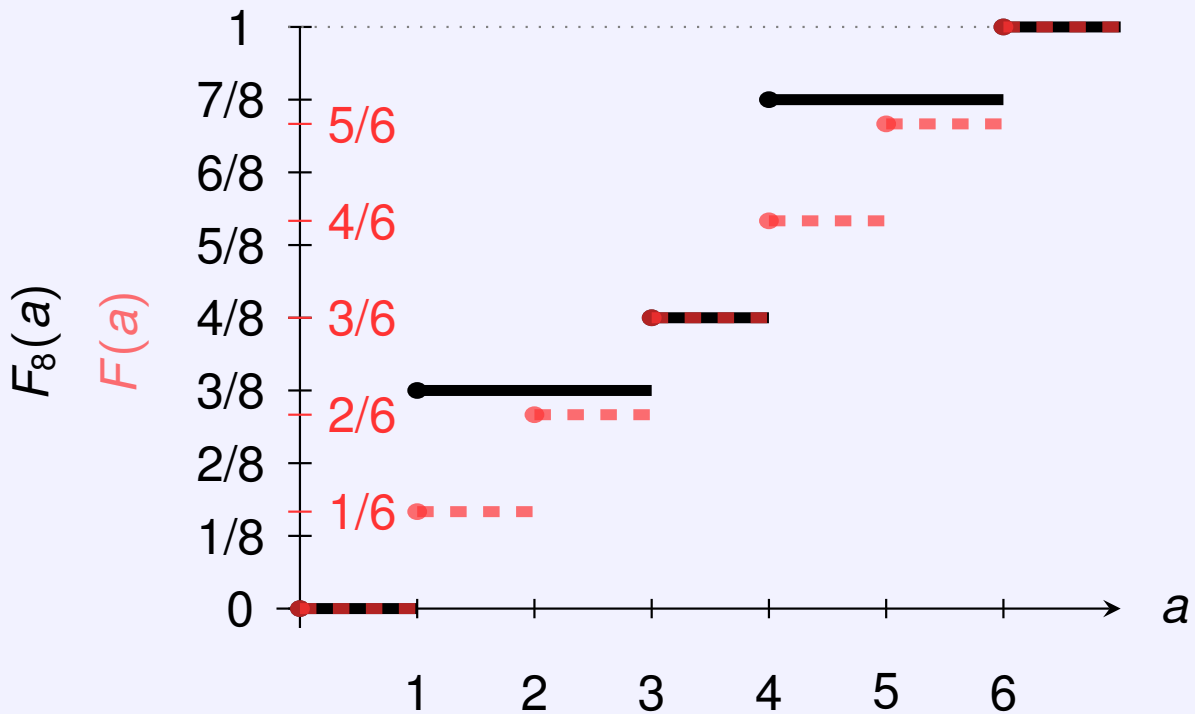
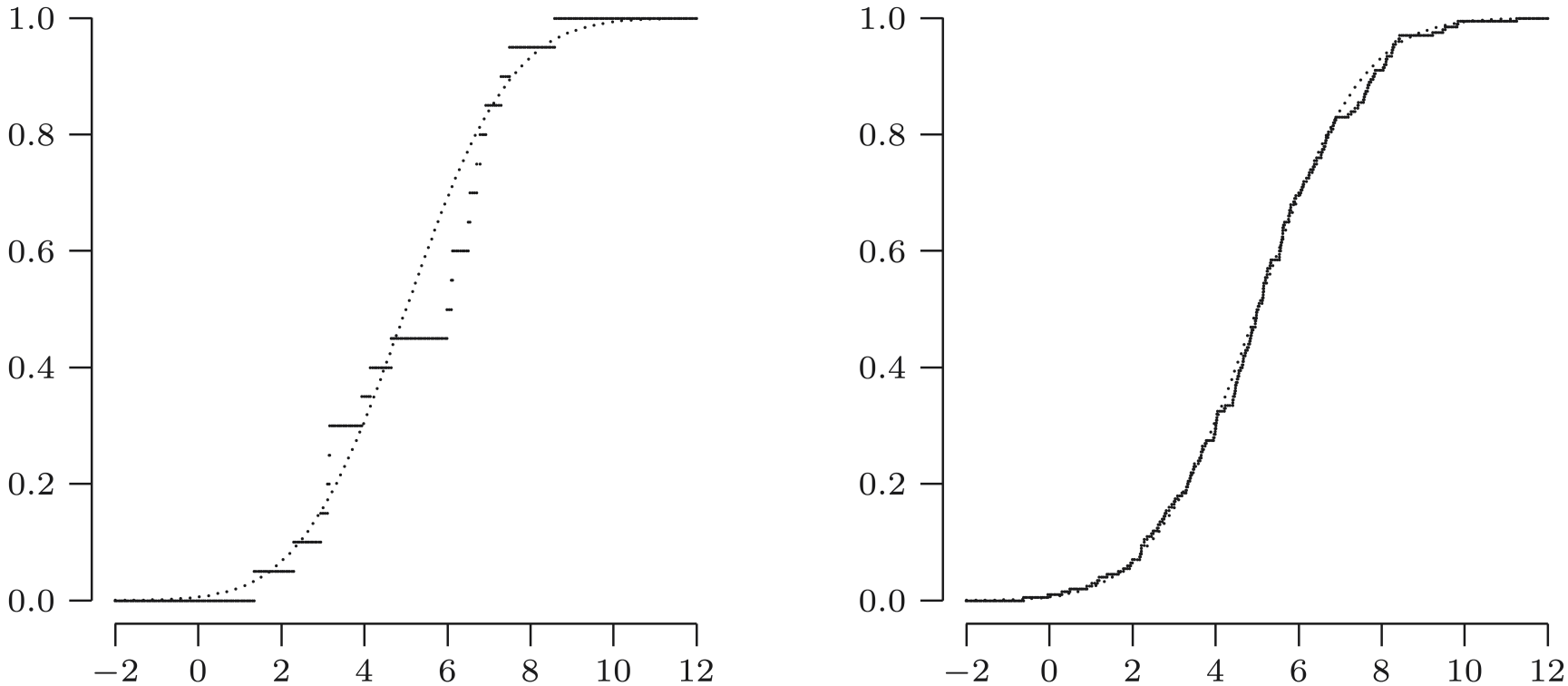


Empirical Distribution Functions (Example 2/2)



Source: Modern Introduction to Statistics

Figure: Empirical Distribution Functions of samples from a Normal Distribution $\mathcal{N}(5, 4)$ ($n = 20$ left, $n = 200$ right)

An Example of an Estimation Problem

Scenario

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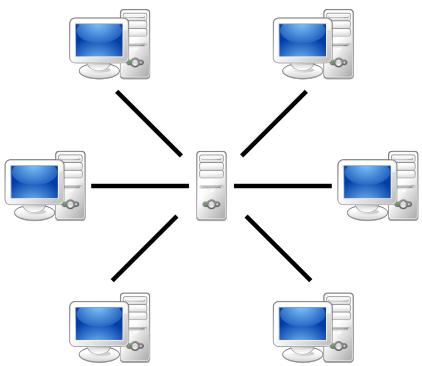

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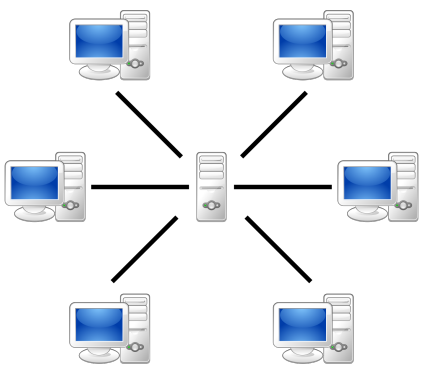

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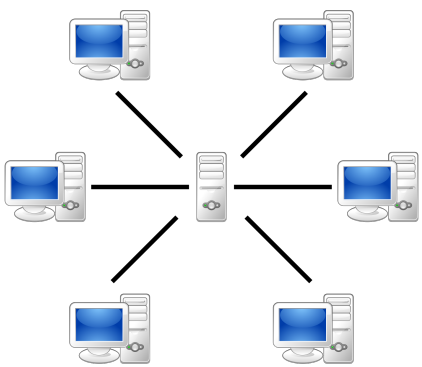

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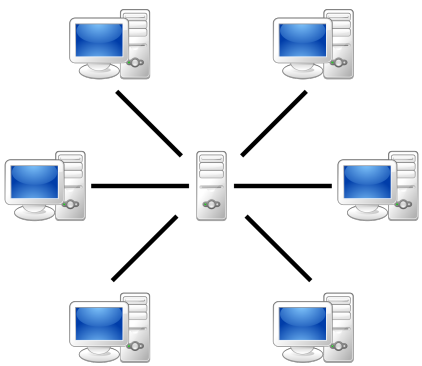

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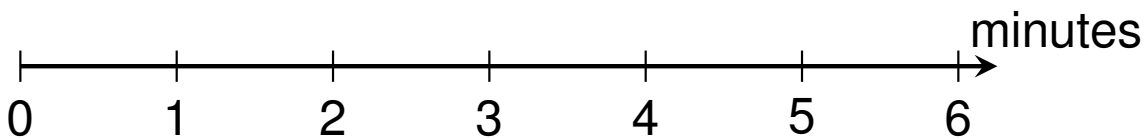

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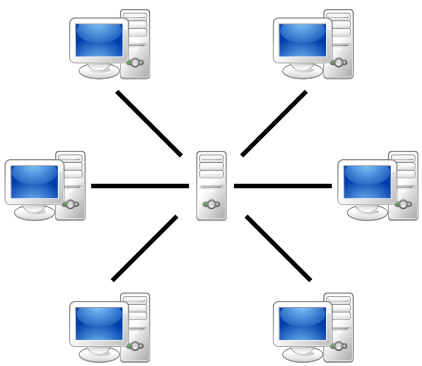

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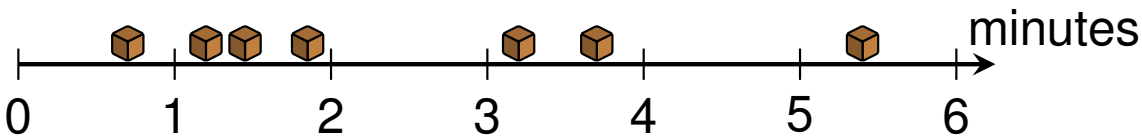

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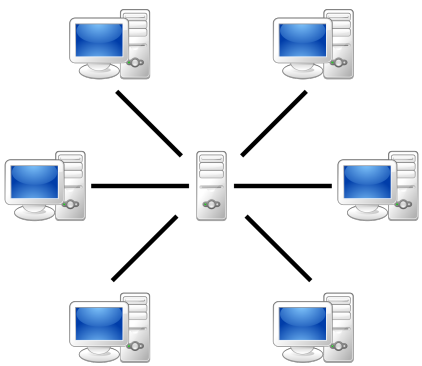

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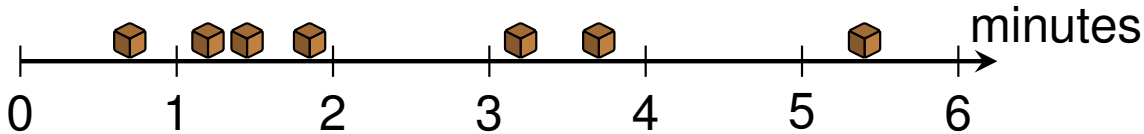

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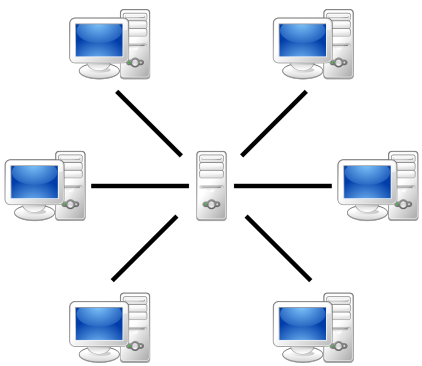

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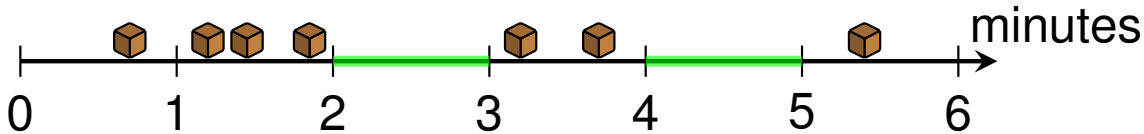

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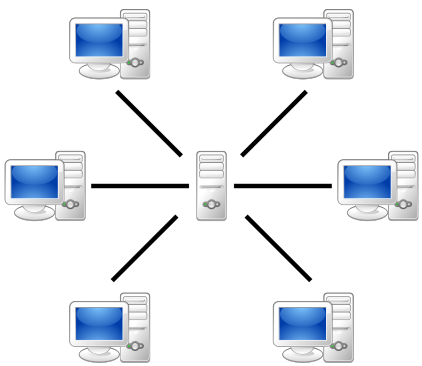

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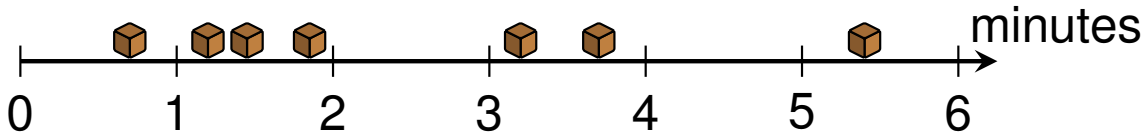

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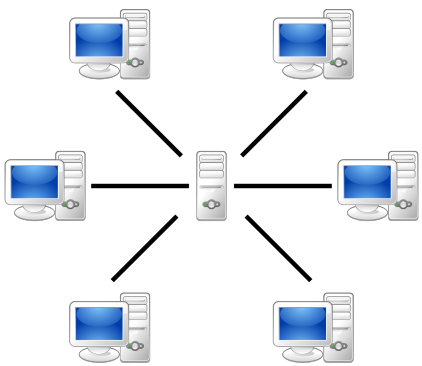

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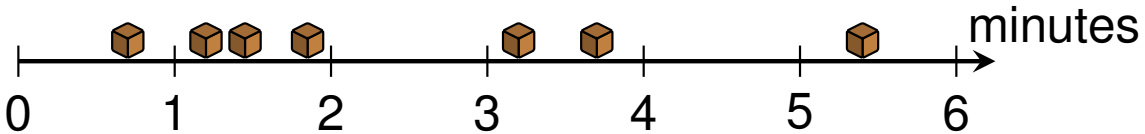

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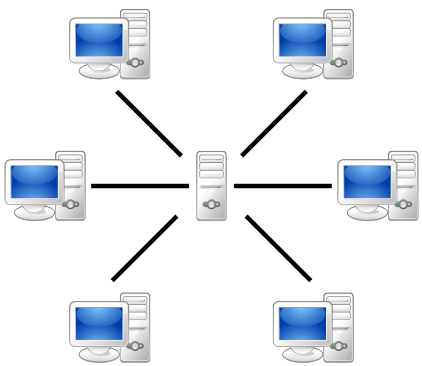

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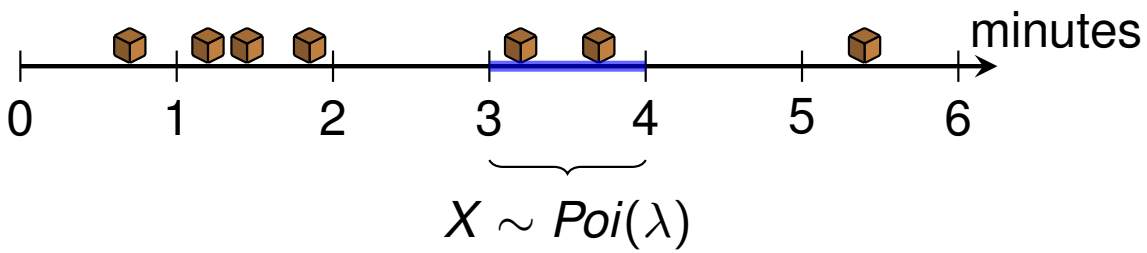

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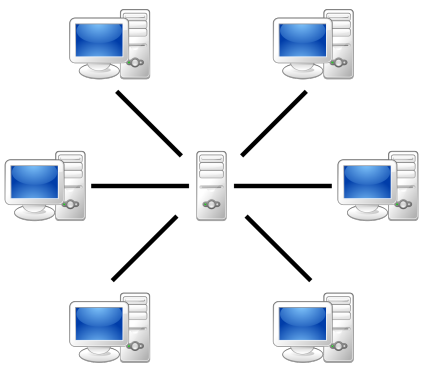

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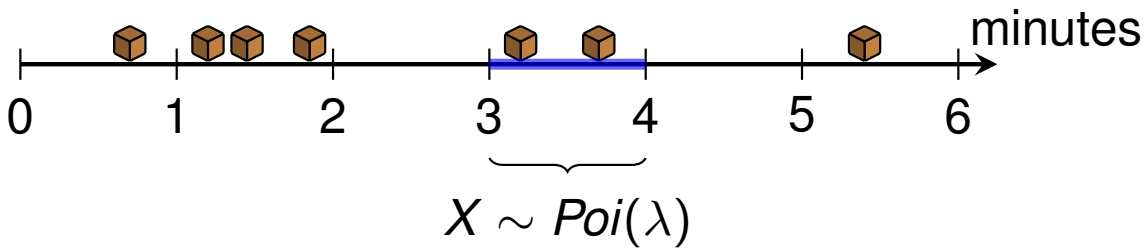

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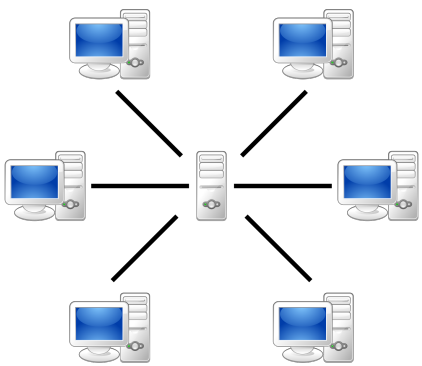

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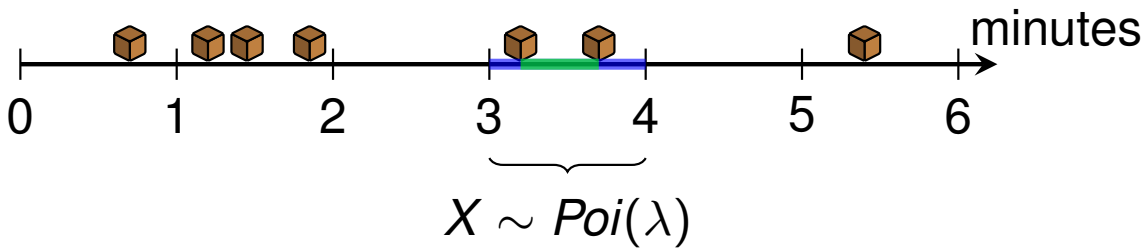

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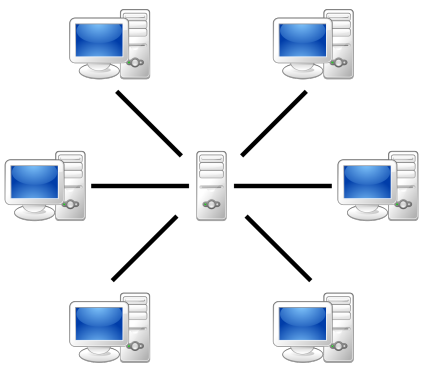

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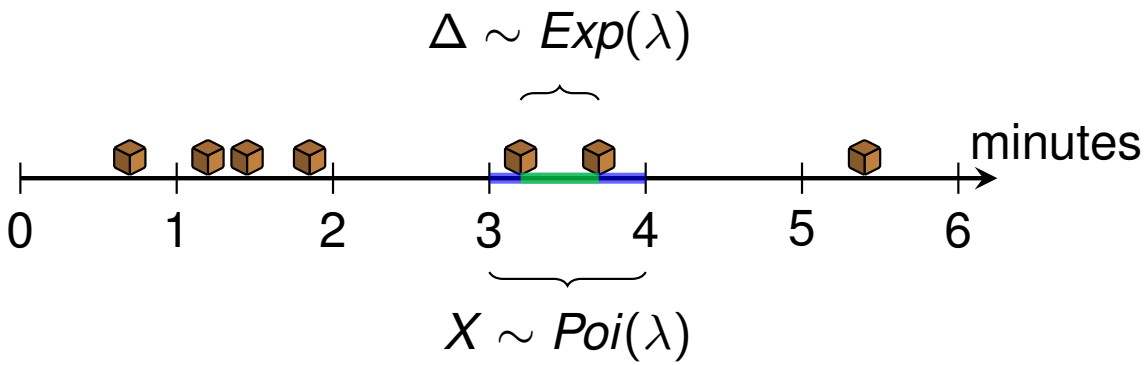

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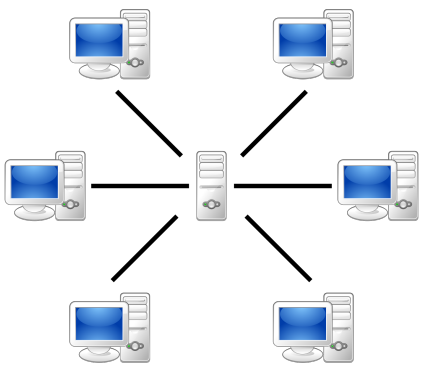

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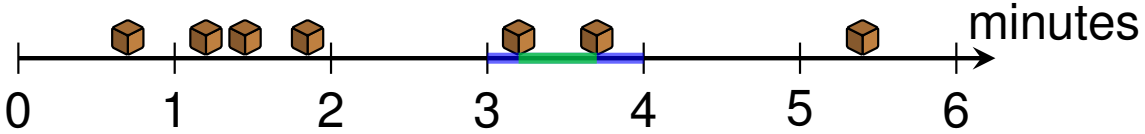
Waiting Time (Lecture 5, Slide 22)

$$\Delta \sim Exp(\lambda)$$



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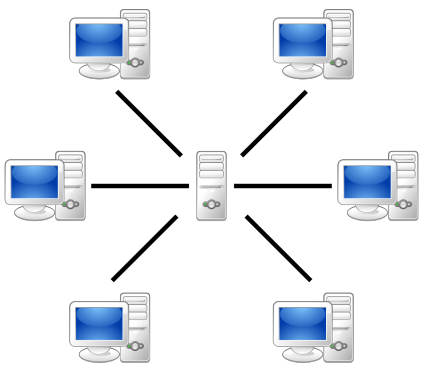
Estimator for λ



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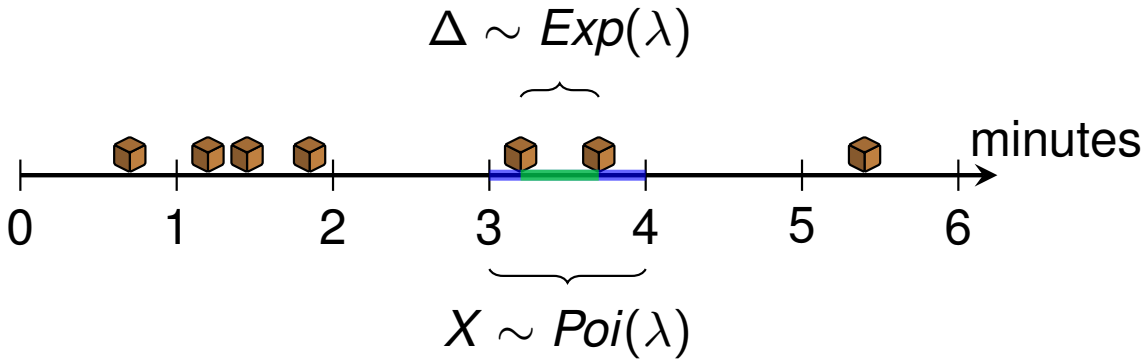

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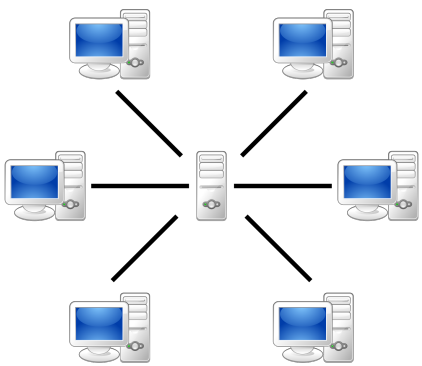

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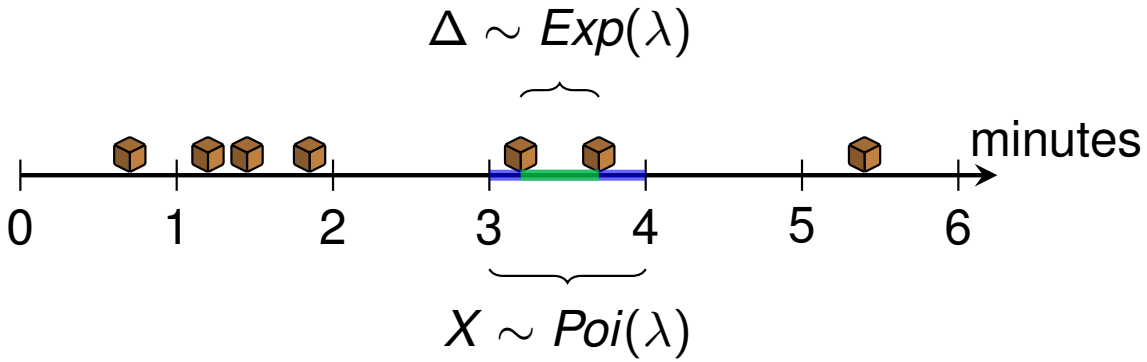

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An estimate is a value t that only depends on the dataset $x_1, x_2, \ldots, x_n$, i.e.,

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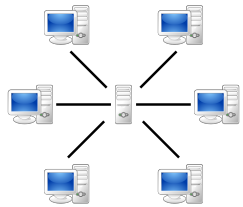
Questions:

- What makes an estimator suitable? unbiased (later: mean squared error)
- Does an unbiased estimator always exist? How to compute it?
- If there are several unbiased estimators, which one to choose?

Outline

Example: Arrival of Packets (1/3)



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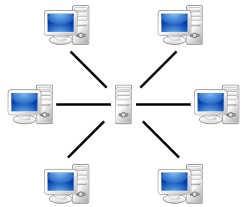

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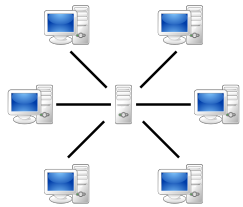

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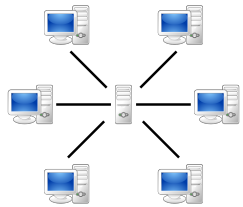

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This suggests to estimate $e^{-\lambda}$ by $e^{-\overline{X}_n}$. Hence our estimator is

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Behaviour of the Estimators

- Suppose we have $n = 30$ and we want to estimate $e^{-\lambda}$
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- Consider the two estimators $h_1(X_1, \ldots, X_n)$ and $h_2(X_1, \ldots, X_n)$.

How **good** are these two estimators?

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For most values of λ, both estimators will never return the exact value of $e^{-\lambda}$ on the basis of 30 observations.

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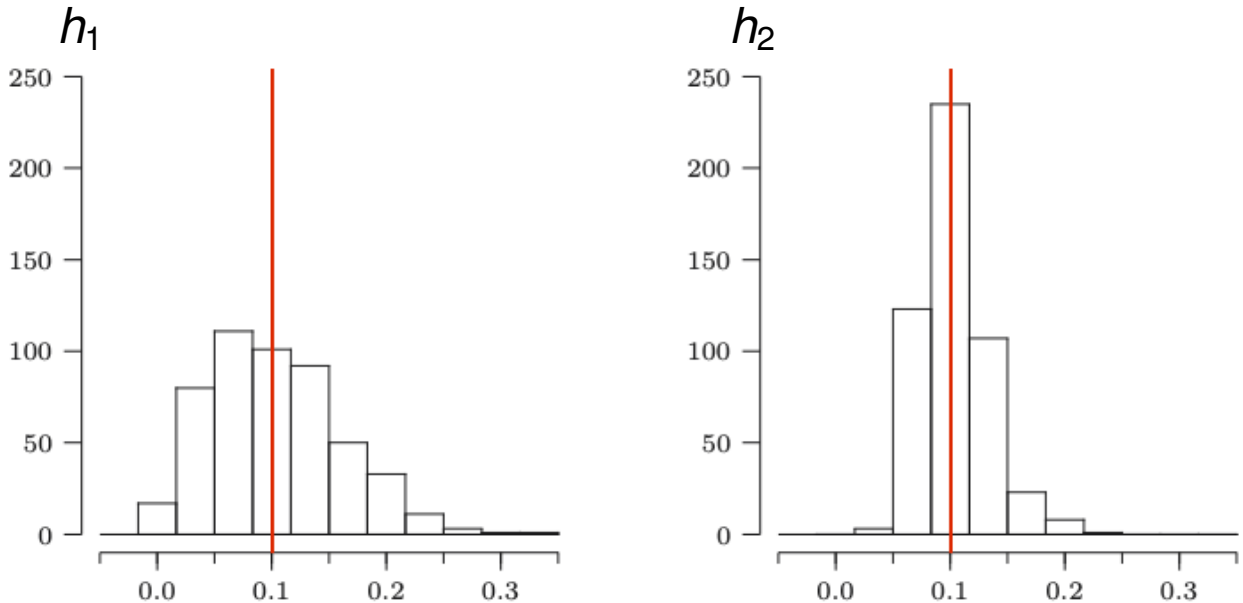

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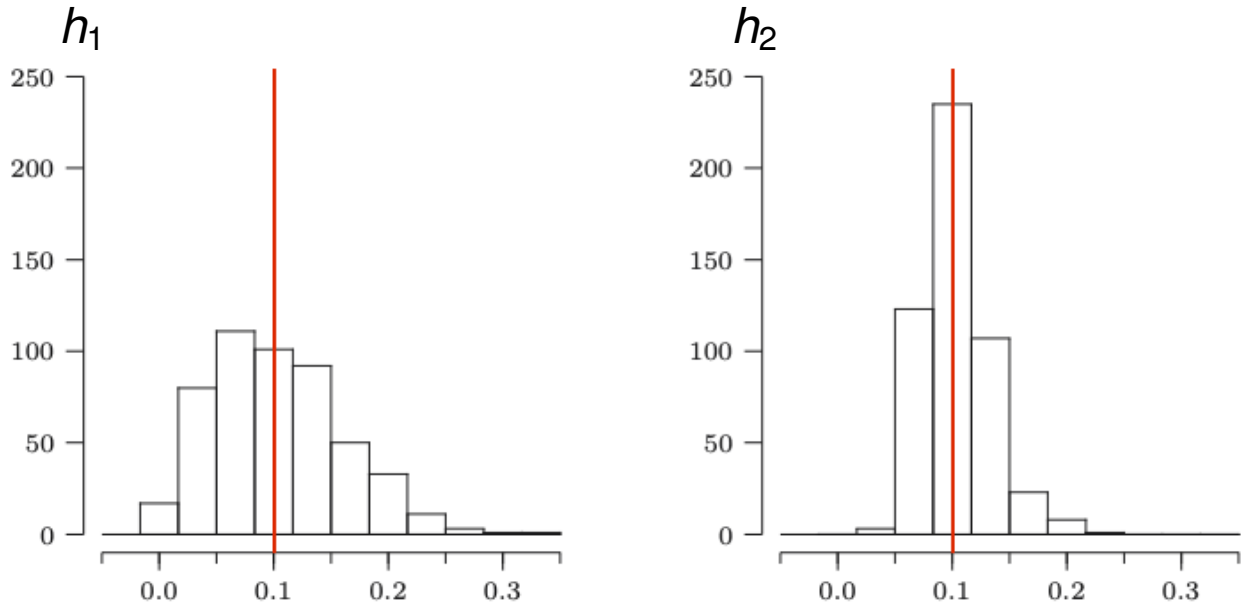

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Both estimators concentrate around the true value 0.1, but the second estimator appears to be more concentrated.

Unbiased Estimators and Bias

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An estimator T is called an unbiased estimator for the parameter θ if

$$\mathbf{E}[T] = \theta,$$

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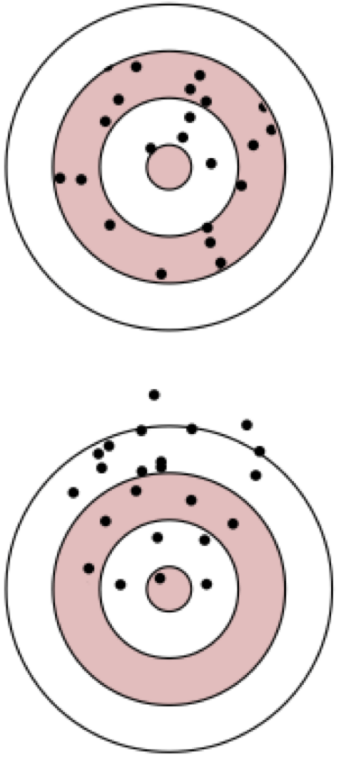

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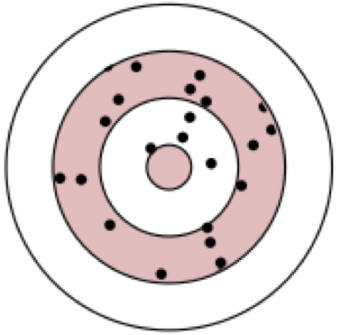

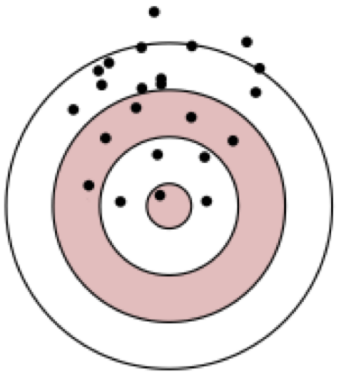

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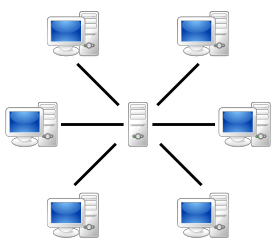
Which of the two estimators h_1, h_2 are unbiased?



Analysis of the Bias of the First Estimator

Example 4a

Is $h_1(X_1, X_2, \ldots, X_n) = \frac{Y_1+Y_2+\cdots+Y_n}{n}$ an unbiased estimator for $e^{-\lambda}$?

Answer

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Bias of the Second Estimator (and Jensen's Inequality)

Example 4b

Is $h_2(X_1, X_2, \ldots, X_n) = e^{-\overline{X}_n}$ an unbiased estimator for $e^{-\lambda}$?

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Is $h_2(X_1, X_2, \ldots, X_n) = e^{-\overline{X}_n}$ an unbiased estimator for $e^{-\lambda}$?

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No! (recall: $\mathbf{E}\left[X^2\right] \geq \mathbf{E}\left[X\right]^2$)

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Jensen's Inequality

For any random variable X, and any convex function $g : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbf{E}[g(X)] \geq g(\mathbf{E}[X]).$$

If g is strictly convex and X is not constant, then the inequality is strict.

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- This follows by Jensen's inequality, and the inequality is strict since $g : z \mapsto e^{-z}$ is strictly convex and $\overline{X}_n$ is not constant.
- Thus $h_2(X_1, X_2, \ldots, X_n)$ is not unbiased – it has positive bias.

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Asymptotic Bias of the Second Estimator (non-examinable)

Example 4c

$\mathbf{E}[\,h_2(X_1,\ldots,X_n)\,] \overset{n\to\infty}{\longrightarrow} e^{-\lambda}$ (hence it is asymptotically unbiased).

Answer

- Recall $h_2(X_1,\ldots,X_n) = e^{-\overline{X}_n}$. For any $0 \le k \le n$,

$$\mathbf{P}\left[\,h_2(X_1,\ldots,X_n) = e^{-k/n}\,\right] = \mathbf{P}\left[\,\sum_{i=1}^{n} X_i = k\,\right] = \mathbf{P}[\,Z = k\,],$$

where $Z \sim Pois(n\cdot\lambda)$ (since $Pois(\lambda_1) + Pois(\lambda_2) = Pois(\lambda_1+\lambda_2)$)

$$\Rightarrow \qquad \mathbf{P}\left[\,h_2(X_1,\ldots,X_n) = e^{-k/n}\,\right] = \frac{e^{-n\lambda}\cdot(n\lambda)^k}{k!}$$

$$\Rightarrow \qquad \mathbf{E}[\,h_2(X_1,\ldots,X_n)\,] = \sum_{k=0}^{\infty} e^{-n\lambda}\cdot\frac{(n\lambda^k)}{k!}\cdot e^{-k/n}$$

By LOTUS

$$= e^{-n\lambda}\cdot e^{n\lambda e^{-1/n}} \sum_{k=0}^{\infty} e^{-n\lambda e^{-1/n}}\cdot\frac{(n\lambda e^{-1/n})^k}{k!}$$

$$= e^{-n\lambda\cdot(1-e^{-1/n})}\cdot 1$$

since $e^x = 1 + x + O(x^2)$ for small x

$$\overset{n\to\infty}{\approx} e^{-n\lambda\cdot(1-1+1/n+O(1/n^2))} = e^{-\lambda+O(\lambda/n)}.$$

Hence in the limit, the positive bias of h_2 diminishes.

Outline

Unbiased Estimator for Expectation and Variance

Unbiased Estimators for Expectation and Variance

Let $X_1, X_2, \ldots, X_n$ be identically distributed samples from a distribution with finite expectation μ and finite variance σ^2.

- Then

$$\overline{X}_n := \frac{X_1 + X_2 + \cdots + X_n}{n}$$

  is an unbiased estimator for μ.

- Furthermore, for $n \geq 2$,

$$S_n = S_n(X_1, \ldots, X_n) := \frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2$$

  is an unbiased estimator for σ^2.

Example 5

We need to prove: $\mathbf{E}[S_n] = \sigma^2$.

Answer

Multiplying by $n-1$ yields:

$$
\begin{aligned}
(n-1)\cdot S_n &= \sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2 \\
&= \sum_{i=1}^{n}\left(X_i - \mu + \mu - \overline{X}_n\right)^2 \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 + \sum_{i=1}^{n}\left(\overline{X}_n - \mu\right)^2 - 2\sum_{i=1}^{n}(X_i-\mu)\left(\overline{X}_n-\mu\right) \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 + n\left(\overline{X}_n - \mu\right)^2 - 2\left(\overline{X}_n-\mu\right)\cdot n\cdot\left(\overline{X}_n-\mu\right) \\
&= \sum_{i=1}^{n}(X_i-\mu)^2 - n\left(\overline{X}_n - \mu\right)^2 .
\end{aligned}
$$

Let us now take expectations:

By Lec. 8, Slide 21: $\mathbf{E}\left[(\overline{X}_n-\mu)^2\right] = \mathbf{V}\left[\overline{X}_n\right] = \sigma^2/n$

$$
\begin{aligned}
(n-1)\cdot\mathbf{E}[S_n] &= \sum_{i=1}^{n}\mathbf{E}\left[(X_i-\mu)^2\right] - n\cdot\mathbf{E}\left[\left(\overline{X}_n-\mu\right)^2\right] \\
&= n\cdot\sigma^2 - n\cdot\sigma^2/n \\
&= (n-1)\cdot\sigma^2 .
\end{aligned}
$$

An Unbiased Estimator may not always exist

Example 6

Suppose that we have one sample $X \sim Bin(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is no unbiased estimator for $1/p$.

Answer

- First a simpler proof which exploits that p might be arbitrarily small
- Intuition: By making p smaller and smaller, we force $\max_{0 \le k \le n} T(k)$, $k \in \{0, 1, \ldots, n\}$ to become bigger and bigger
- Formal Argument:
  - Fix any estimator $T(X)$
  - Define $M := \max_{0 \le k \le n} T(k)$. Then,

$$\begin{aligned} \mathbf{E}[\, T(X) \,] &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot T(k) \\ &\le M \cdot \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = M. \end{aligned}$$

  - Hence this estimator does not work for $p < \frac{1}{M}$, since then $\mathbf{E}[\, T(X) \,] \le M < \frac{1}{p}$ (negative bias!)
- The next proof will work even if $p \in [a, b]$ for $0 < a < b \le 1$.

An Unbiased Estimator may not always exist (cntd. - non-examinable)

Example 6 (cntd.)

Suppose that we have one sample $X \sim Bin(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is no unbiased estimator for $1/p$.

Answer

- Suppose there exists an unbiased estimator with $\mathbf{E}[\,T(X)\,] = 1/p$.
- Then

$$\begin{aligned} 1 &= p \cdot \mathbf{E}[\,T(X)\,] \\ &= p \cdot \sum_{k=0}^{n} \mathbf{P}[\,X = k\,] \cdot T(k) \\ &= p \cdot \sum_{k=0}^{n} \binom{n}{k} p^k \cdot (1-p)^{n-k} \cdot T(k) \end{aligned}$$

- Last term is a polynomial of degree $n+1$ with constant term zero

  $\Rightarrow p \cdot \mathbf{E}[\,T(X)\,] - 1$ is a (non-zero) polynomial of degree $\leq n+1$

  $\Rightarrow$ this polynomial has at most $n+1$ roots

  $\Rightarrow \mathbf{E}[\,T(X)\,]$ can be equal to $1/p$ for at most $n+1$ values of p, and thus cannot be an unbiased.

Introduction to Probability

Lecture 11: Estimators (Part II)

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

Easter 2026

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Outline

Estimating Population Size (First Model)

- Suppose we have a sample of a few serial numbers (IDs) of some product
- We assume IDs are running from 1 to an unknown parameter N (so $N = \theta$)
- Each of the IDs is drawn without replacement from the discrete uniform distribution over $\{1, 2, \ldots, N\}$

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  - their number must satisfy $n \leq N$

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Construct an unbiased estimator T_1 using the sample mean.

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- Thus we obtain an unbiased estimator by

$$T_1 := 2 \cdot \overline{X}_n - 1.$$

Example: Odd Behaviour of T_1

- Suppose $n = 5$

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- Achieving unbiasedness alone is not a good strategy
- Improvement: find an estimator which always returns a value at least $\max(X_1, X_2, \ldots, X_n)$

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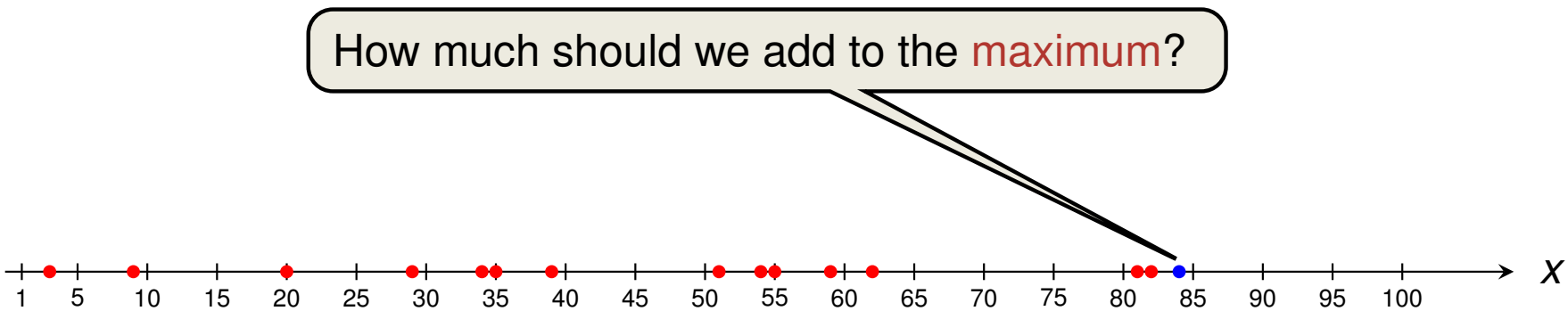

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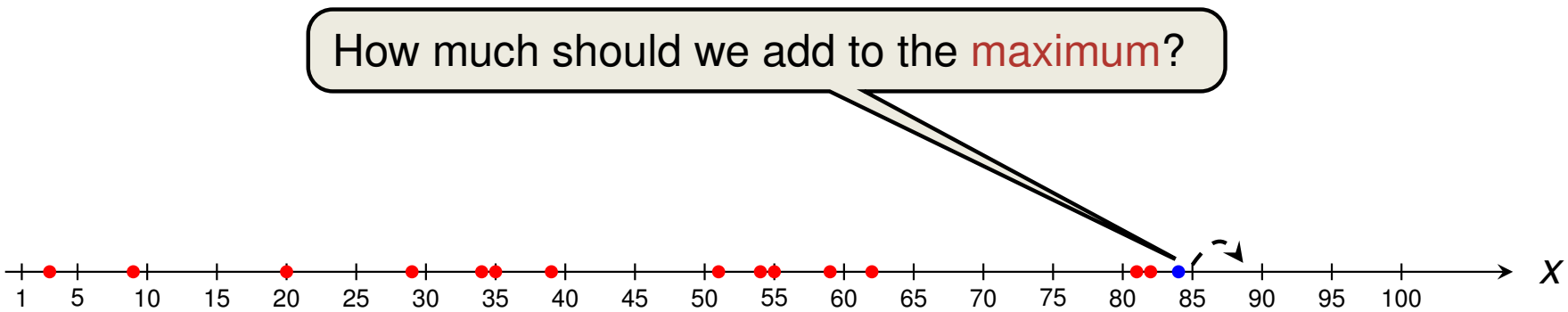

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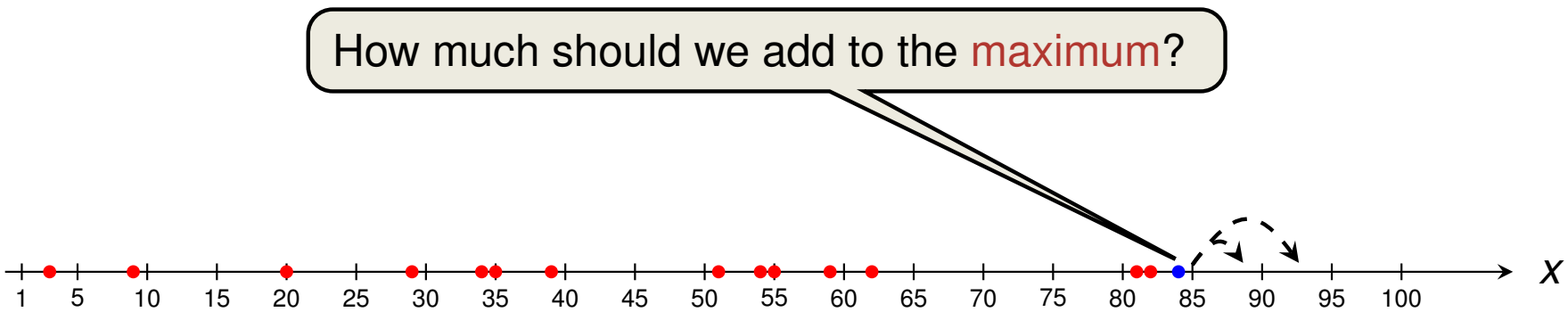

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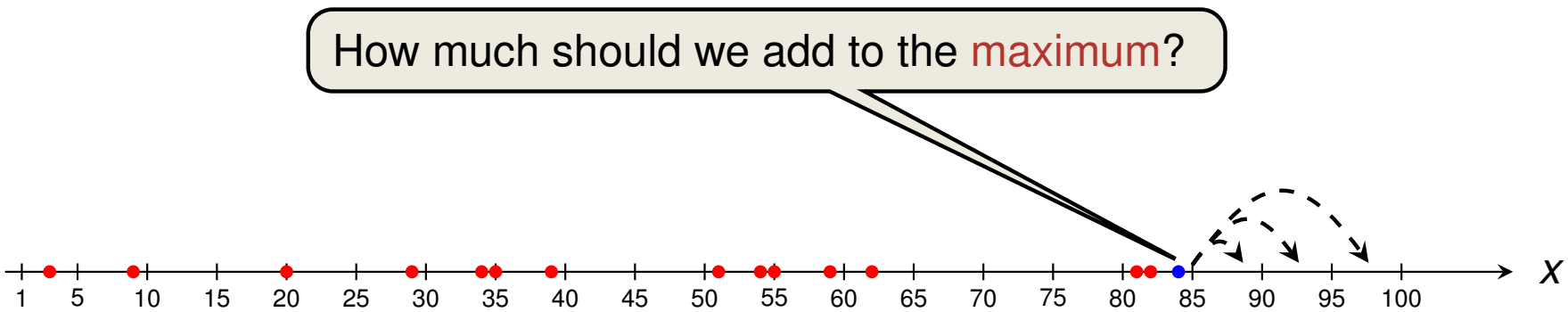

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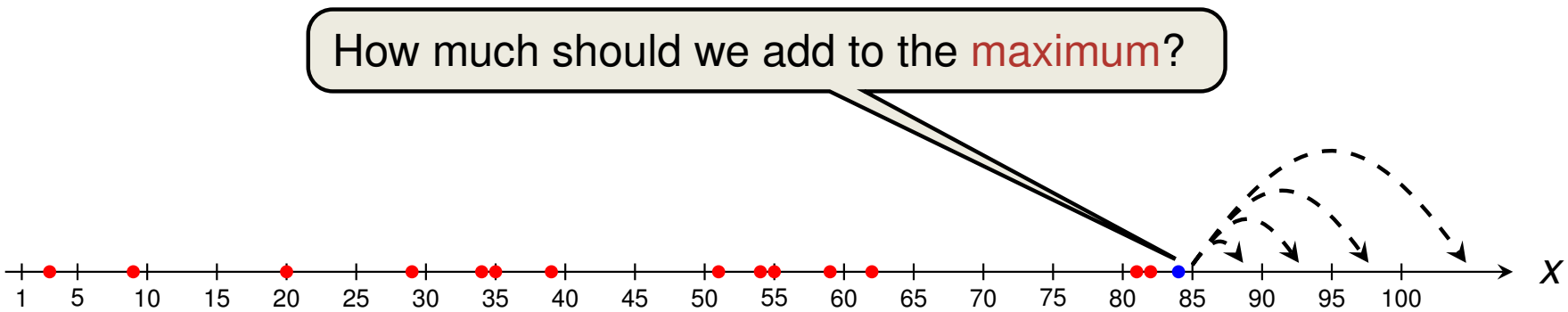

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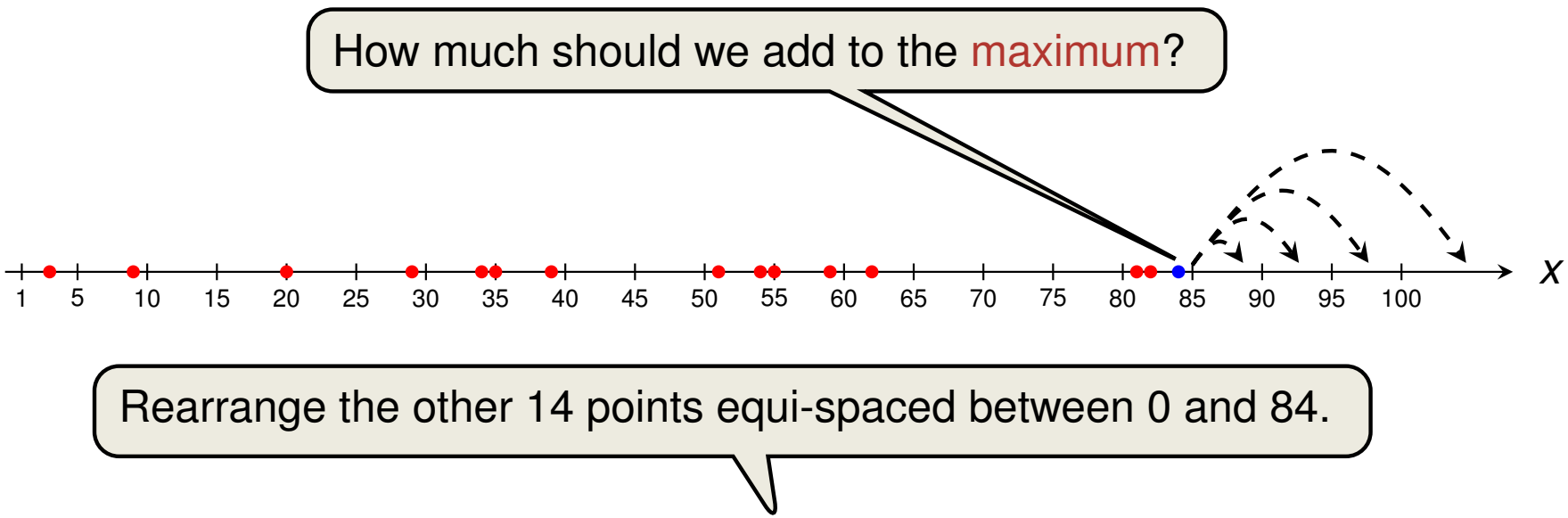

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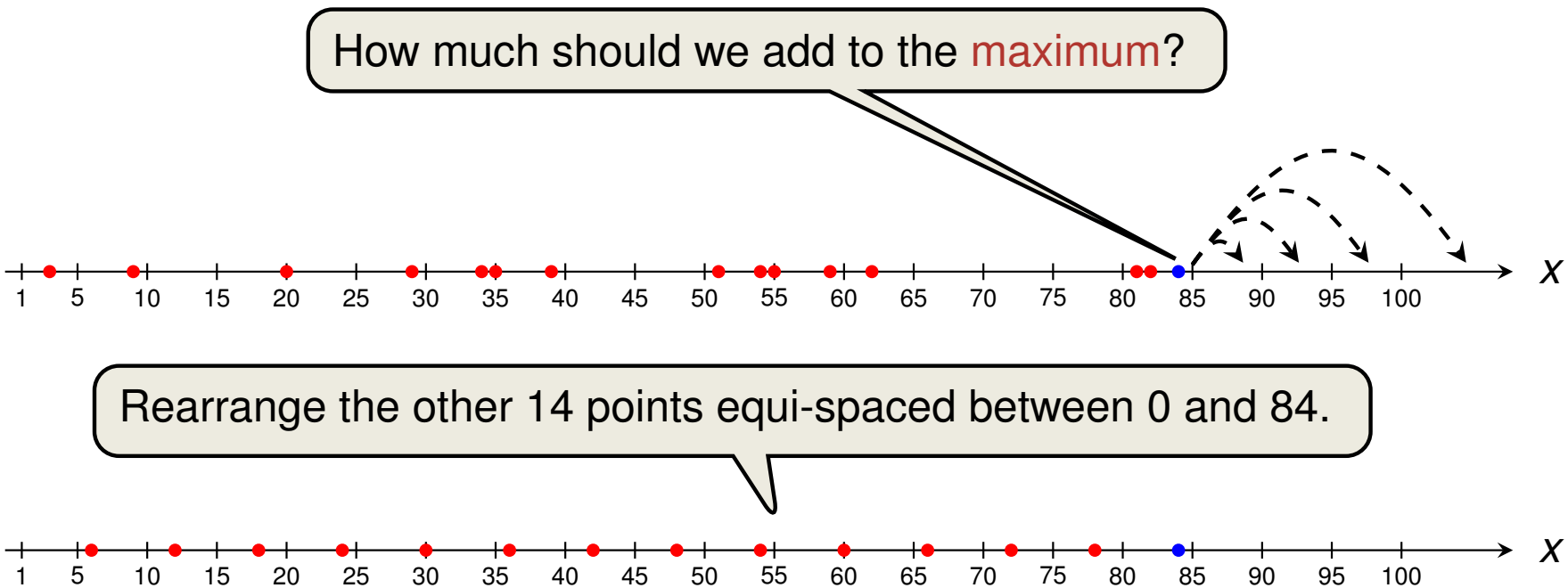

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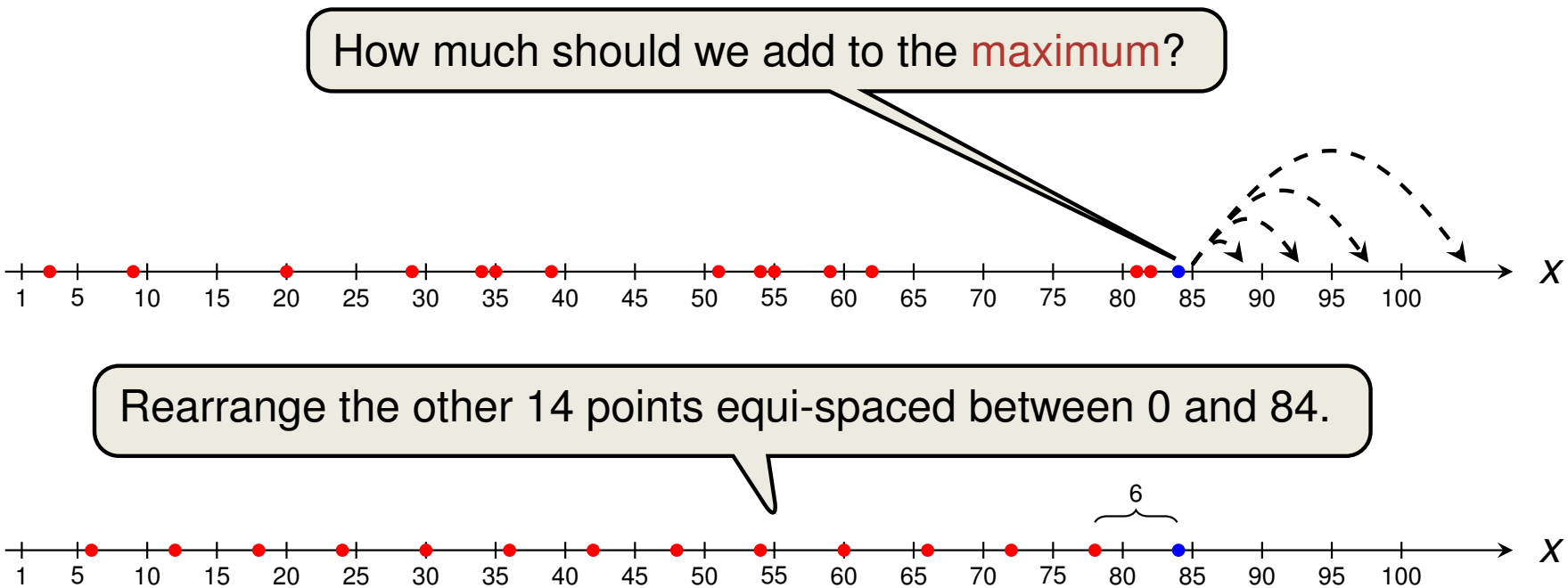

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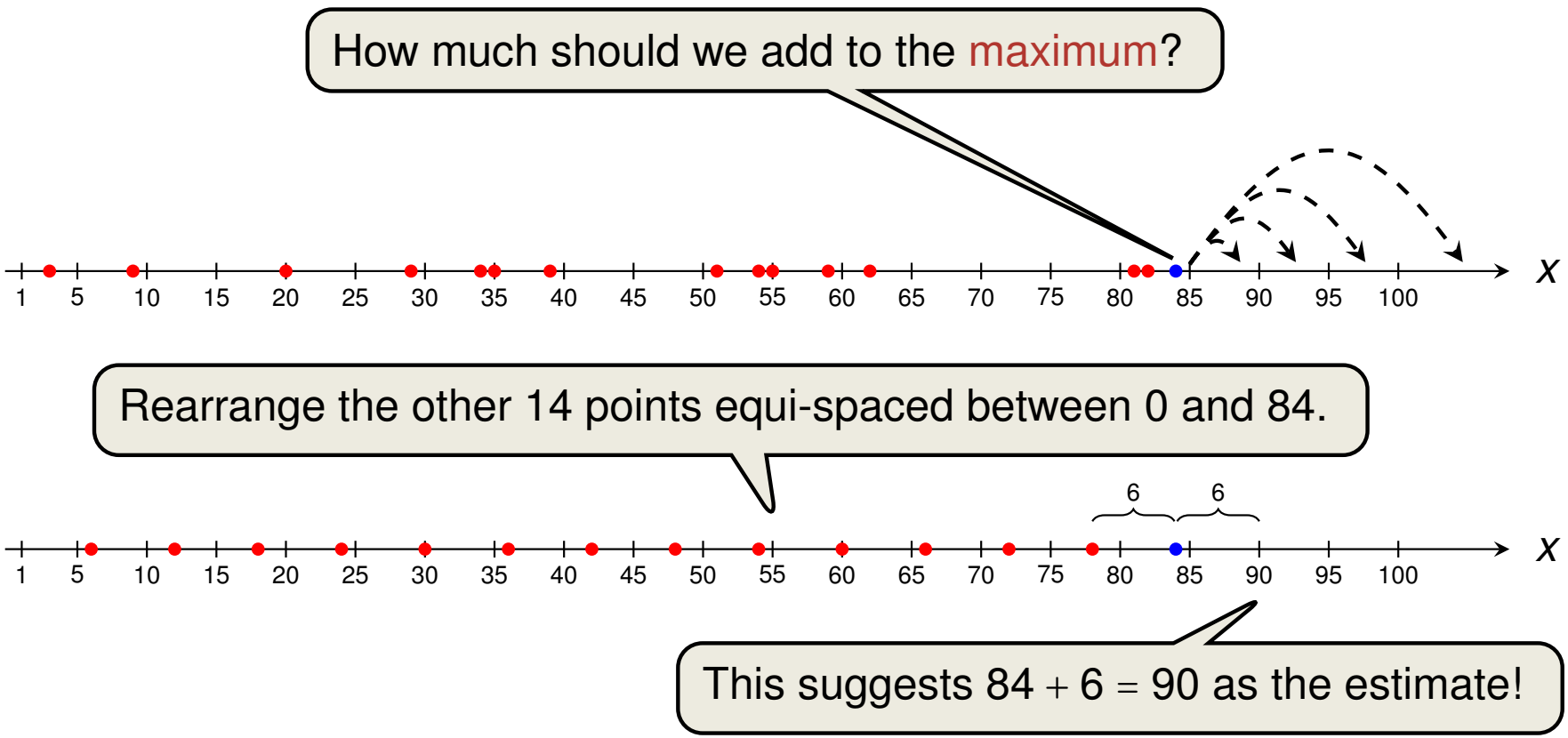

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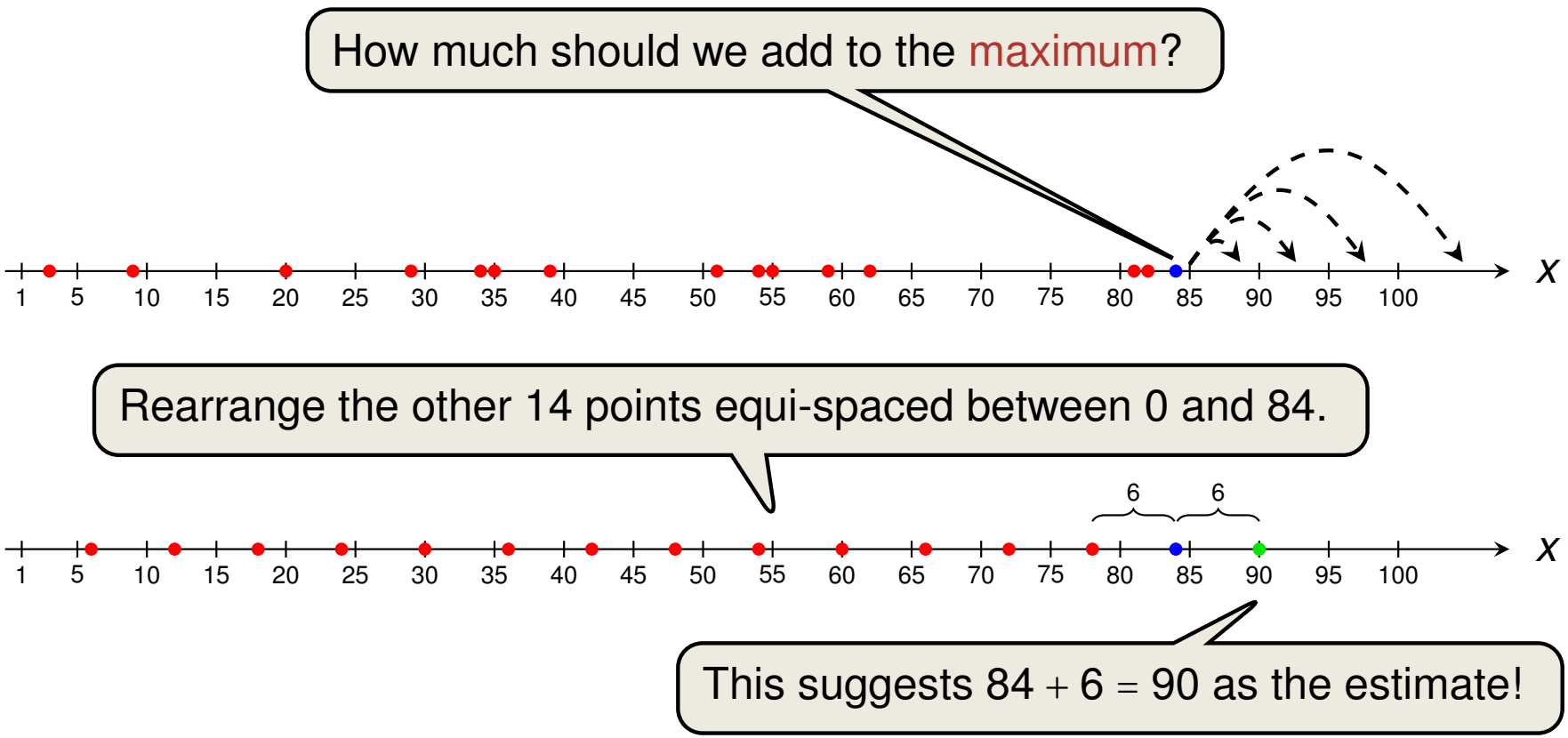

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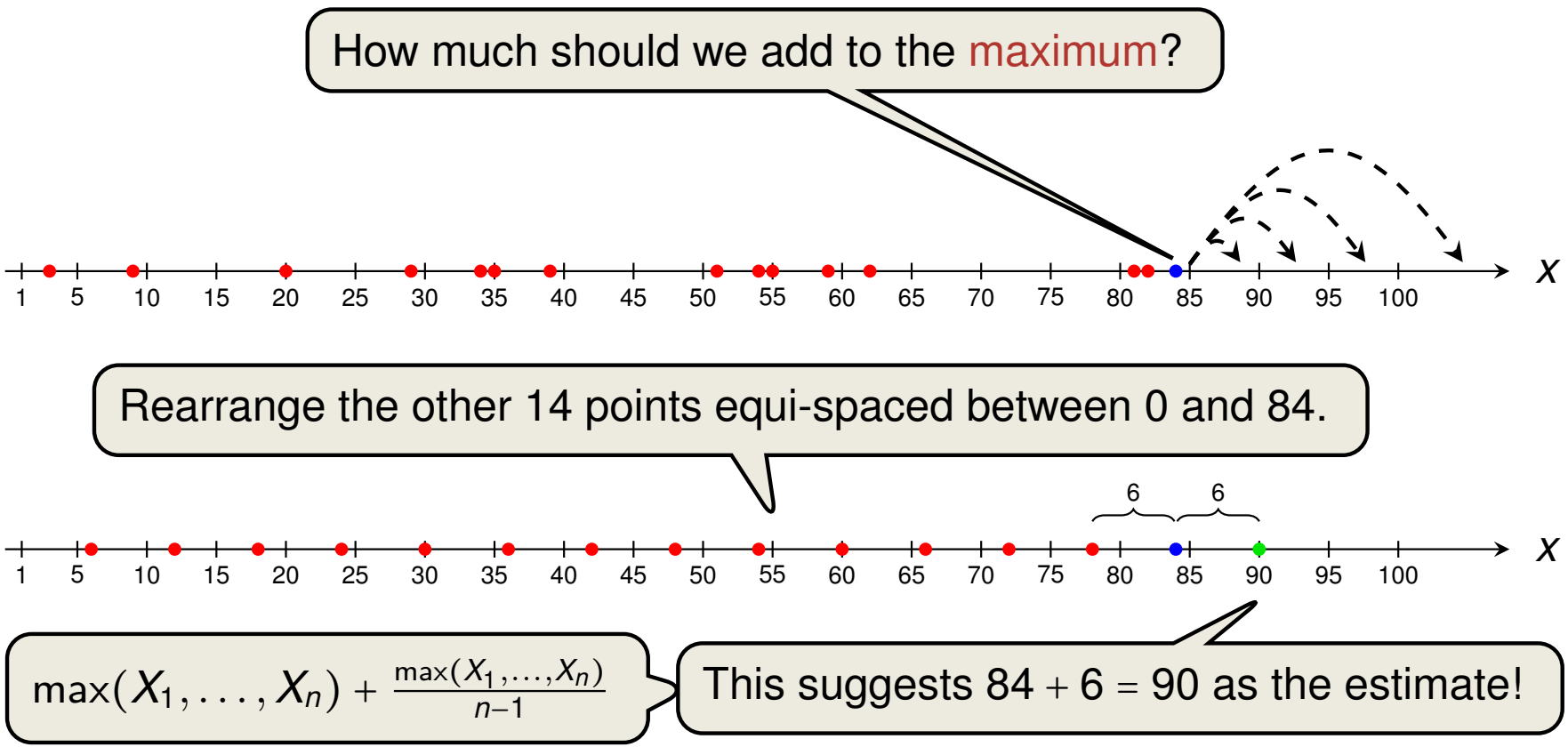

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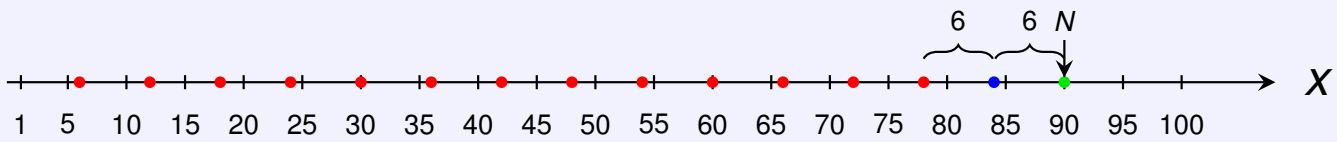

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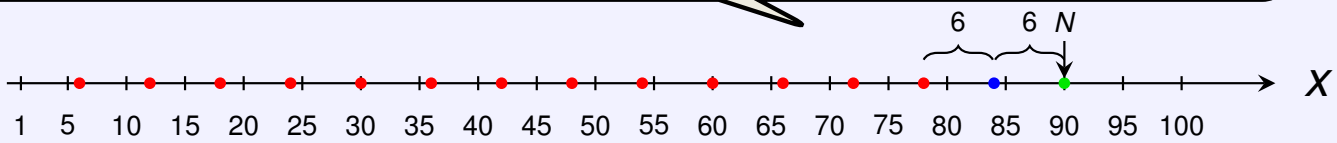

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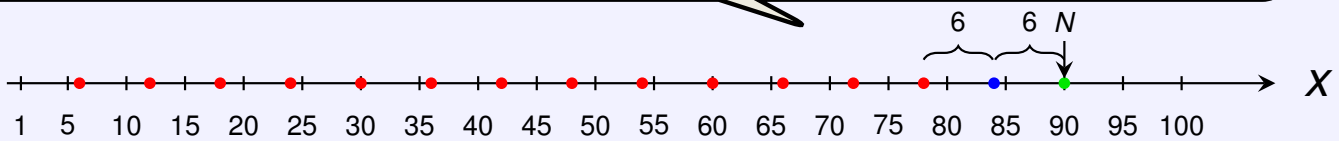

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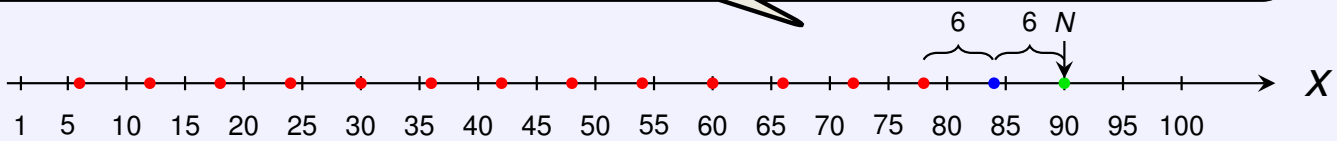

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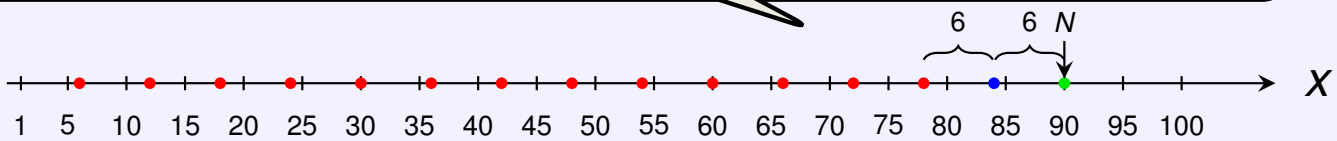

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- For our samples before, we get $t_2 = \frac{16}{15} \cdot 84 - 1 = 88.6$.

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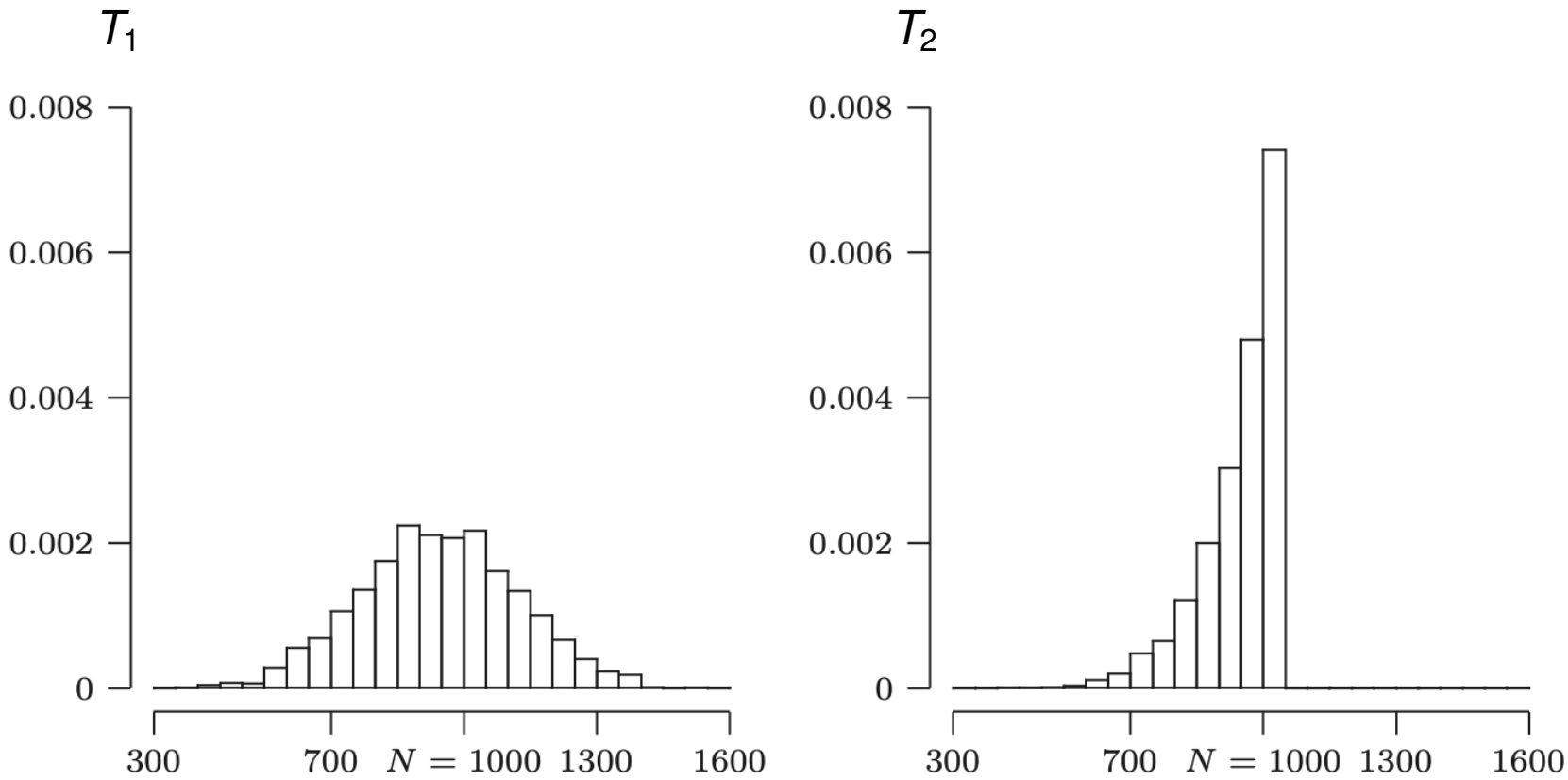

Source: Modern Introduction to Statistics

Figure: Histogram of 2000 values for T_1 and T_2, when $N = 1000$ and $n = 10$.

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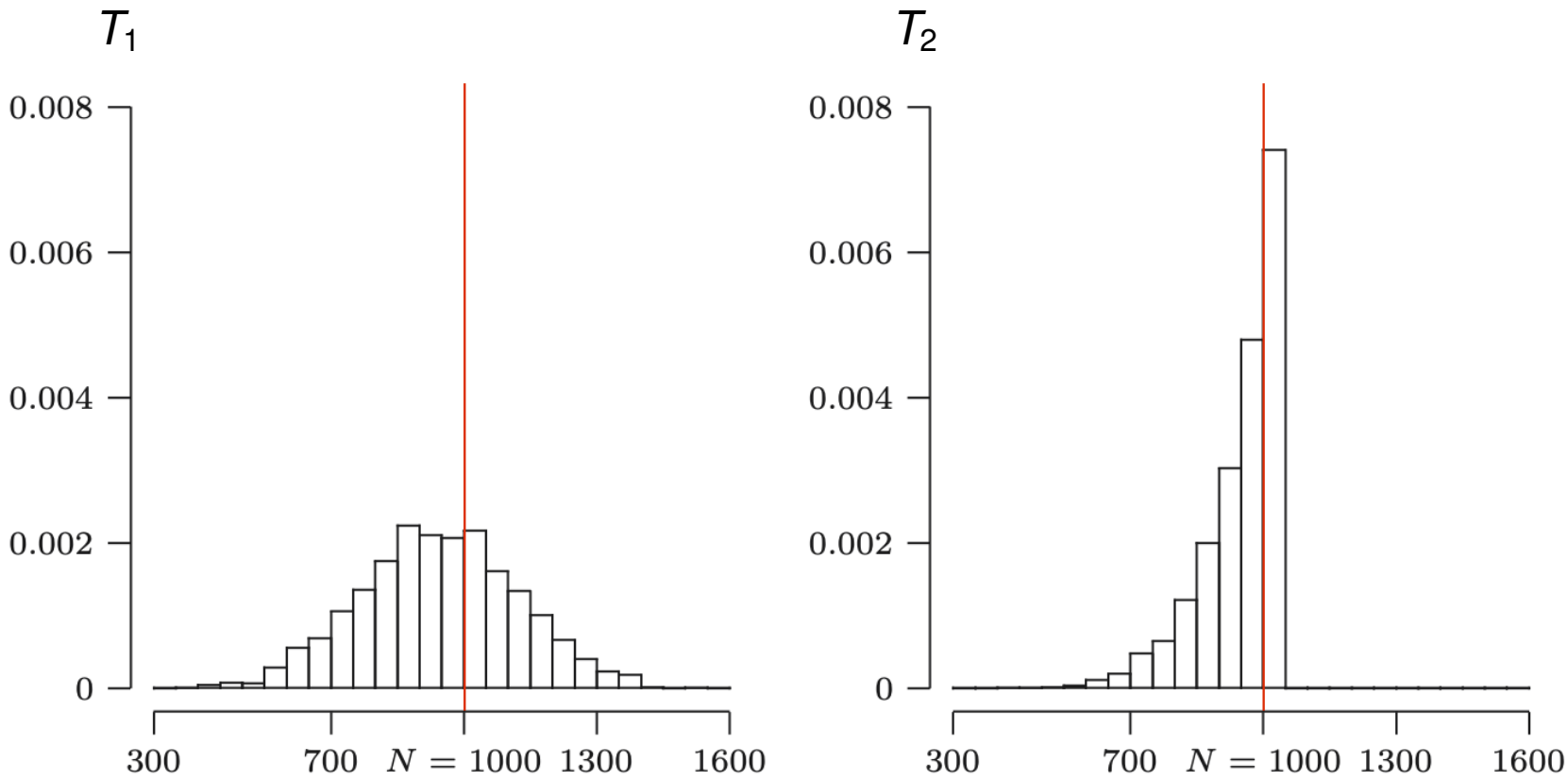

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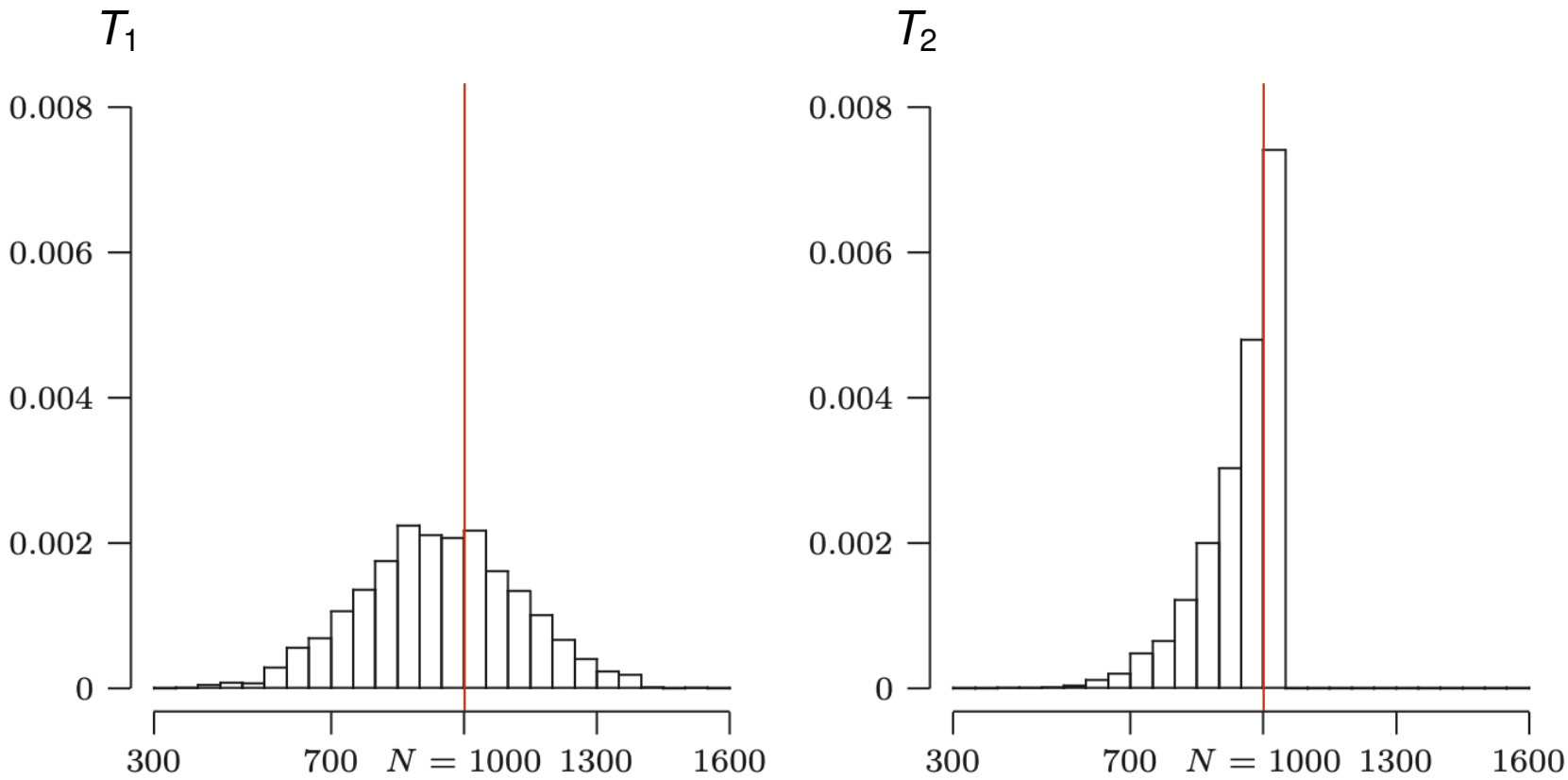

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Can we find a quantity that captures the superiority of T_2 over T_1?

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We need to prove: $\mathbf{MSE}[T] = (\mathbf{E}[T] - \theta)^2 + \mathbf{V}[T]$.

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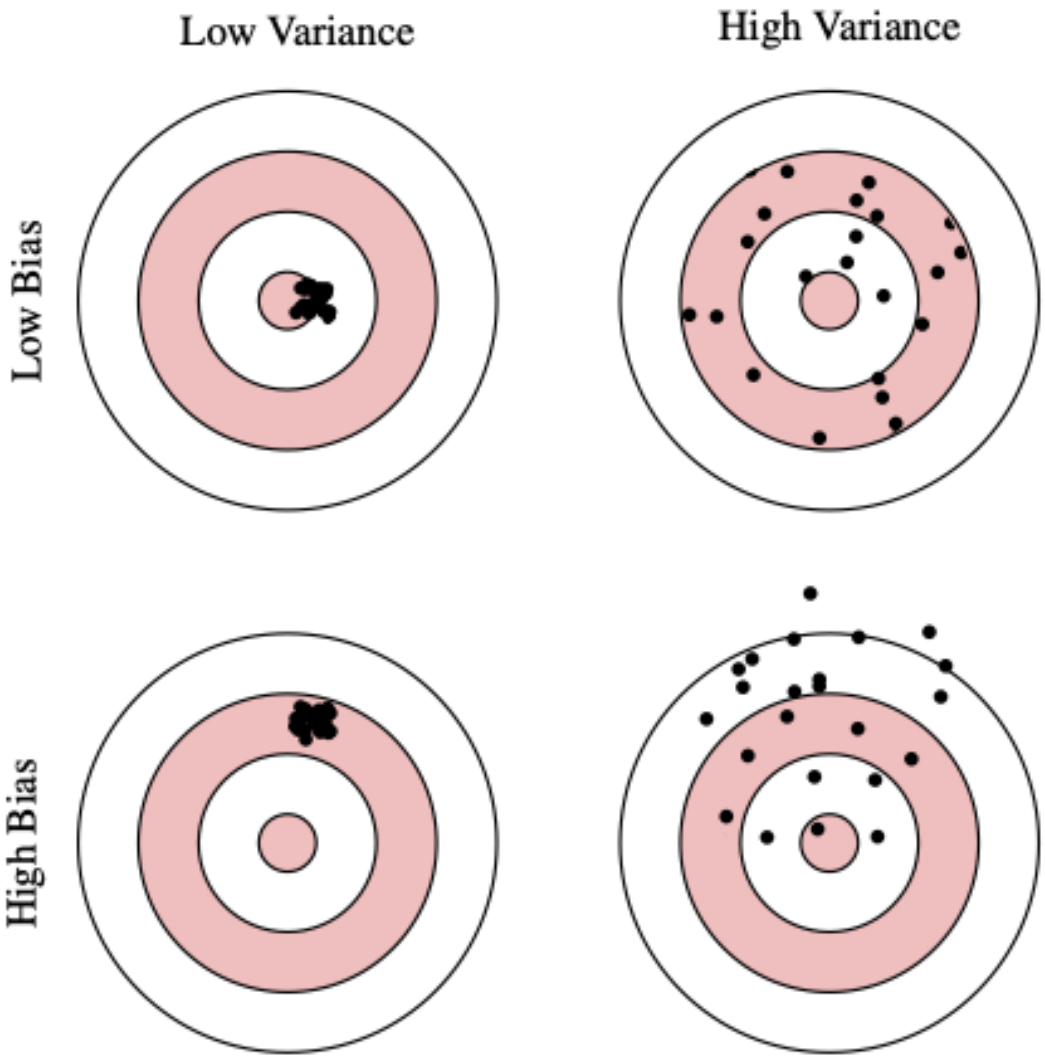
Bias-Variance Decomposition: Illustration



Source: Edwin Leuven (Point Estimation)

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- Rearranging and simplifying gives

$$\mathbf{V}[T_1] = \frac{(N+1)(N-n)}{3n}.$$

Analysis of the MSE for T_2 (non-examinable)

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It holds that **MSE** $[\ T_2\ ] = \Theta\left(\frac{N^2}{n^2}\right)$, where $T_2 = \frac{n+1}{n} \cdot \max(X_1, \ldots, X_n) - 1$.

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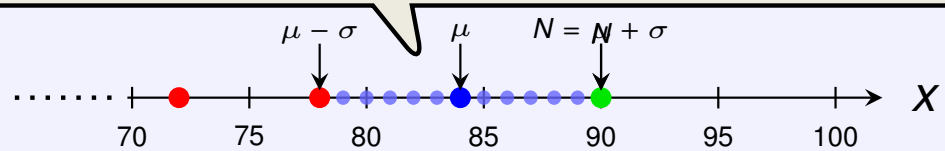

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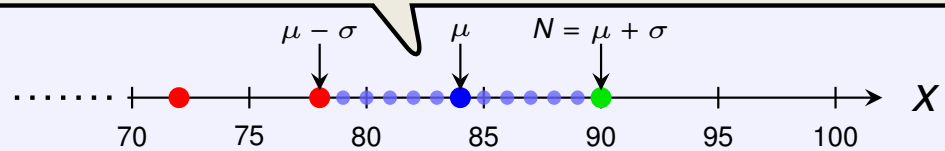

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- $\mathbf{MSE}[\,T_2\,]$ is much lower than $\mathbf{MSE}[\,T_1\,] = \Theta\left(\frac{N^2}{n}\right)$, i.e., $\frac{\mathbf{MSE}[\,T_1\,]}{\mathbf{MSE}[\,T_2\,]} = \frac{n+2}{3}$
- $\Rightarrow$ confirms simulations suggesting that T_2 is better than T_1!
- can be shown T_2 is the best unbiased estimator, i.e., it minimises MSE.

Outline

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Similar idea applies to situations where elements are not labelled before we see them first time (Mark & Recapture Method)

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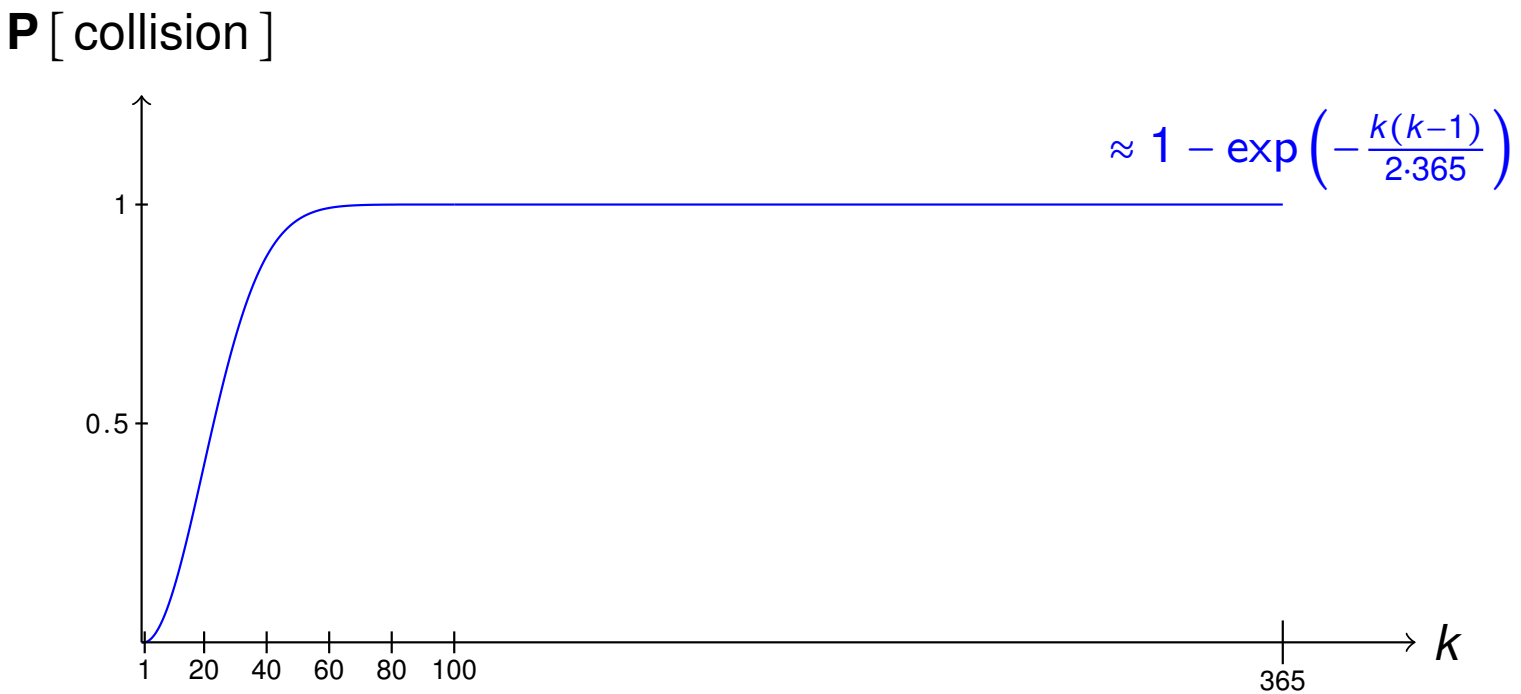

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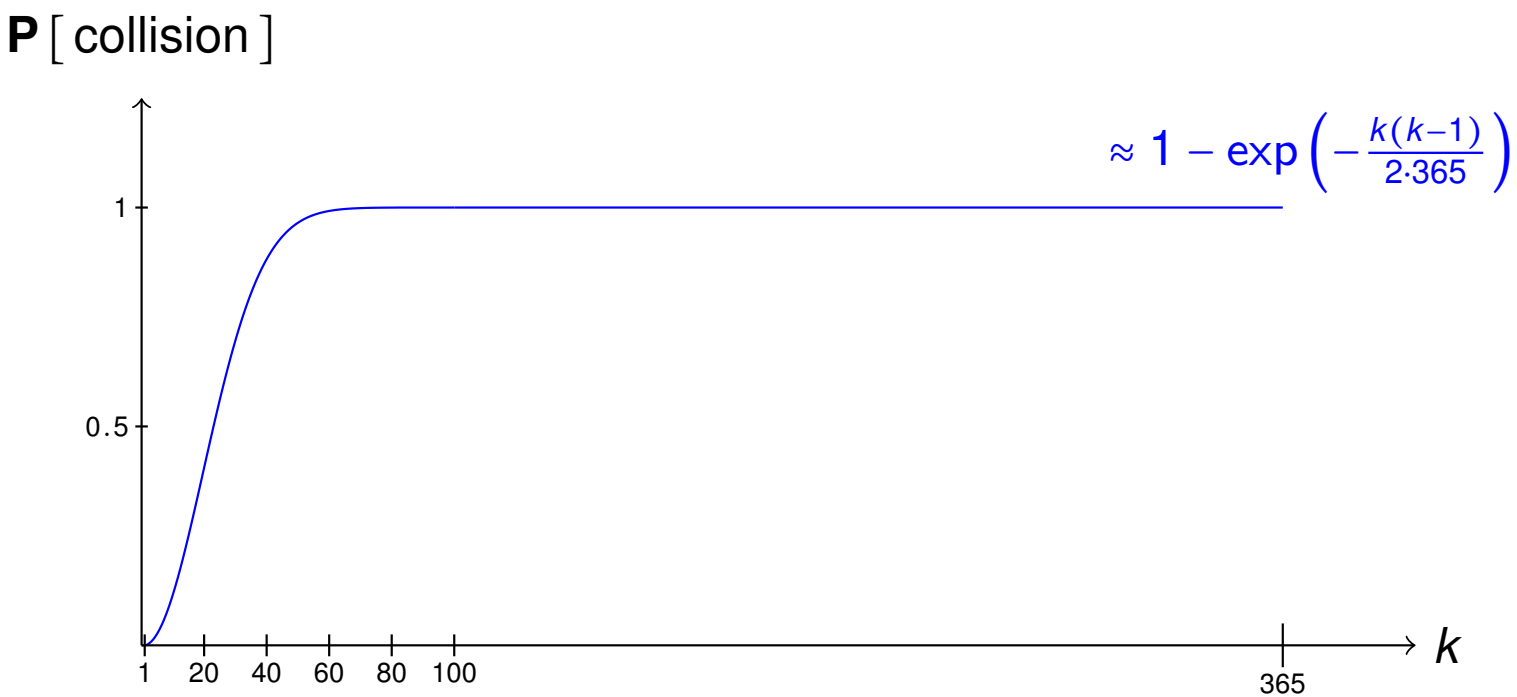

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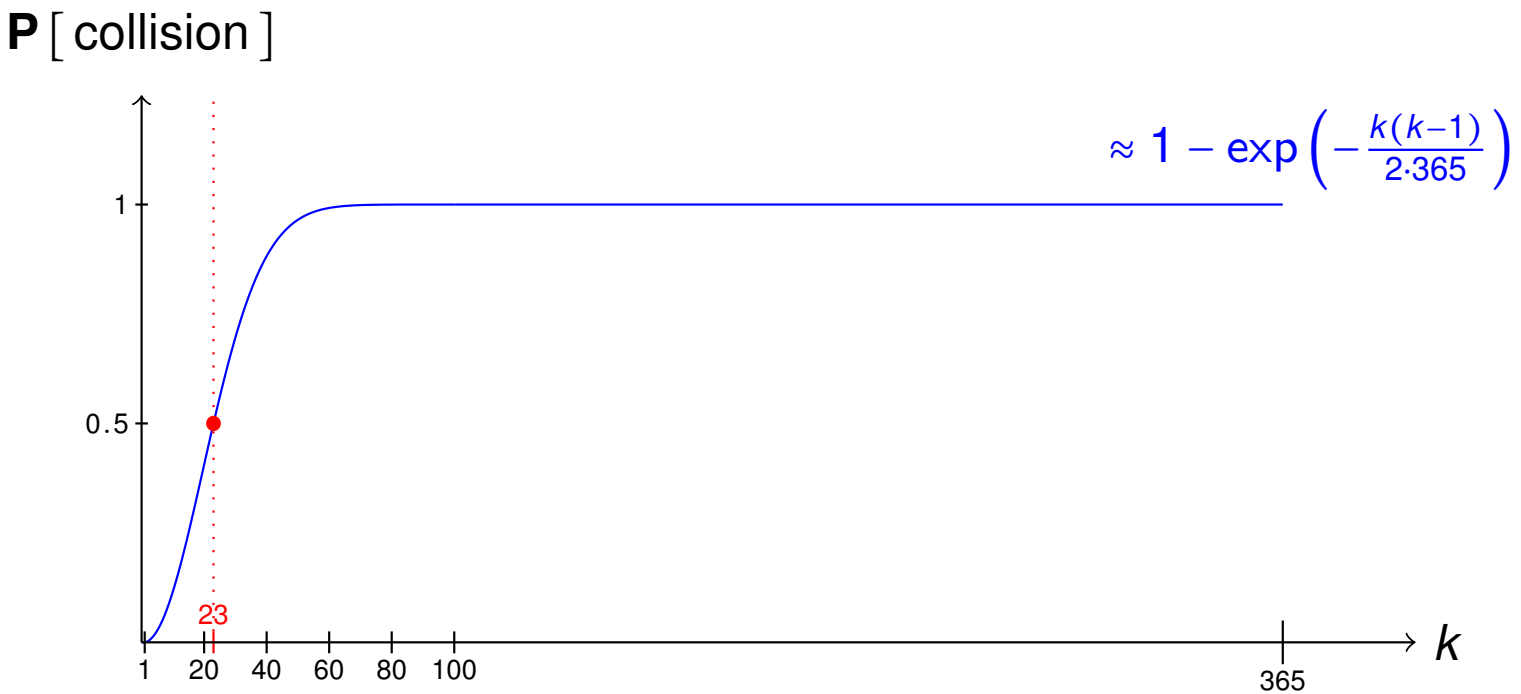

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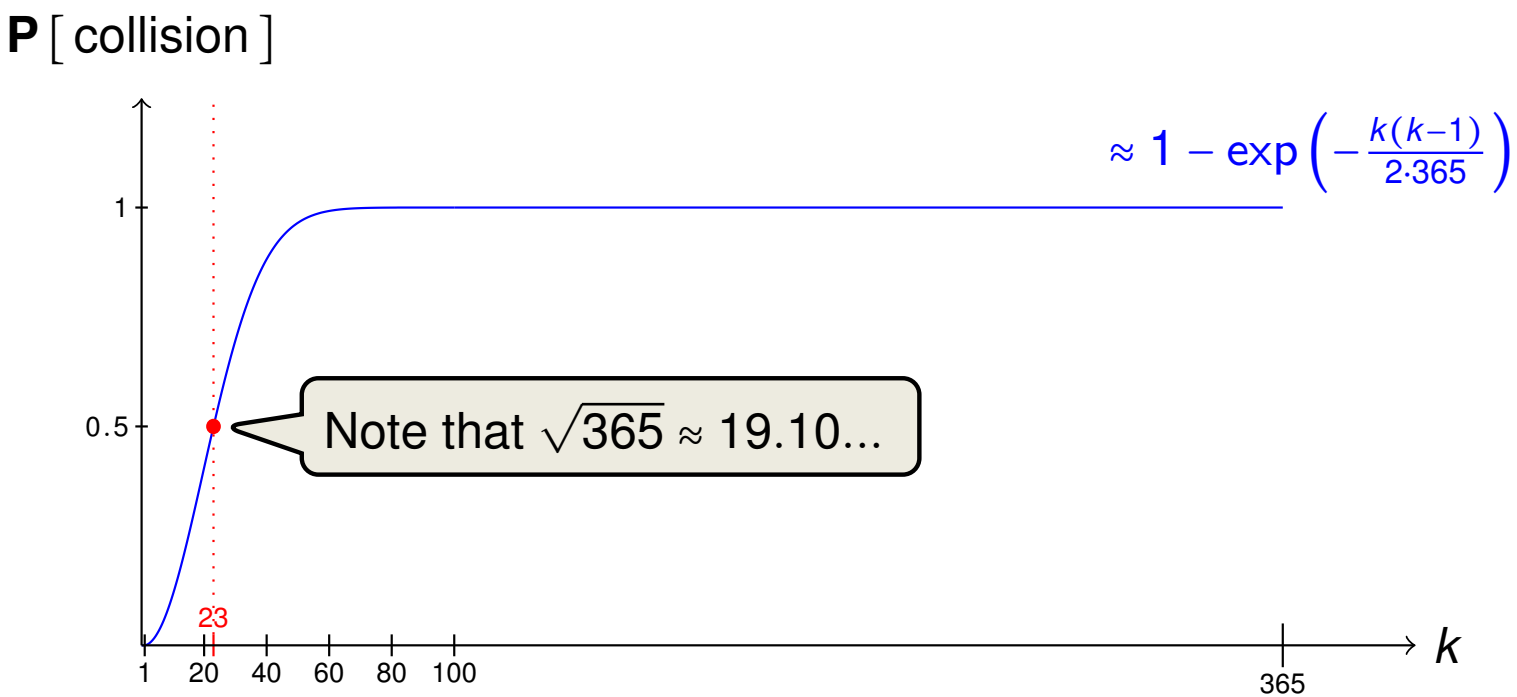

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Exercise: Prove a bound of $\leq 2 \cdot \sqrt{N}$

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One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[\,T\,] = |S|$ for any finite, non-empty set S.

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Estimation via Collision: Getting the Estimator Unbiased

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Introduction to Probability

Lecture 12: Online Algorithms

Mateja Jamnik, Thomas Sauerwald

University of Cambridge, Department of Computer Science and Technology
email: {mateja.jamnik,thomas.sauerwald}@cl.cam.ac.uk

Easter 2026

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Outline

Introduction: Dice Game

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This boils down to finding a threshold from which we `STOP` as soon as a 6 is thrown.

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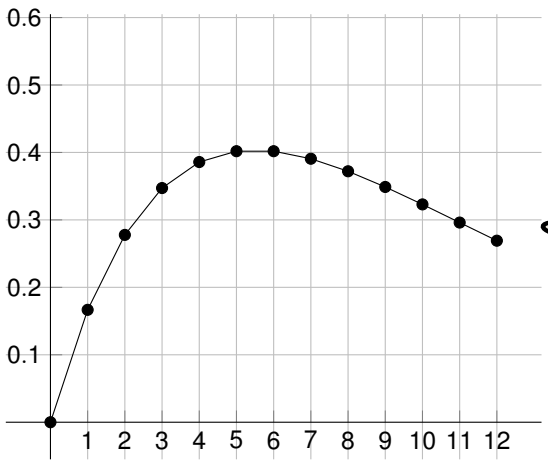

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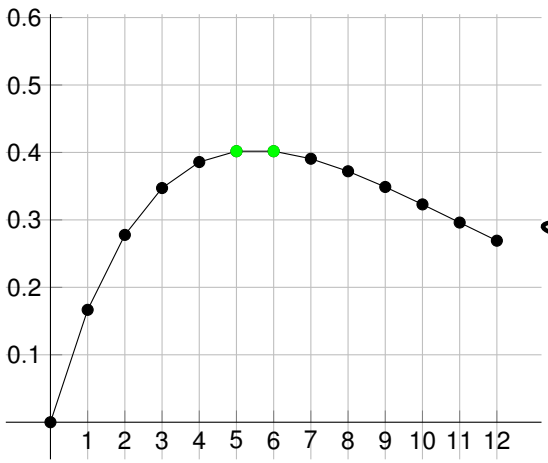

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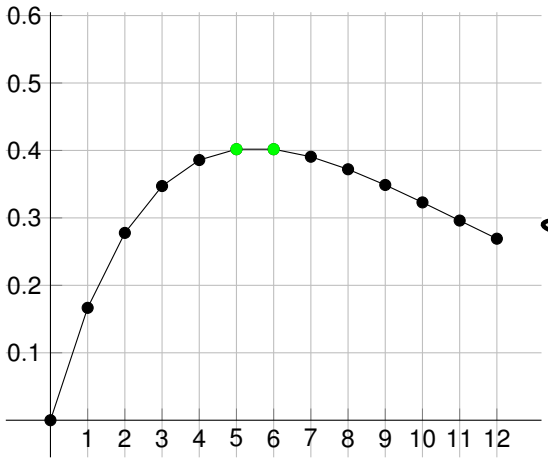

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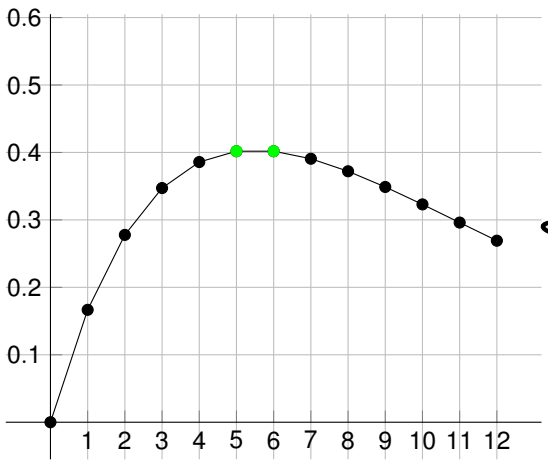

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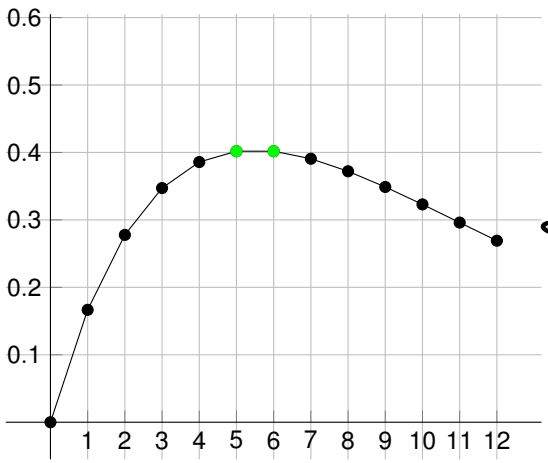

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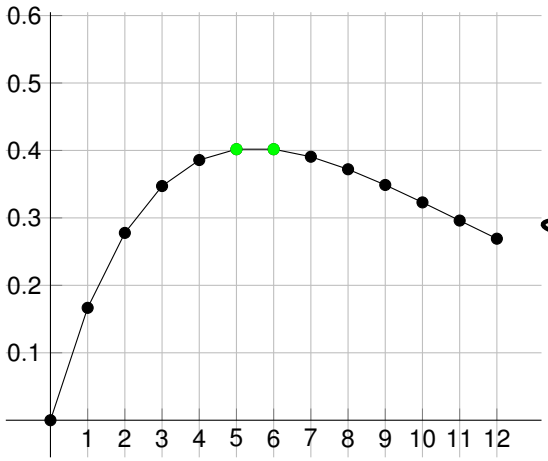

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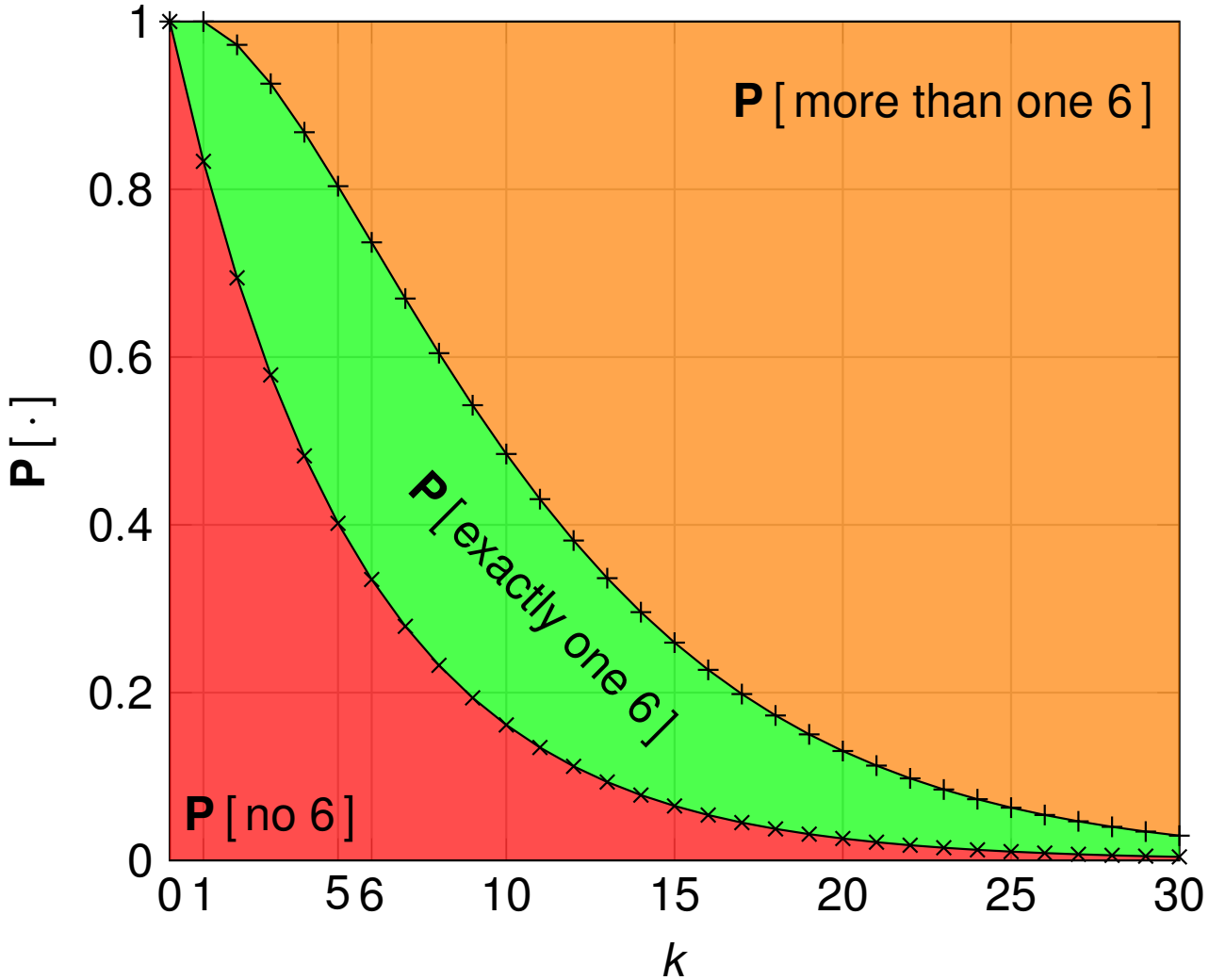
$$\left(\frac{5}{6}\right)^5 \approx 0.40.$$

Illustration of the Three Possibilities



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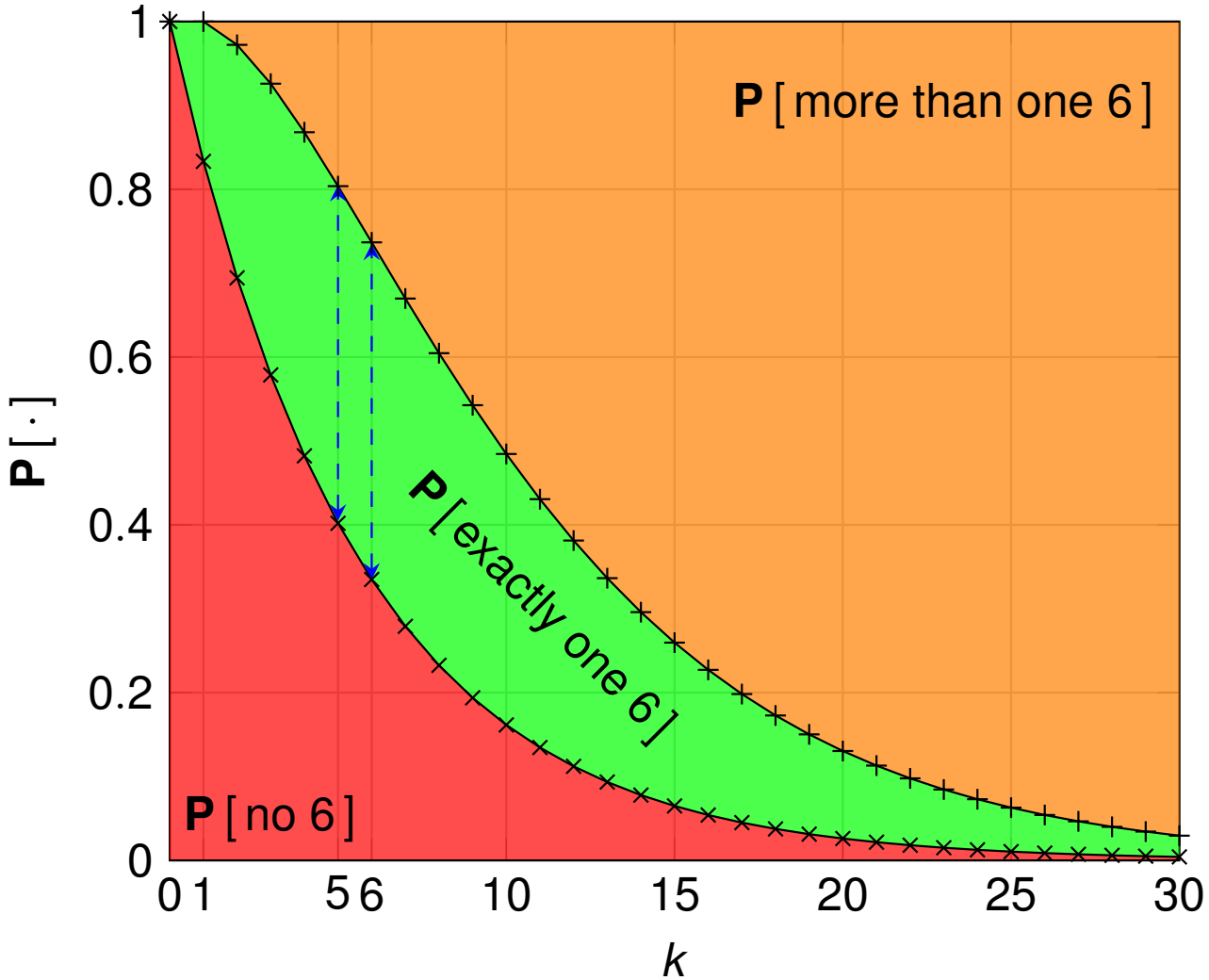

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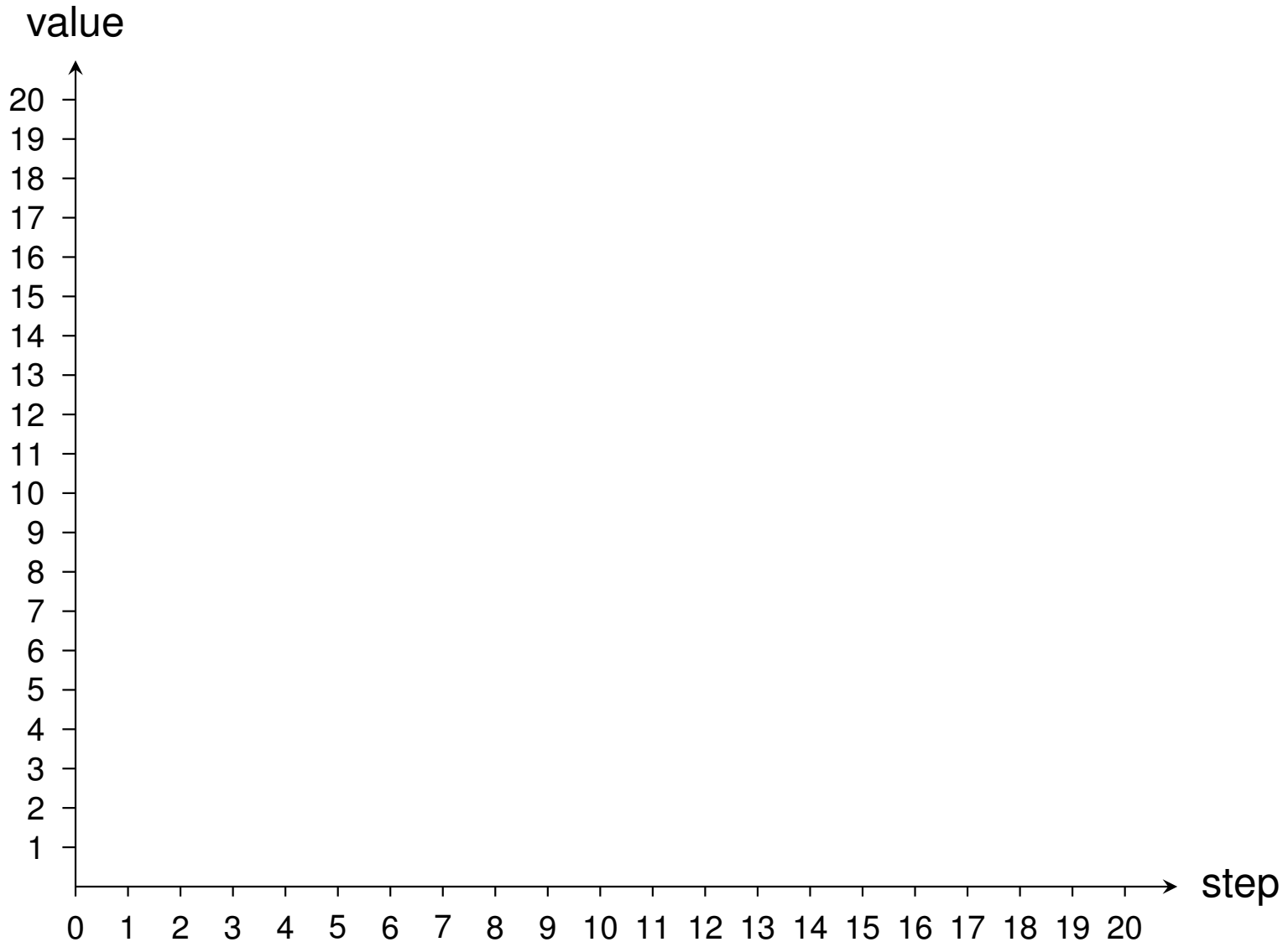
$\Rightarrow$ For our problem we may as well assume that the only information we have when interviewing candidate i is whether that candidate is best among $\{1, \ldots, i\}$ or not.

Illustration ($n = 20$)

unknown permutation:



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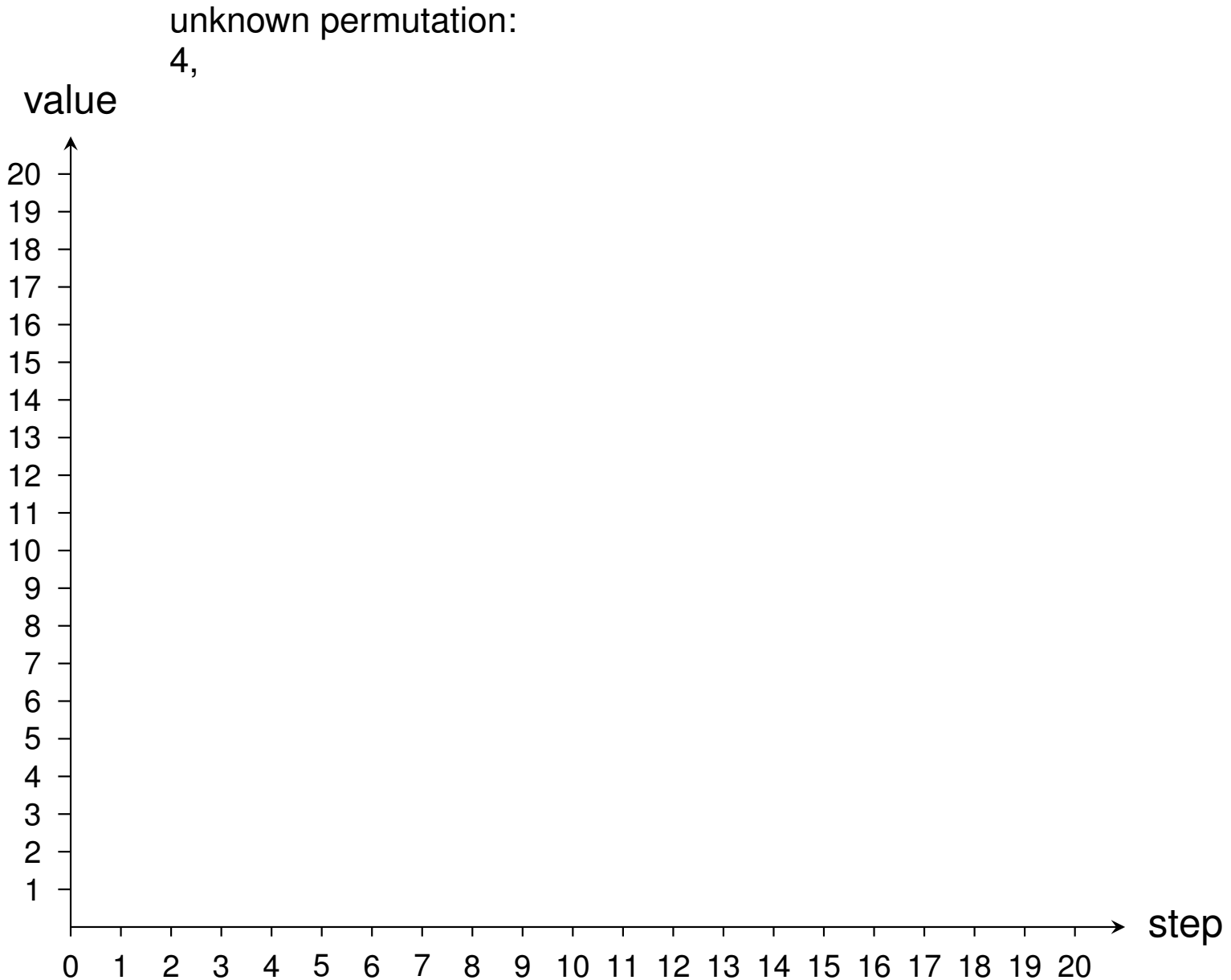

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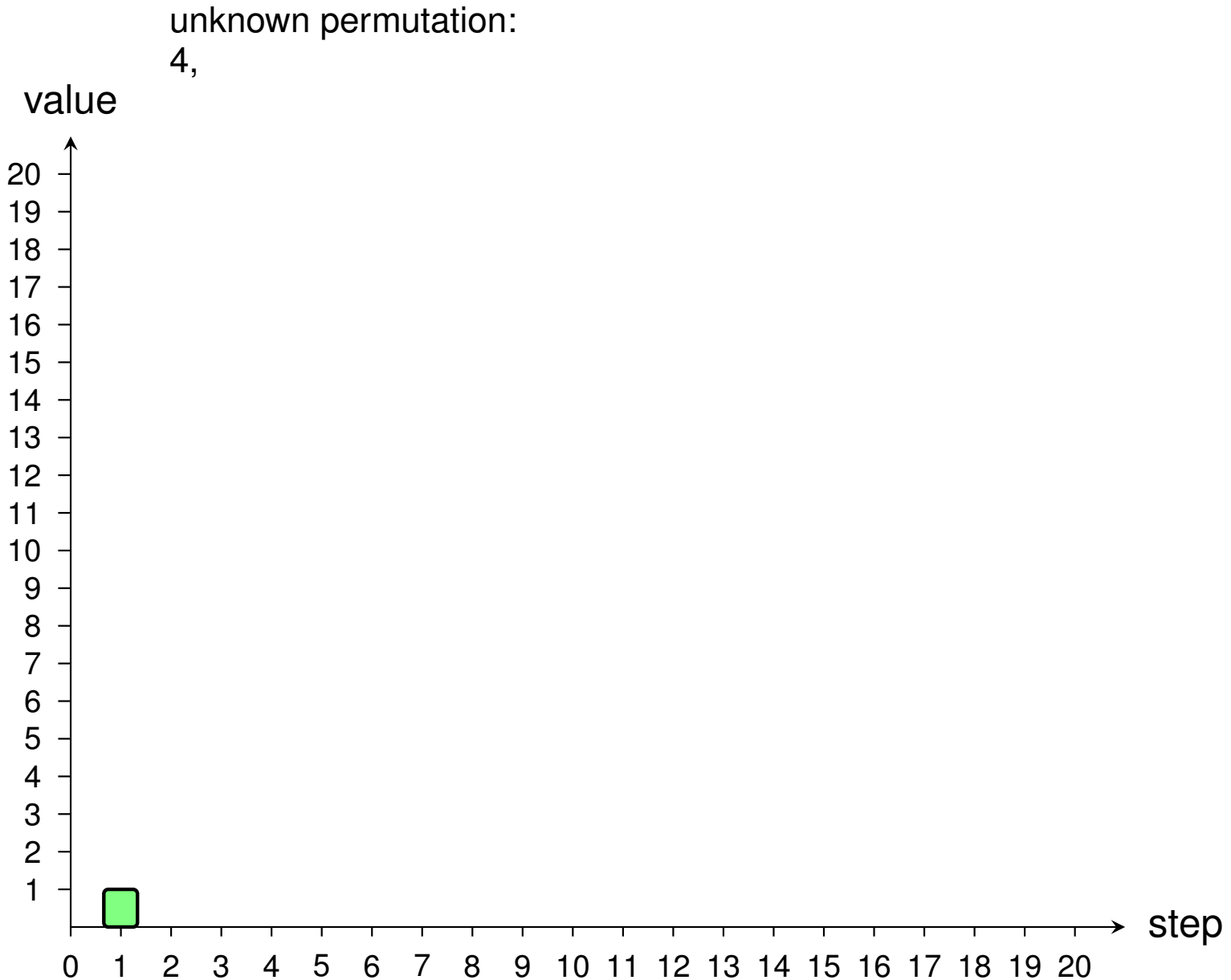

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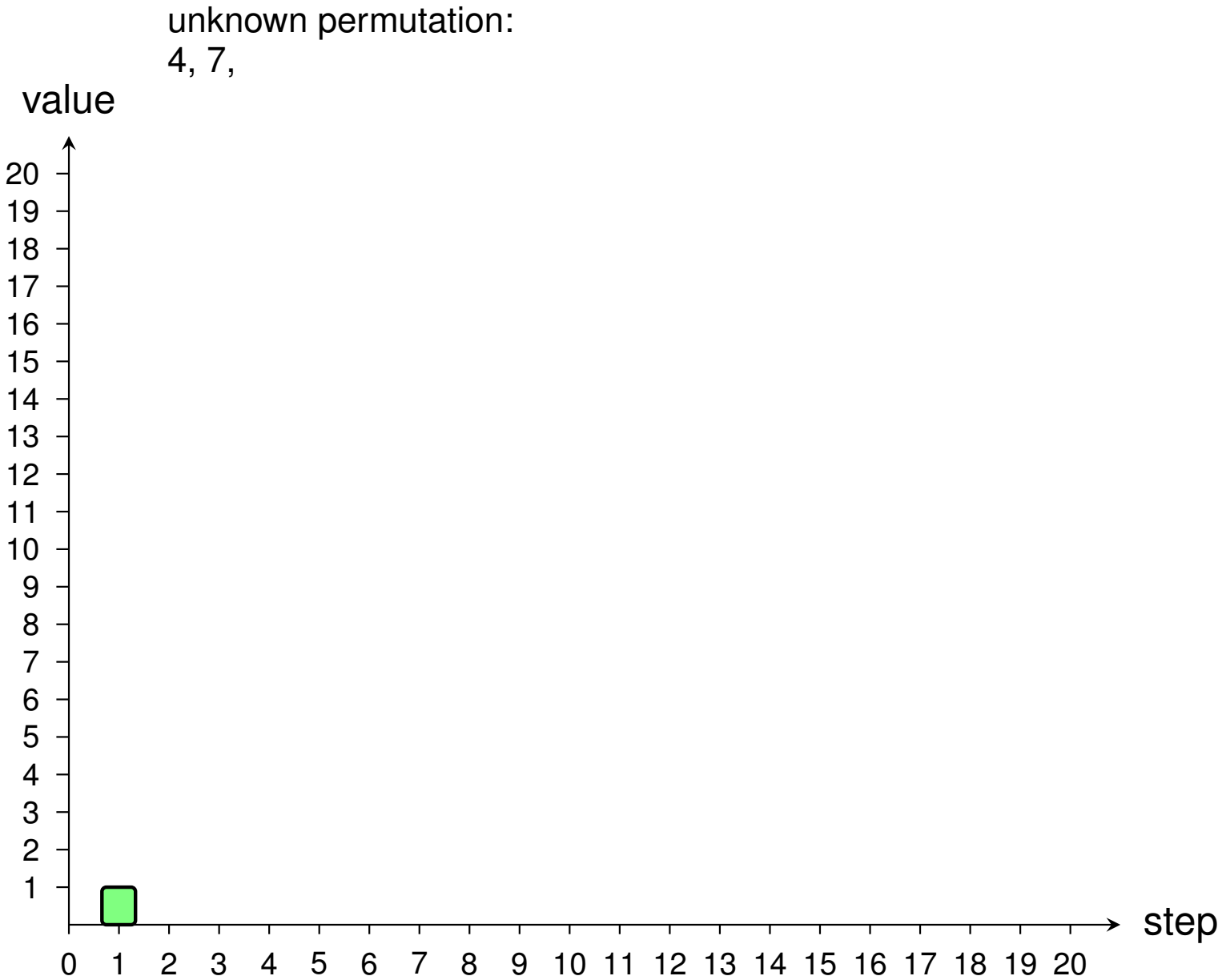

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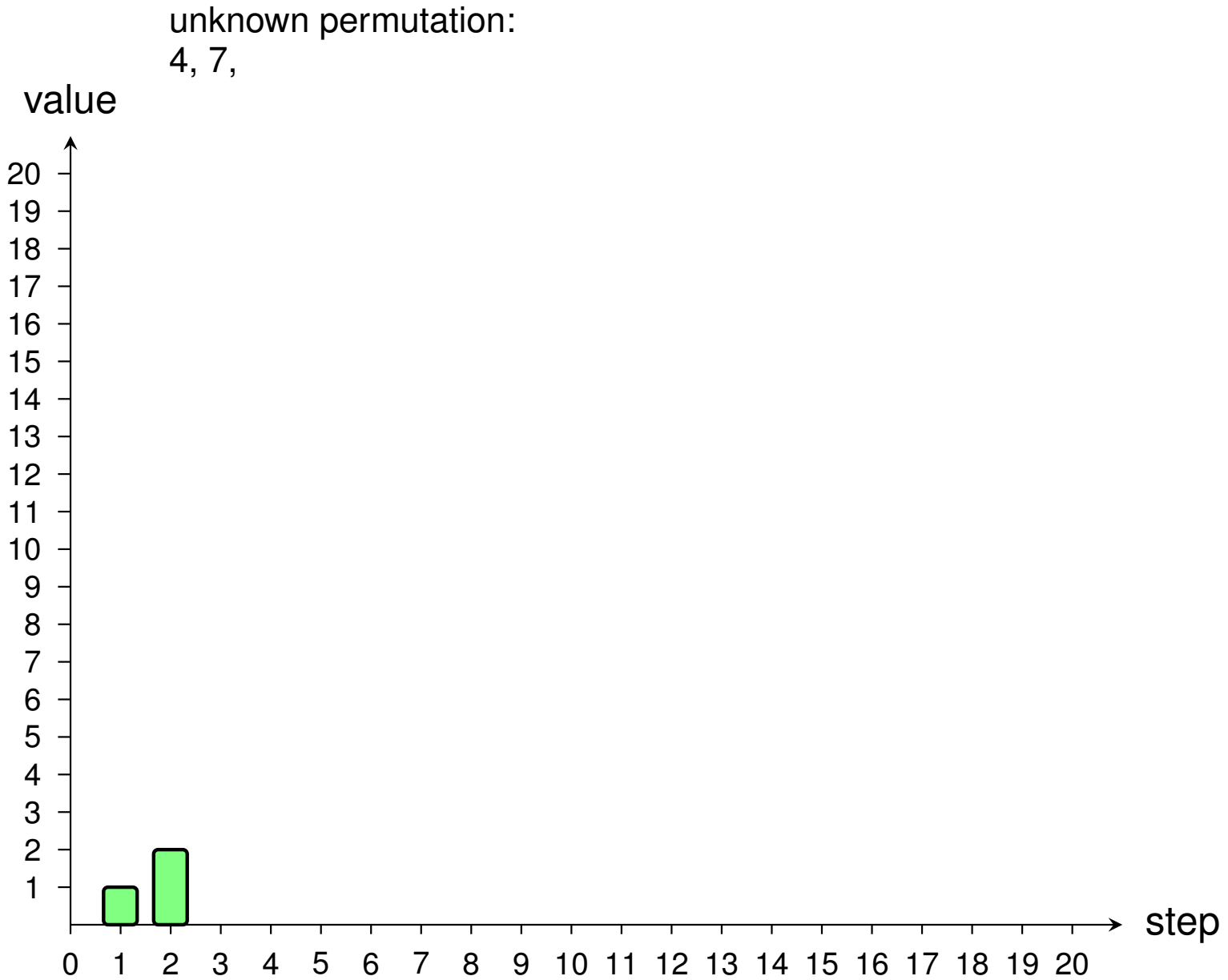

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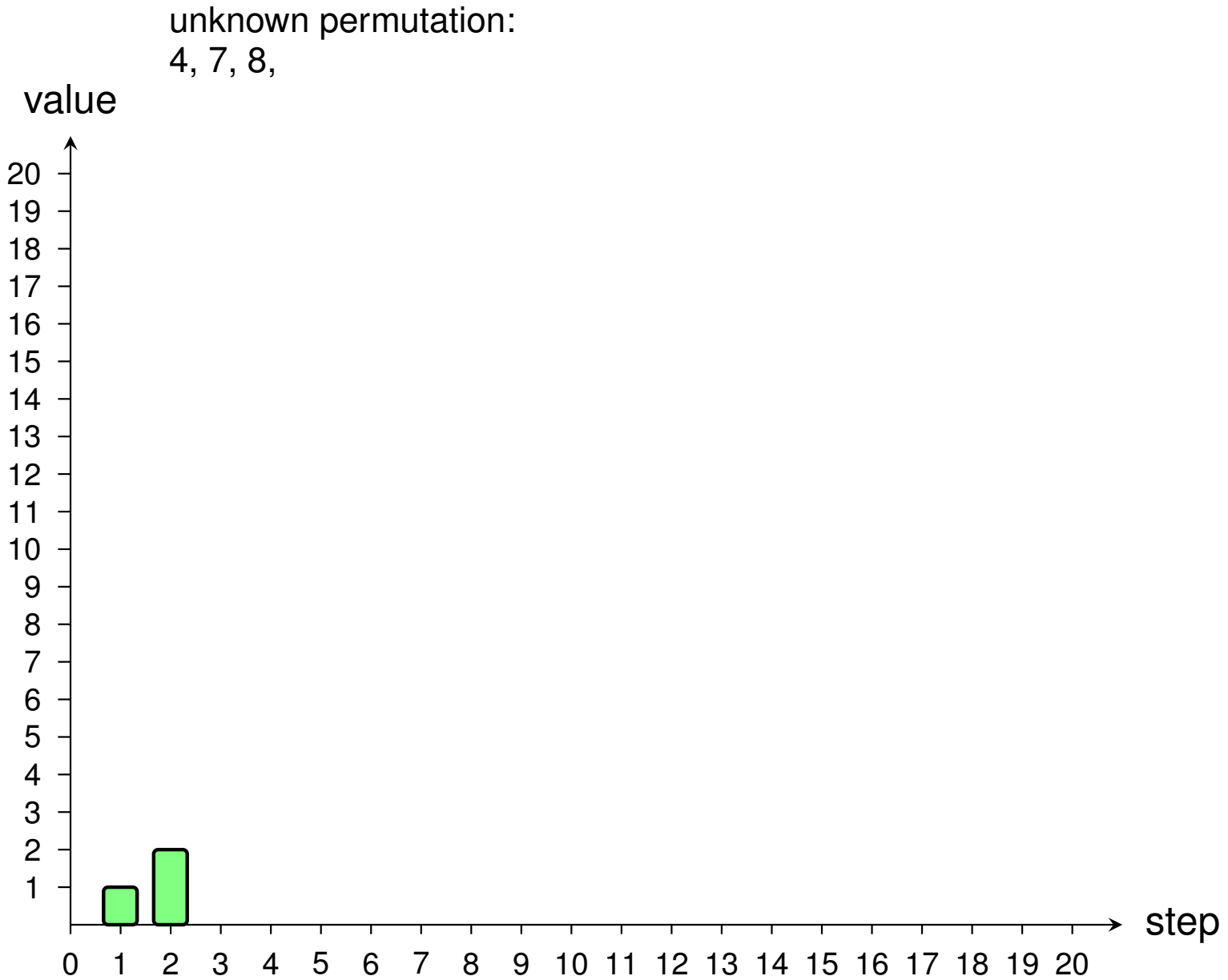

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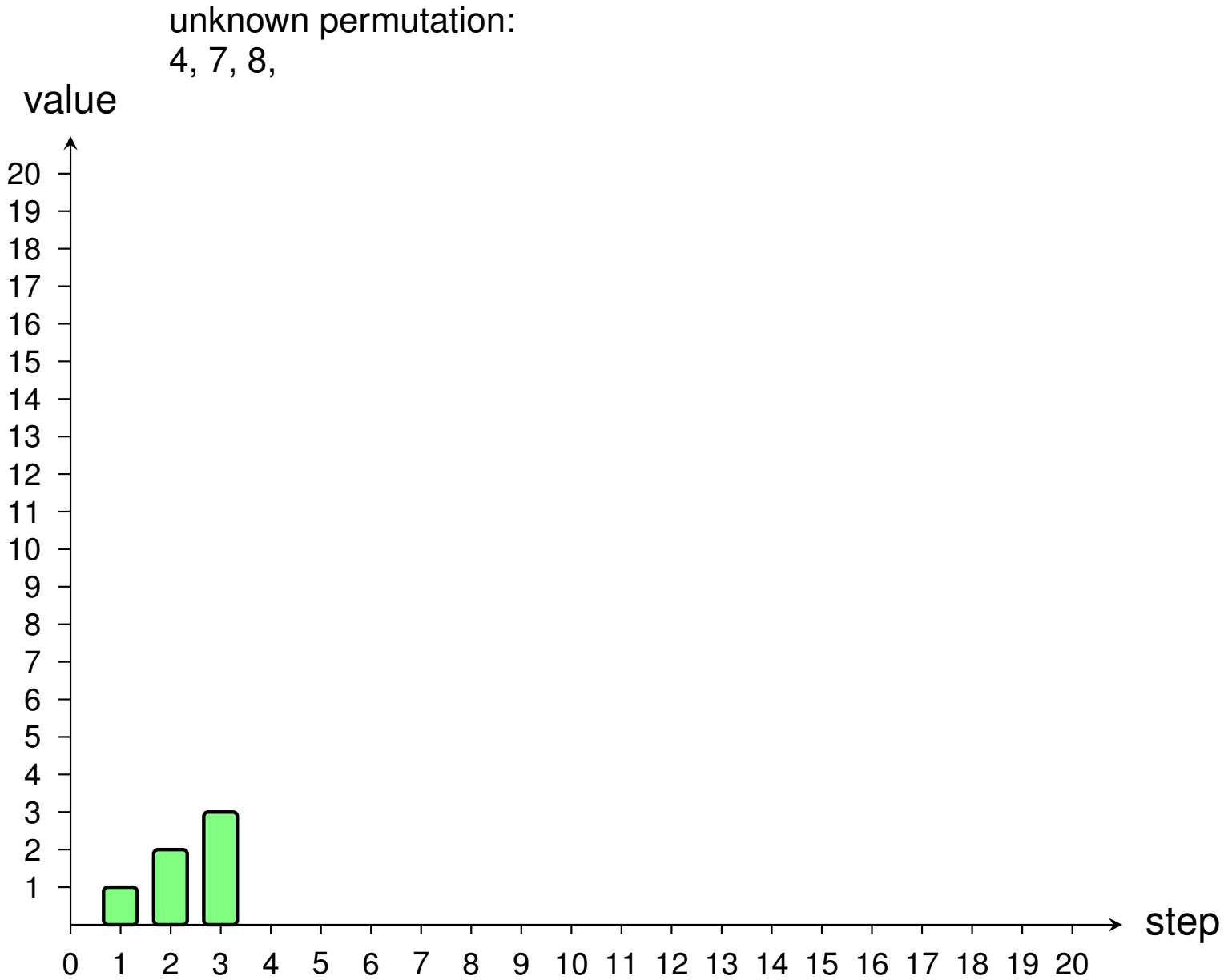

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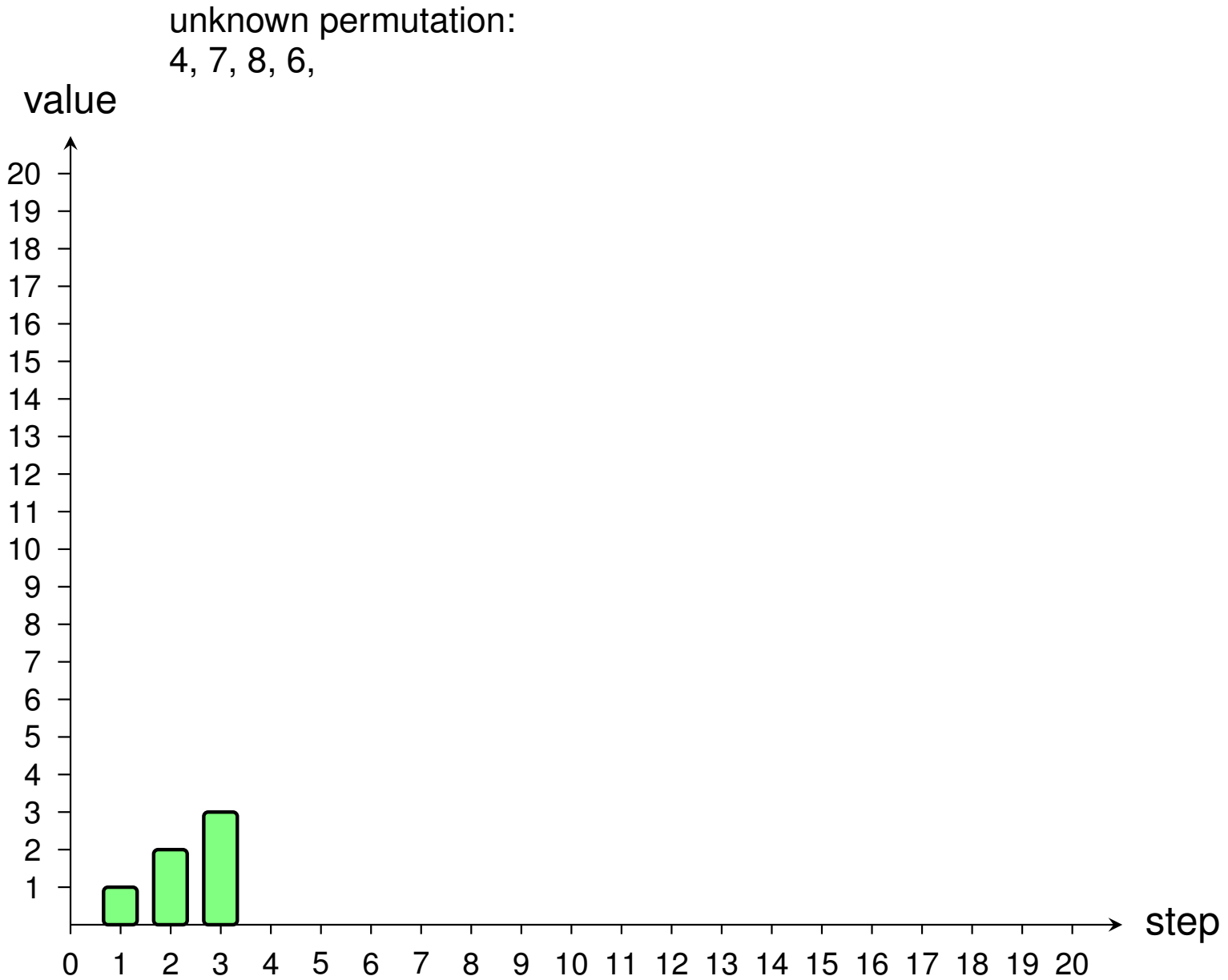

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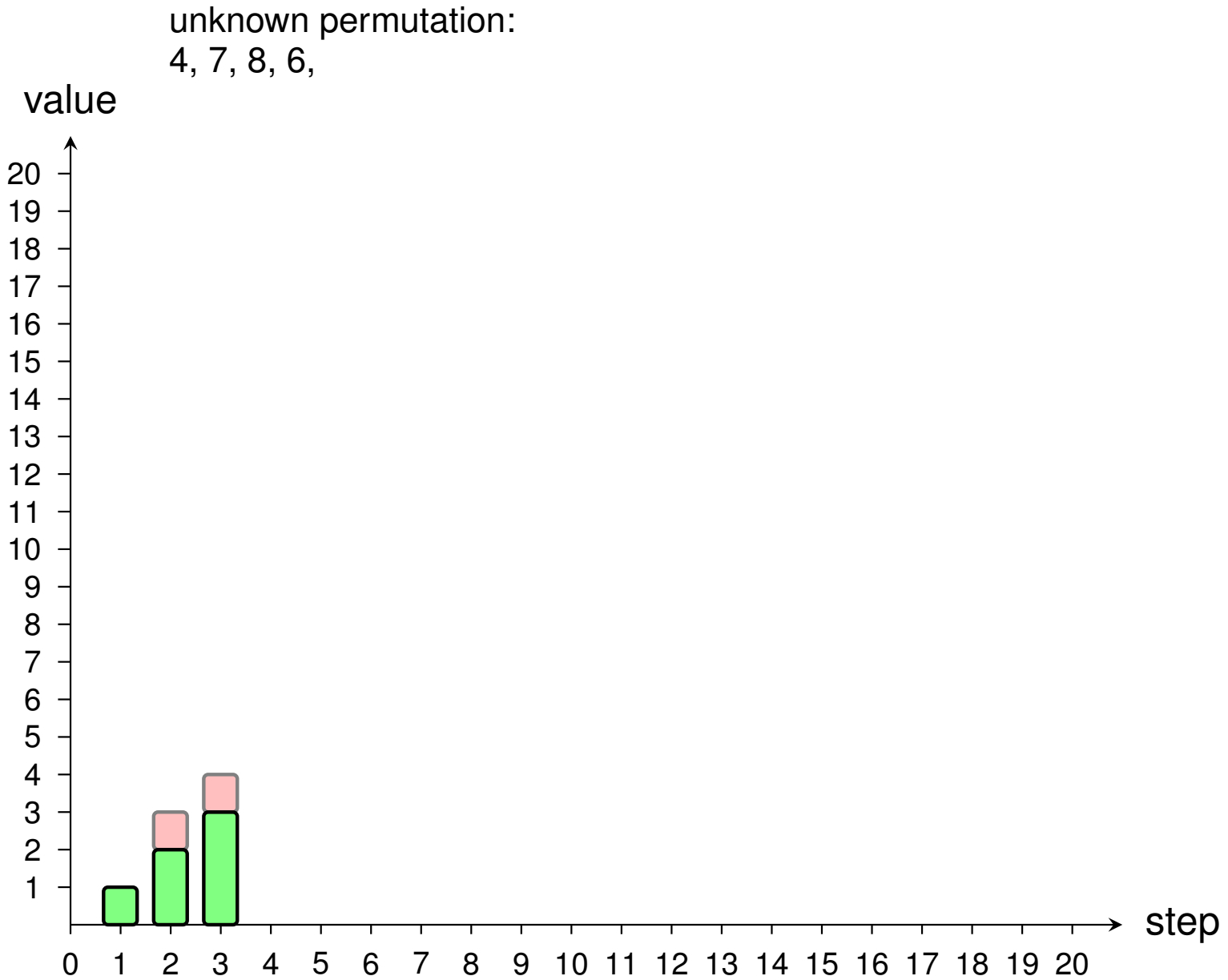

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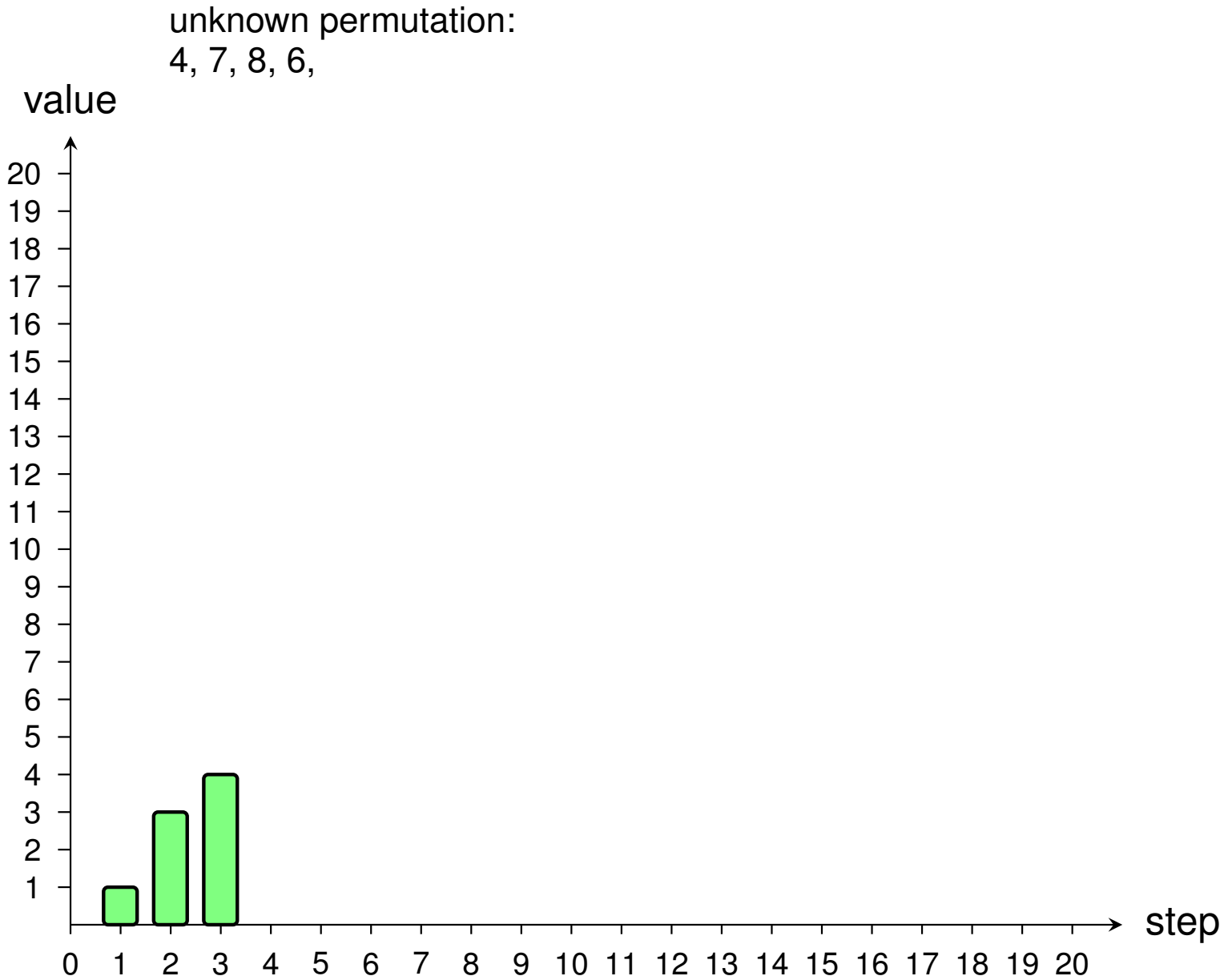

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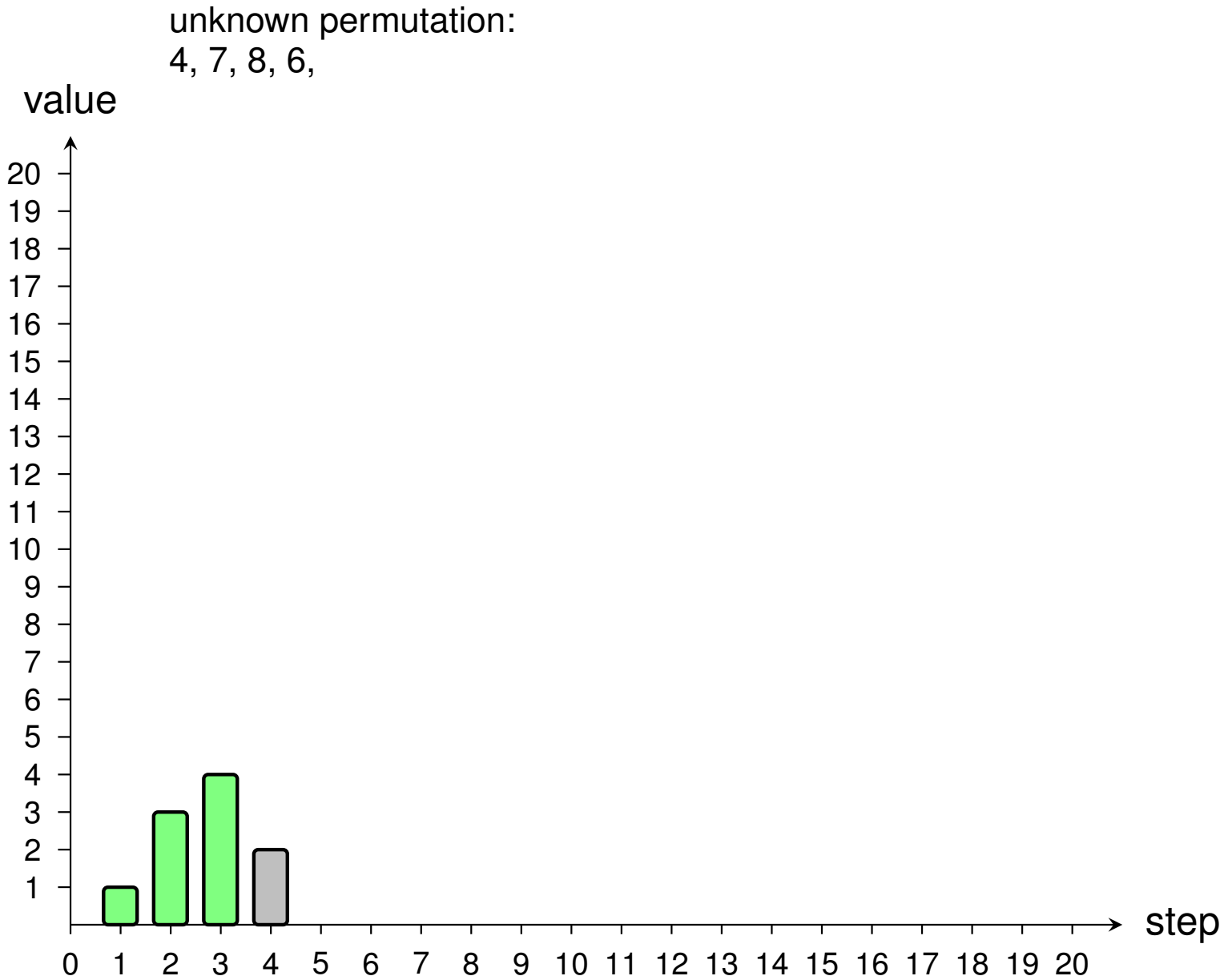

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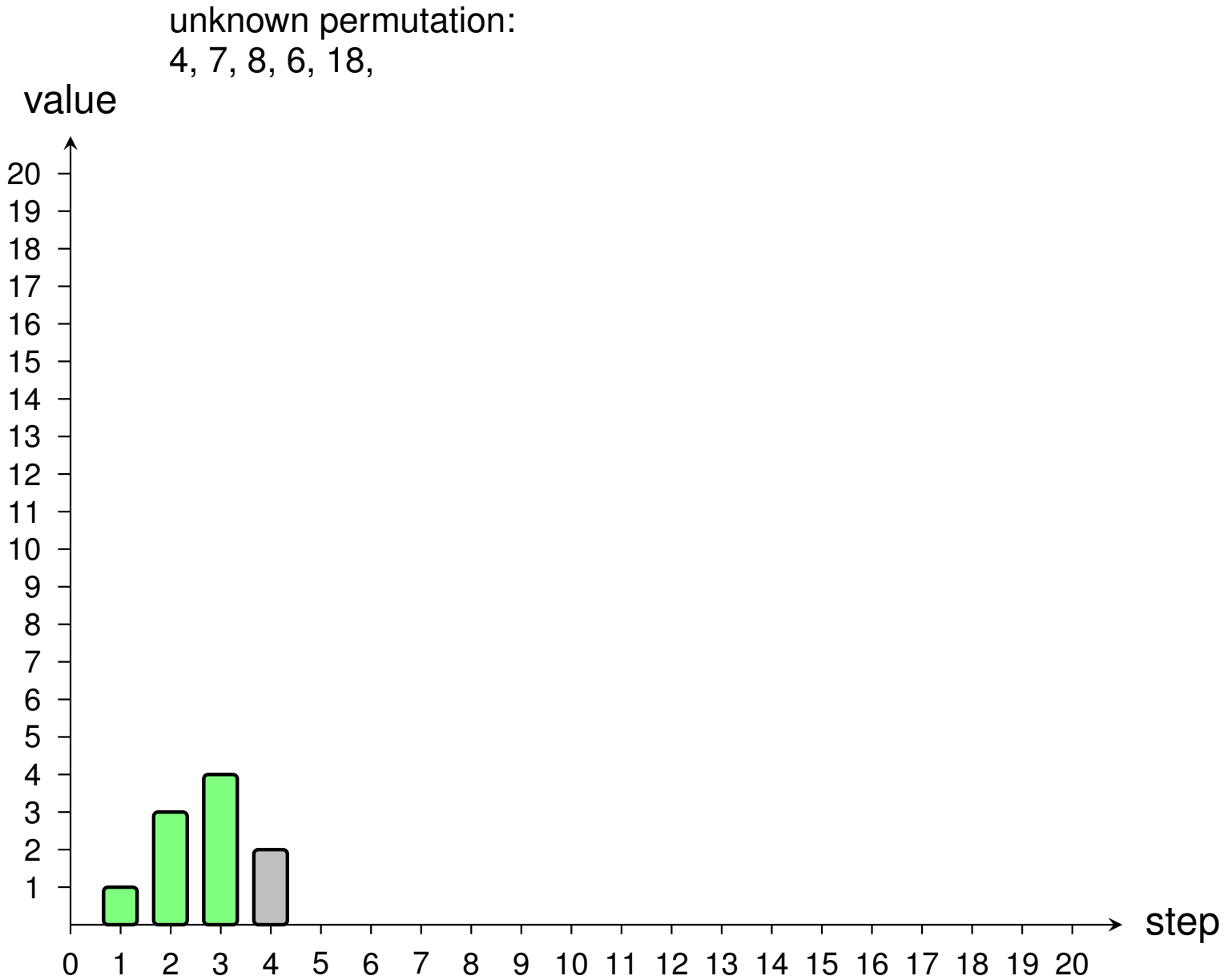

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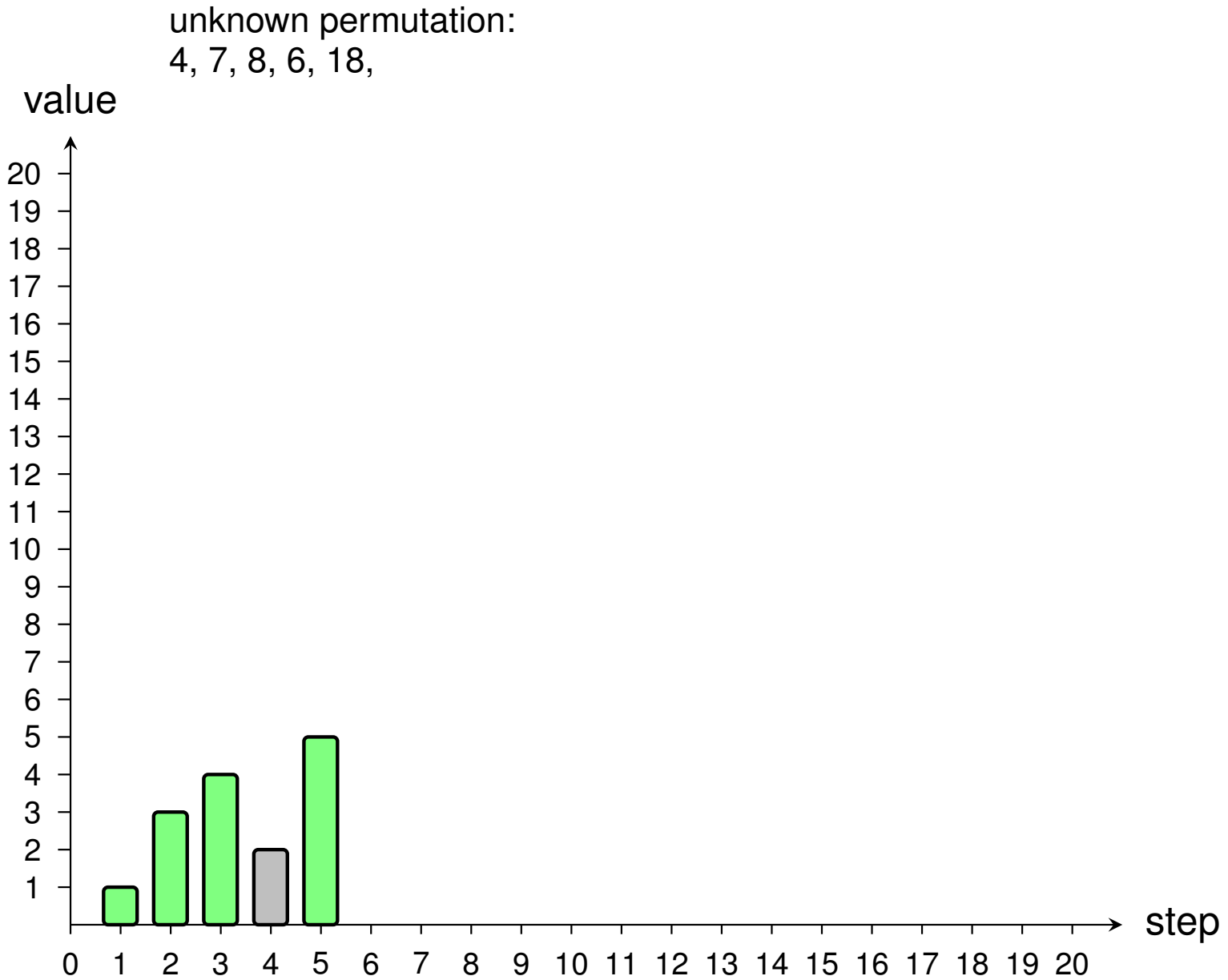

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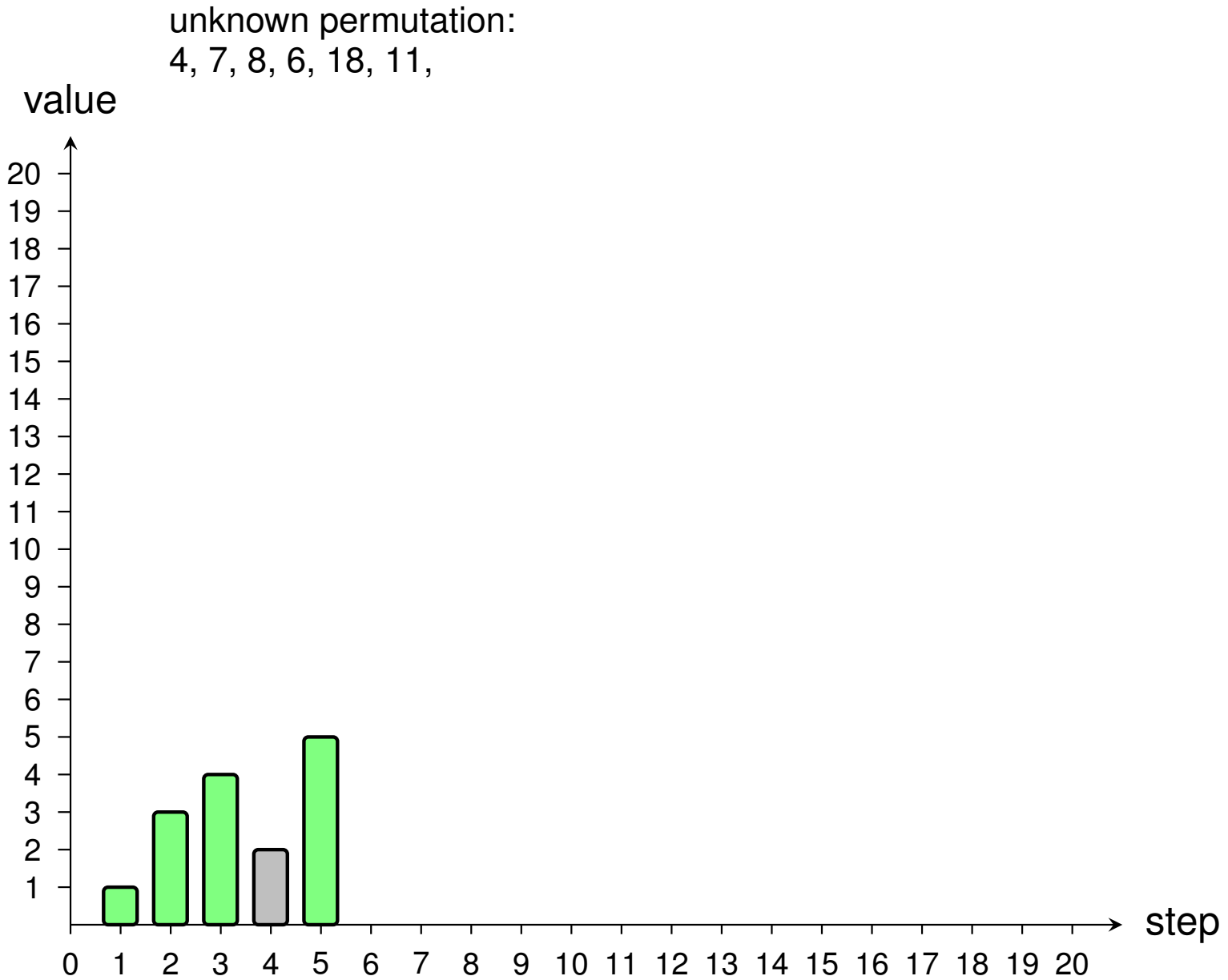

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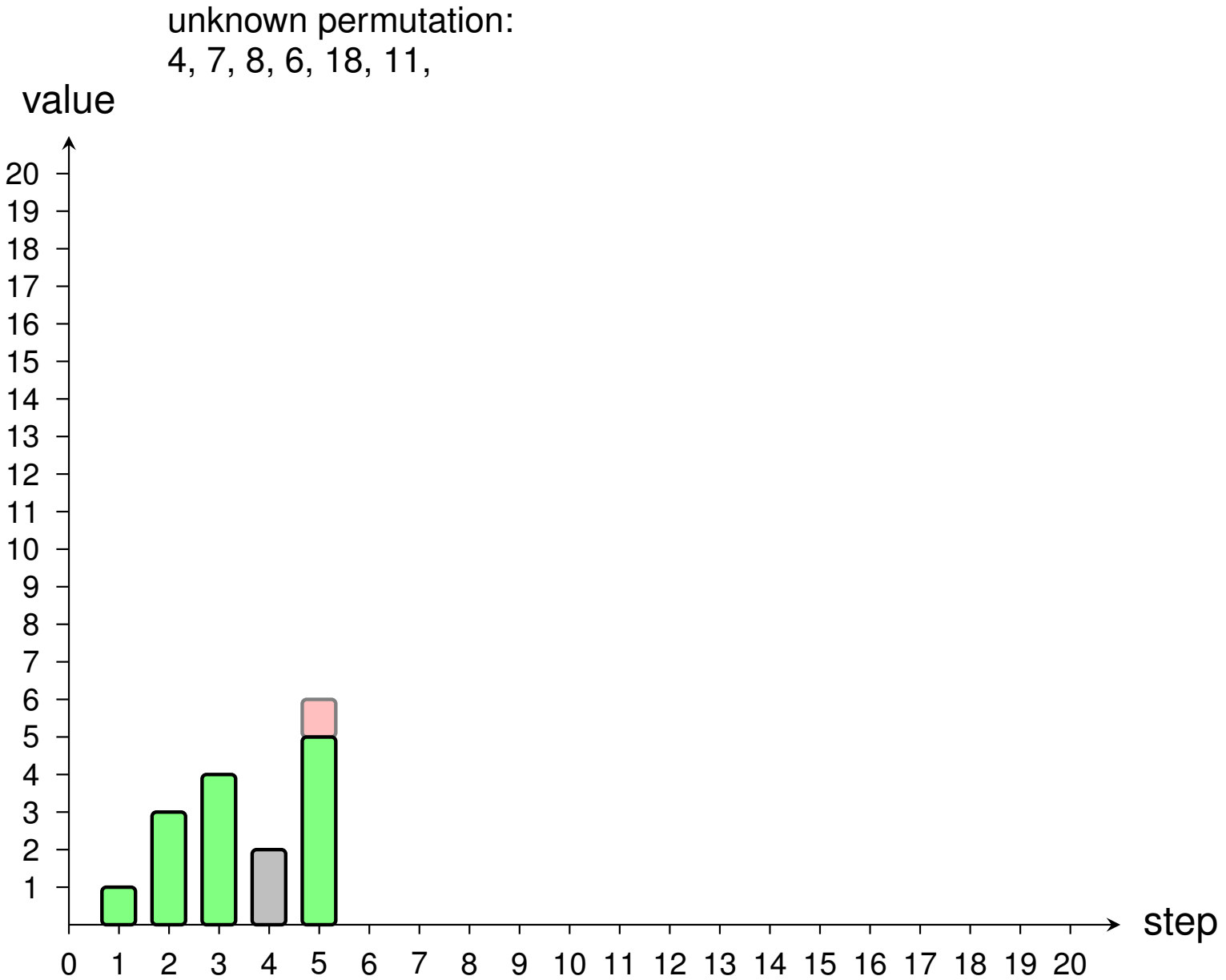

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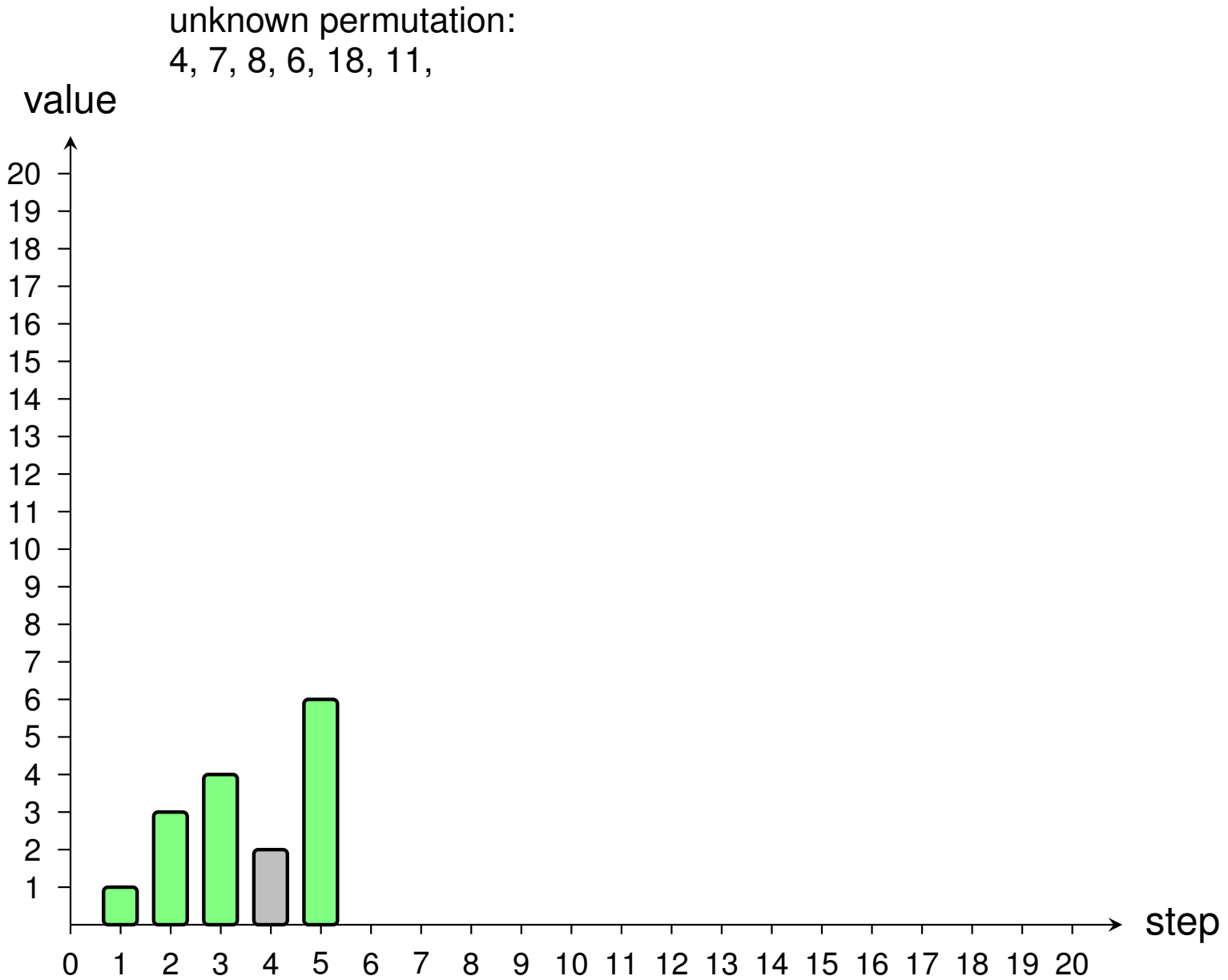

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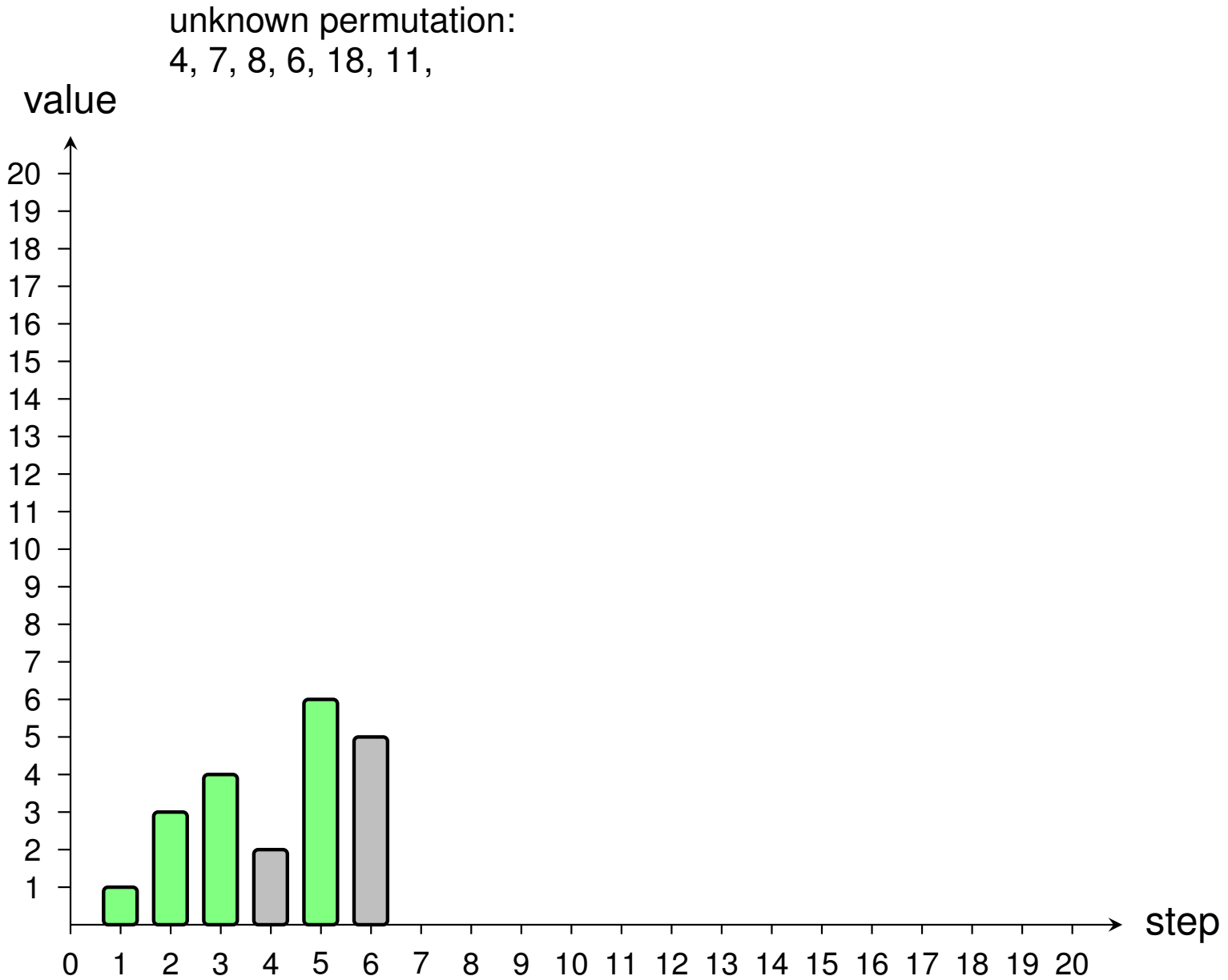

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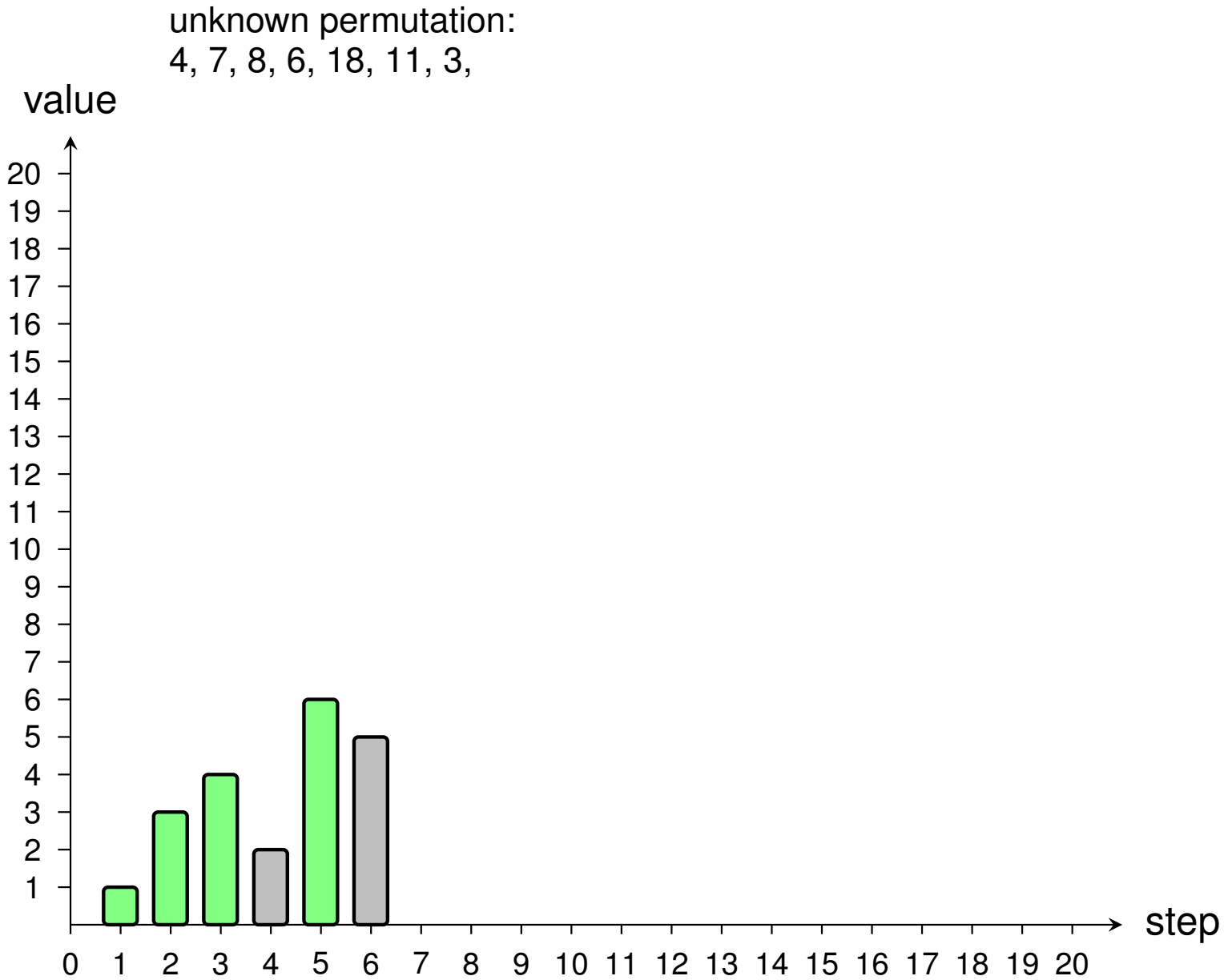

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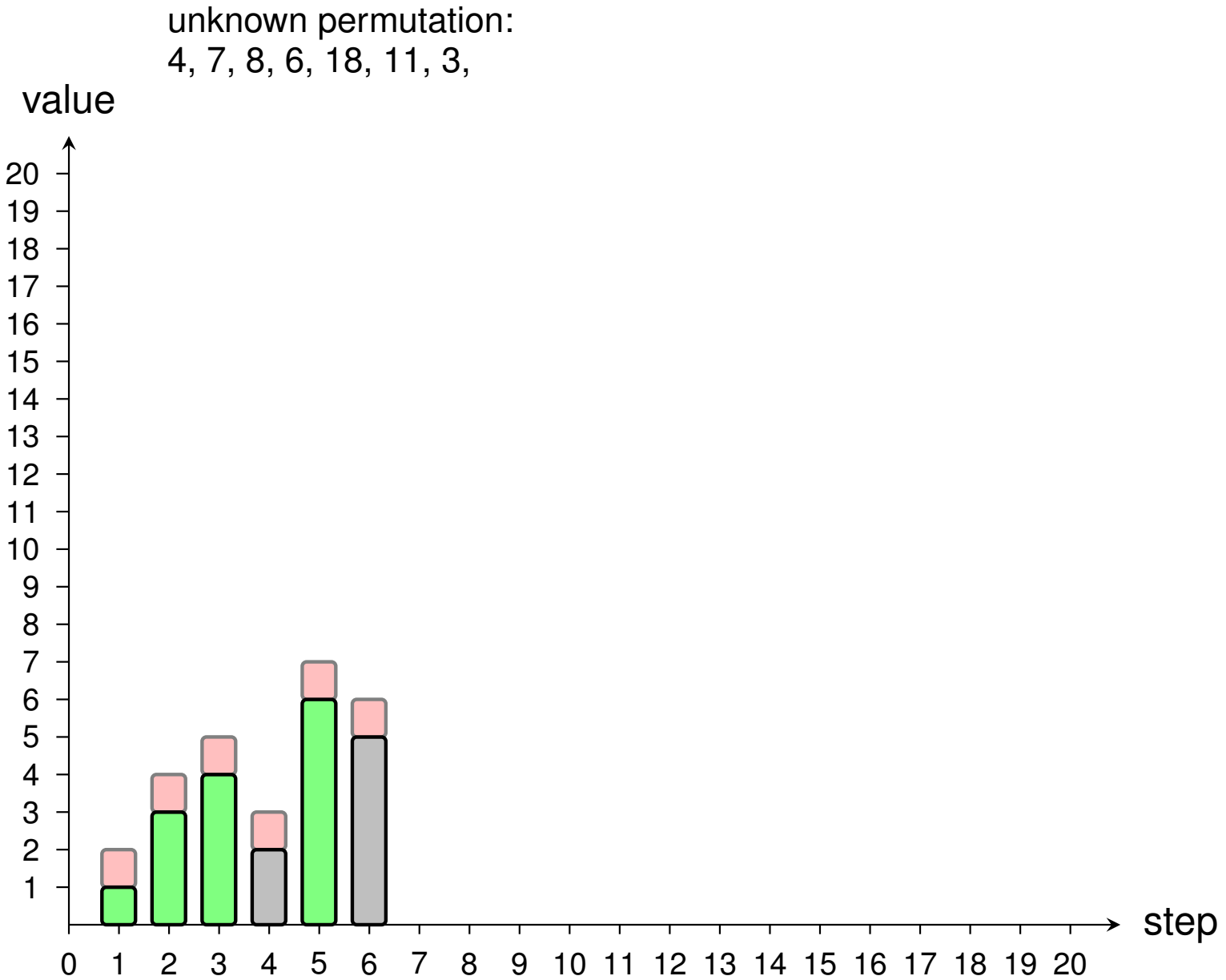

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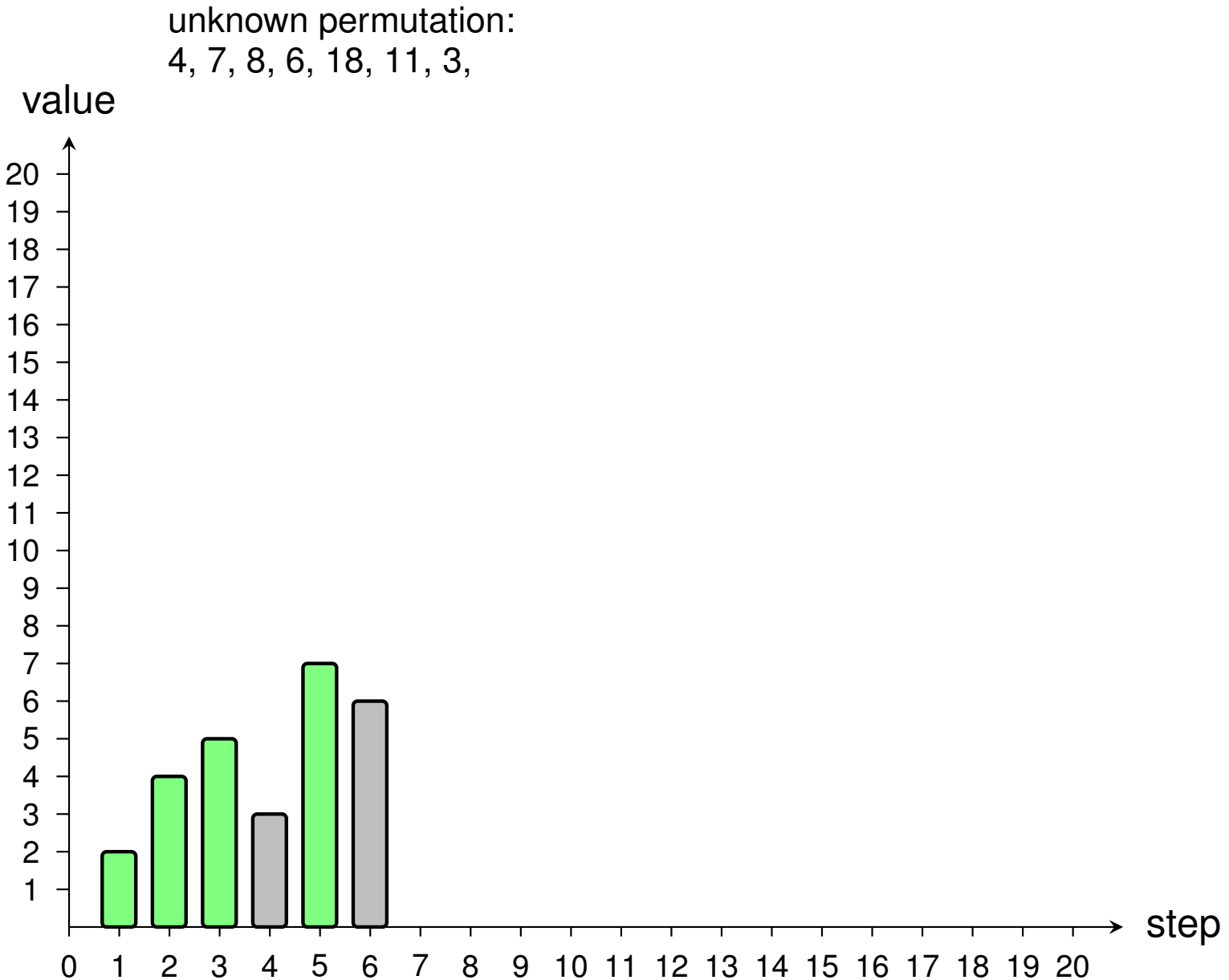

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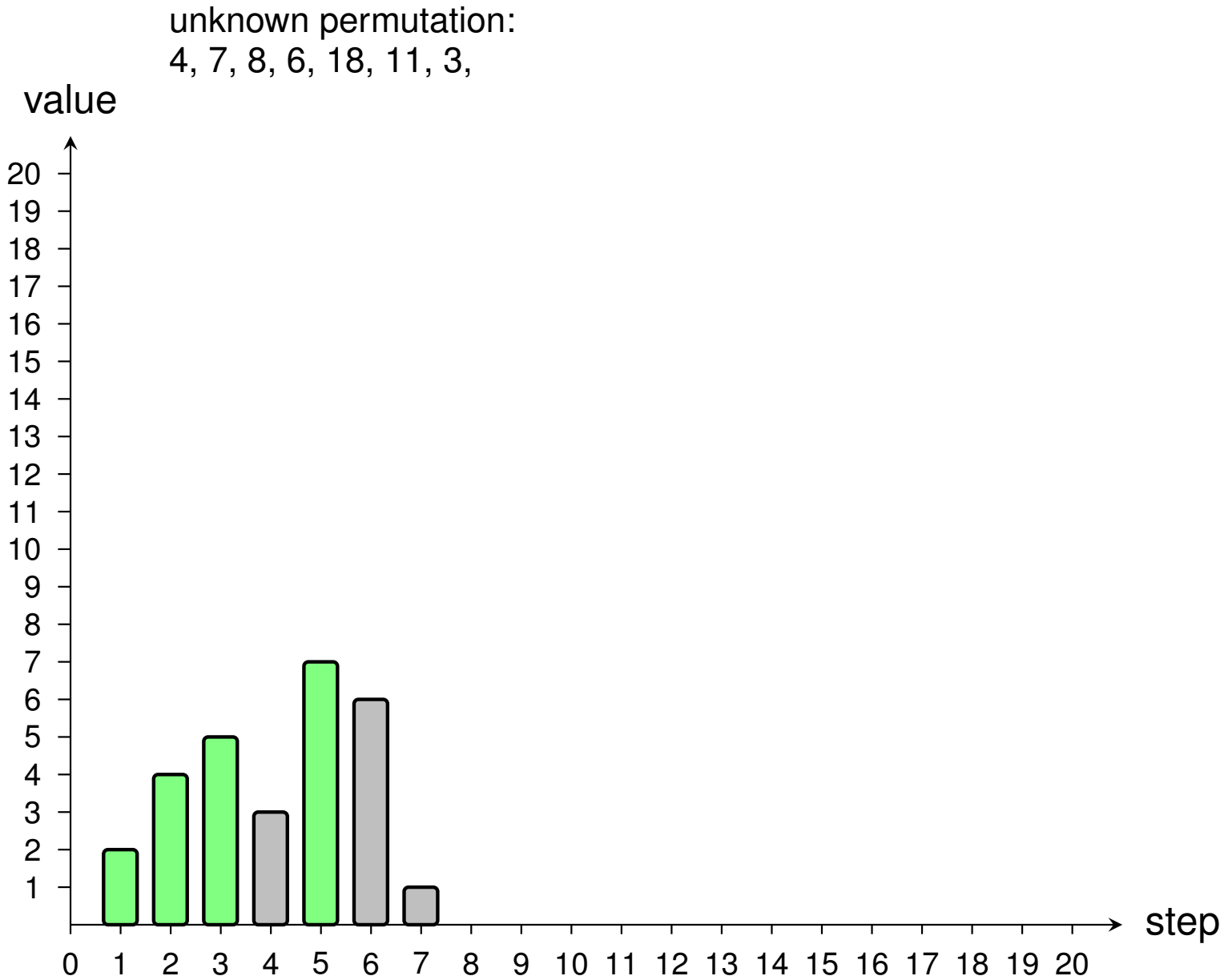

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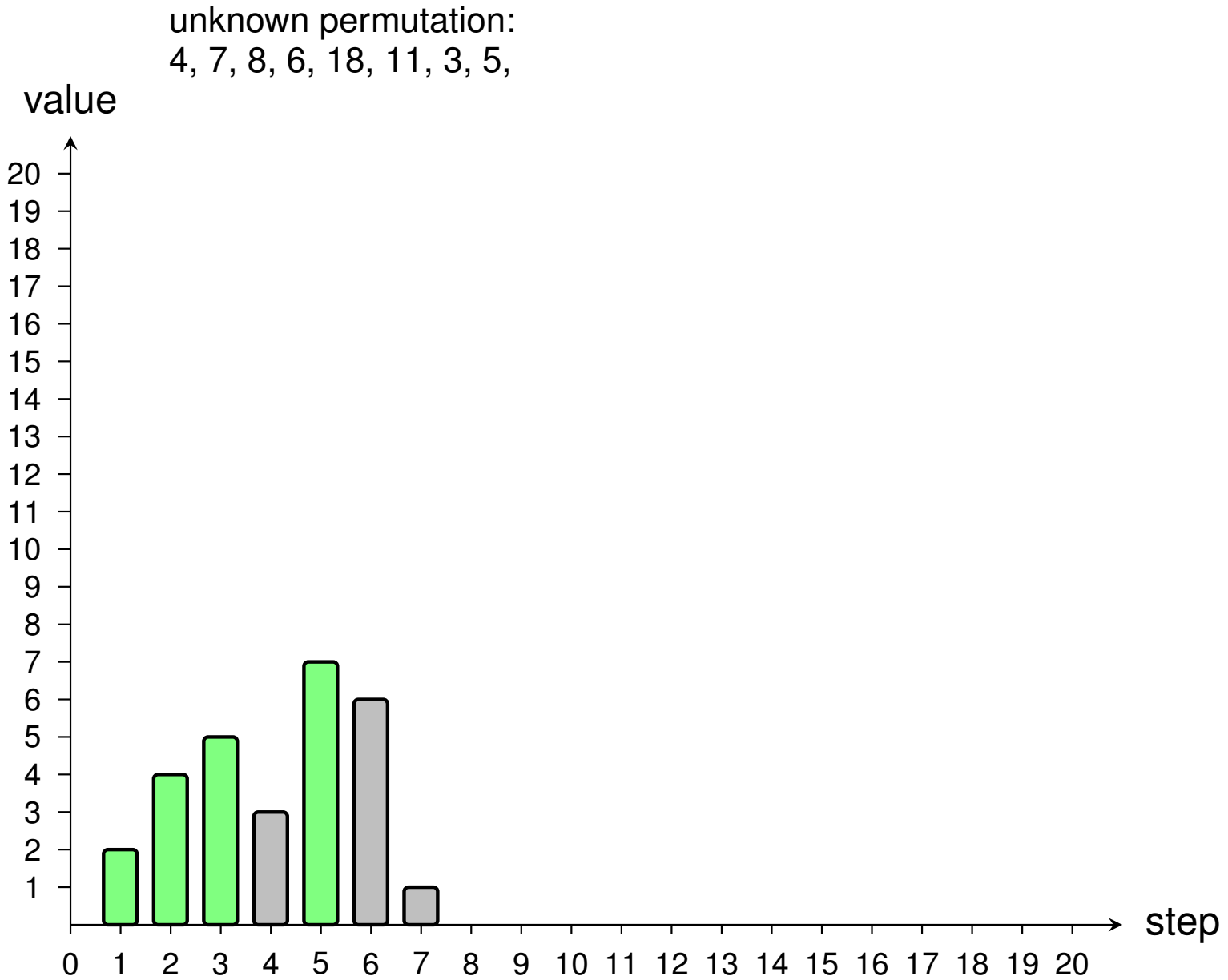

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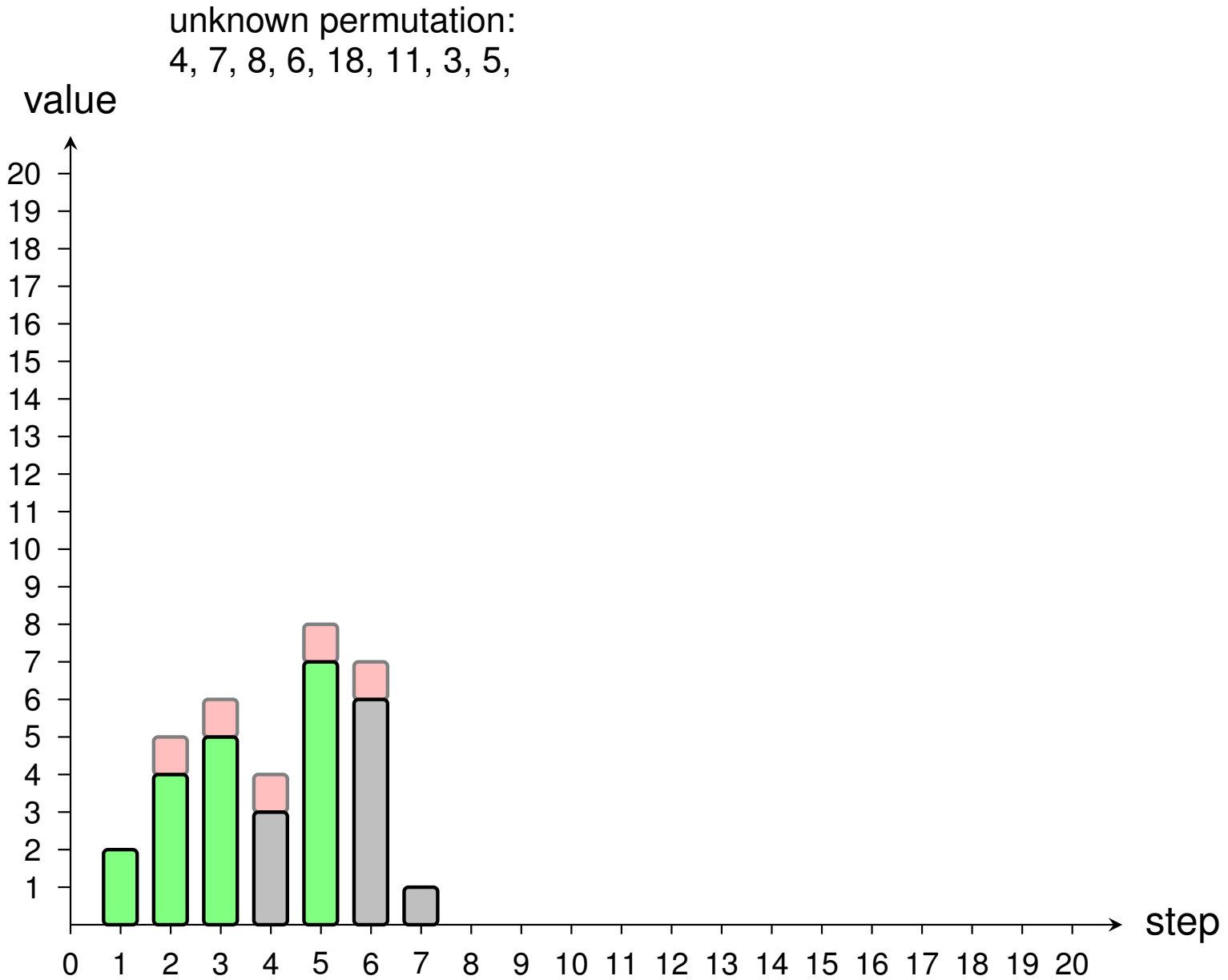

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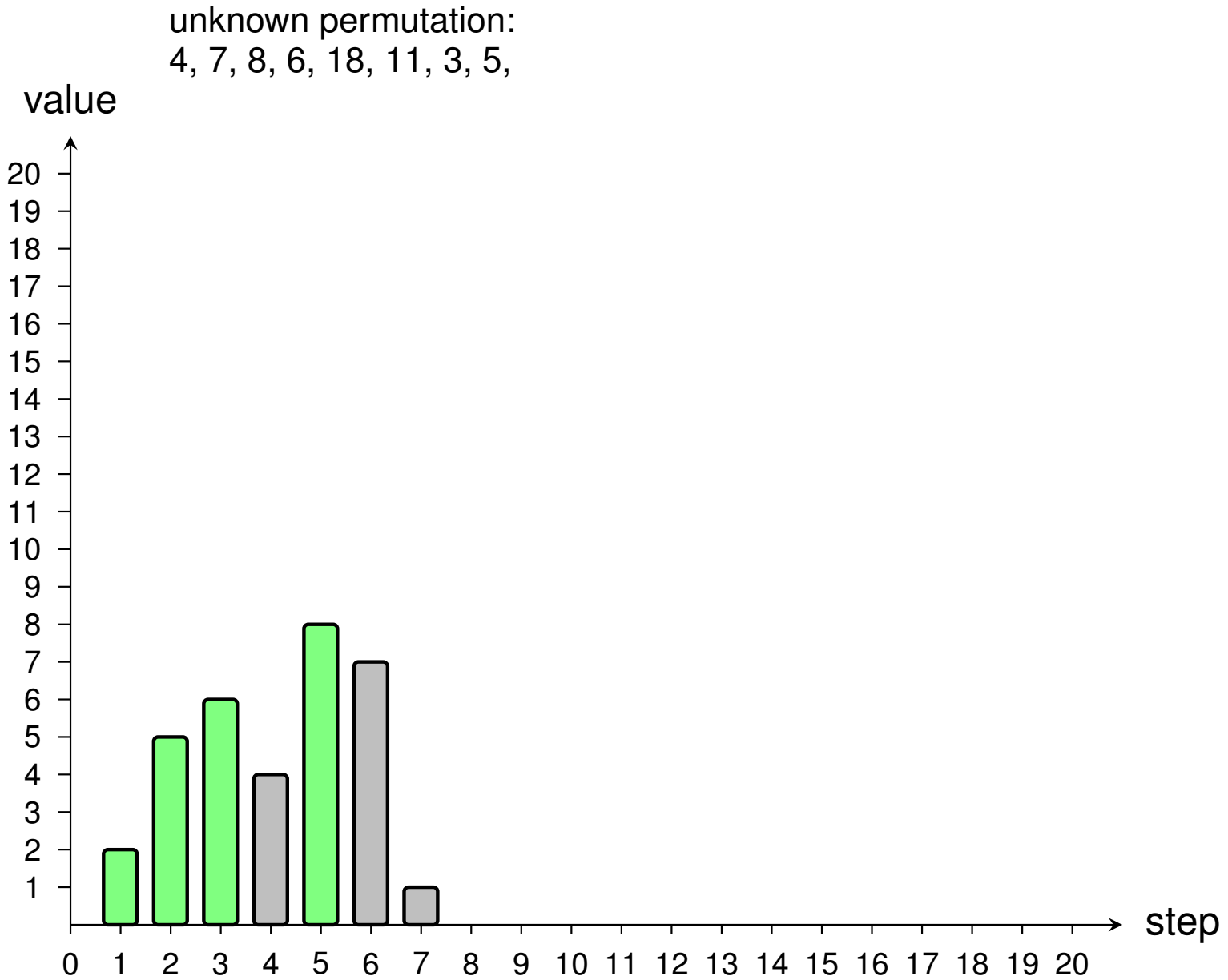

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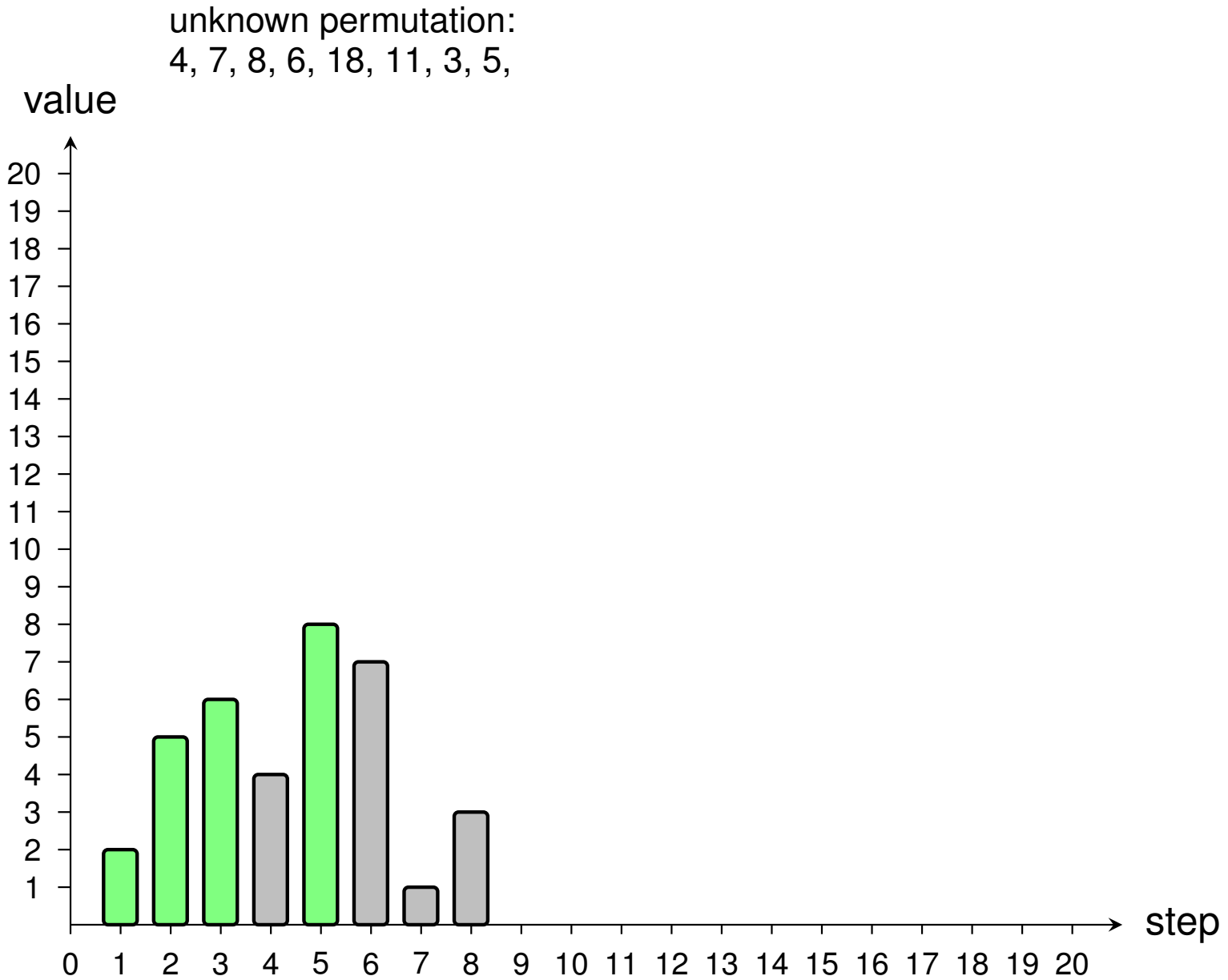

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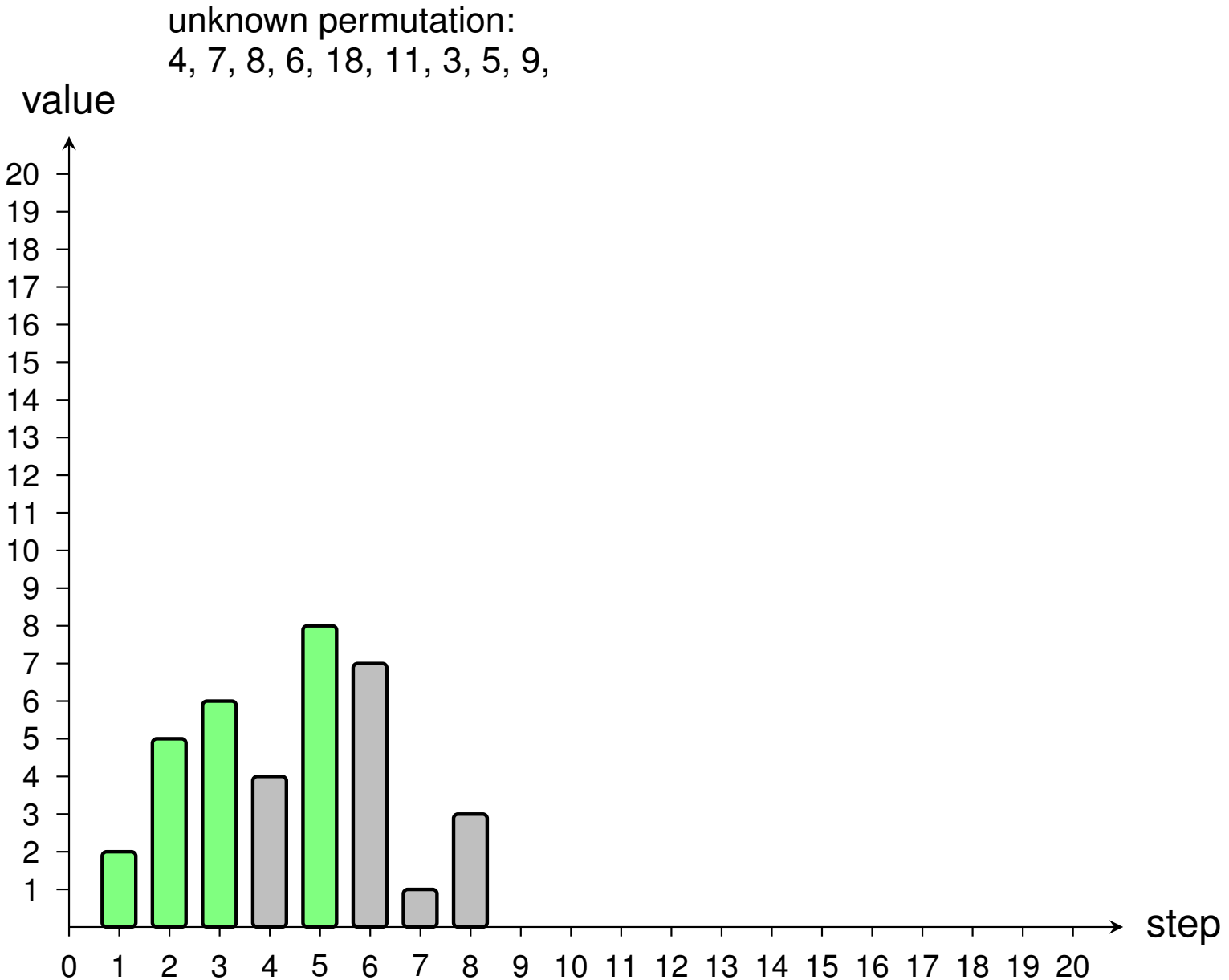

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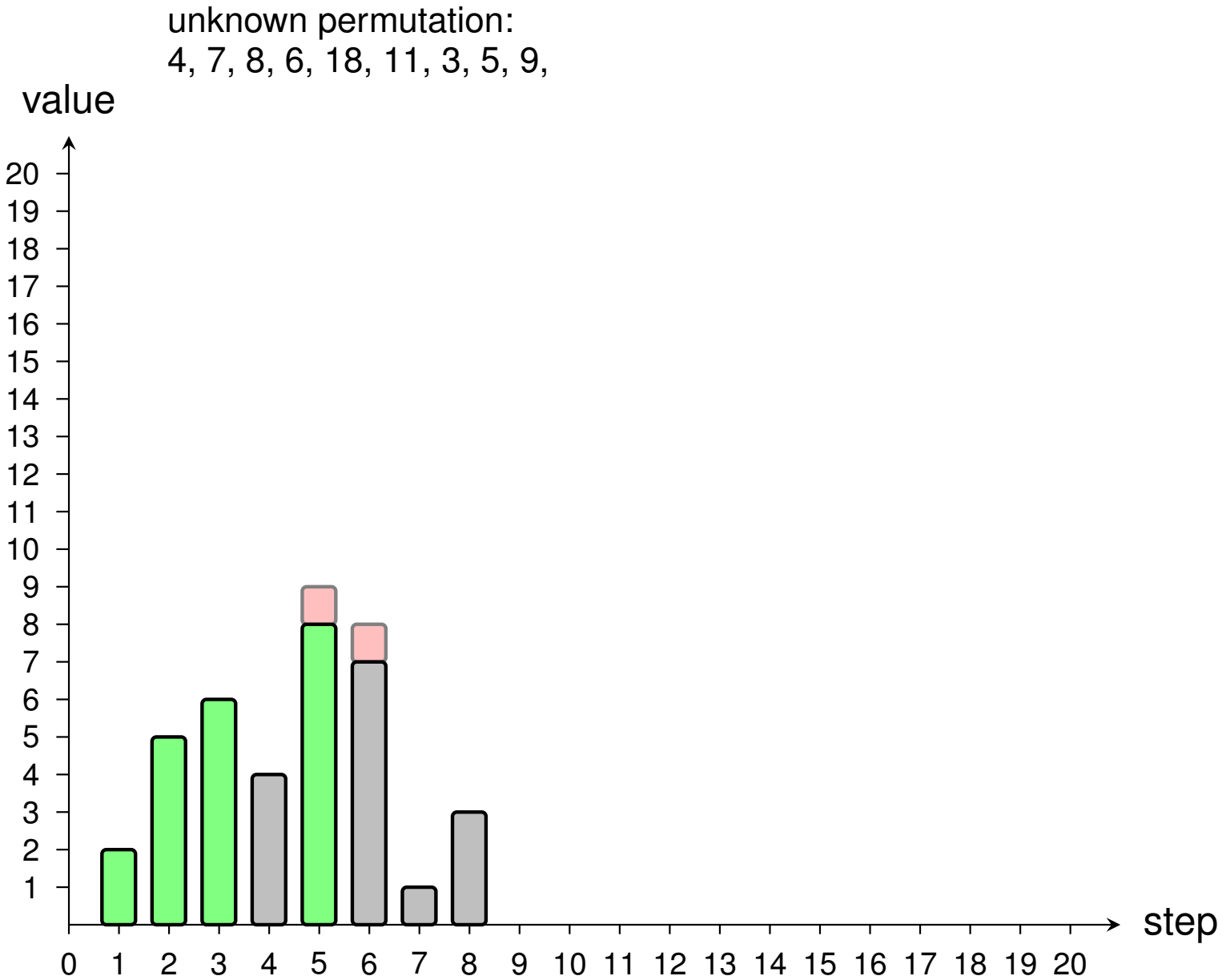

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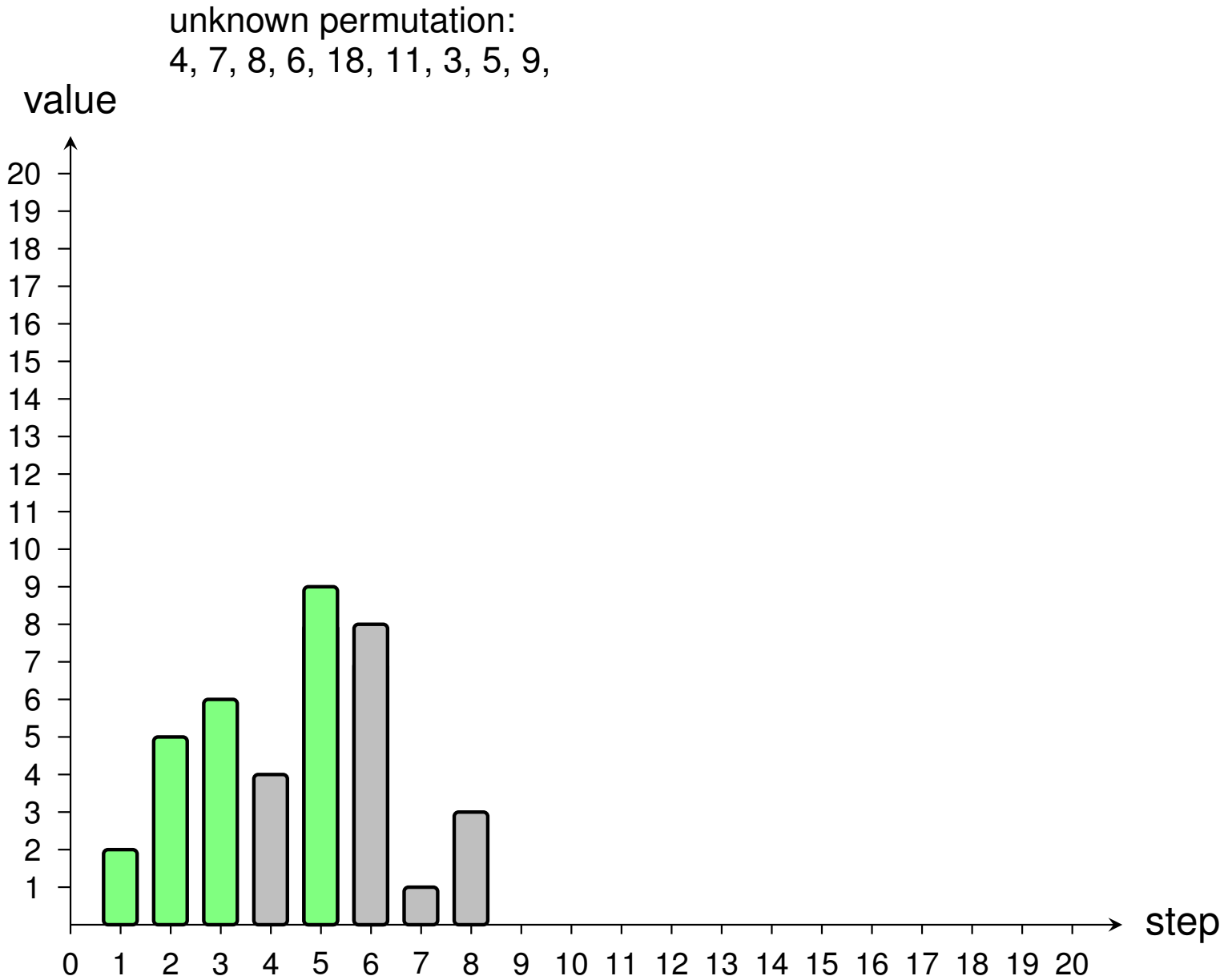

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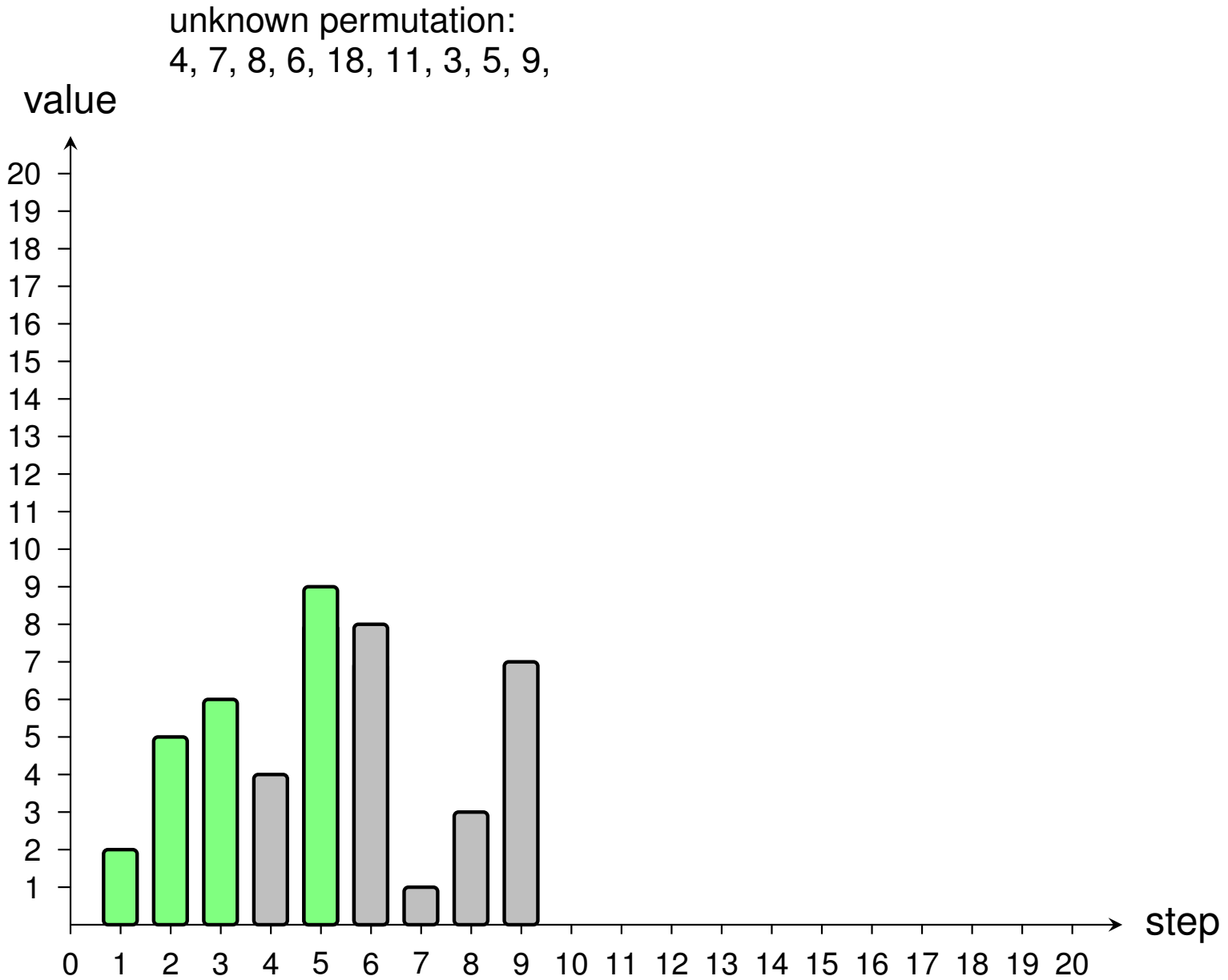

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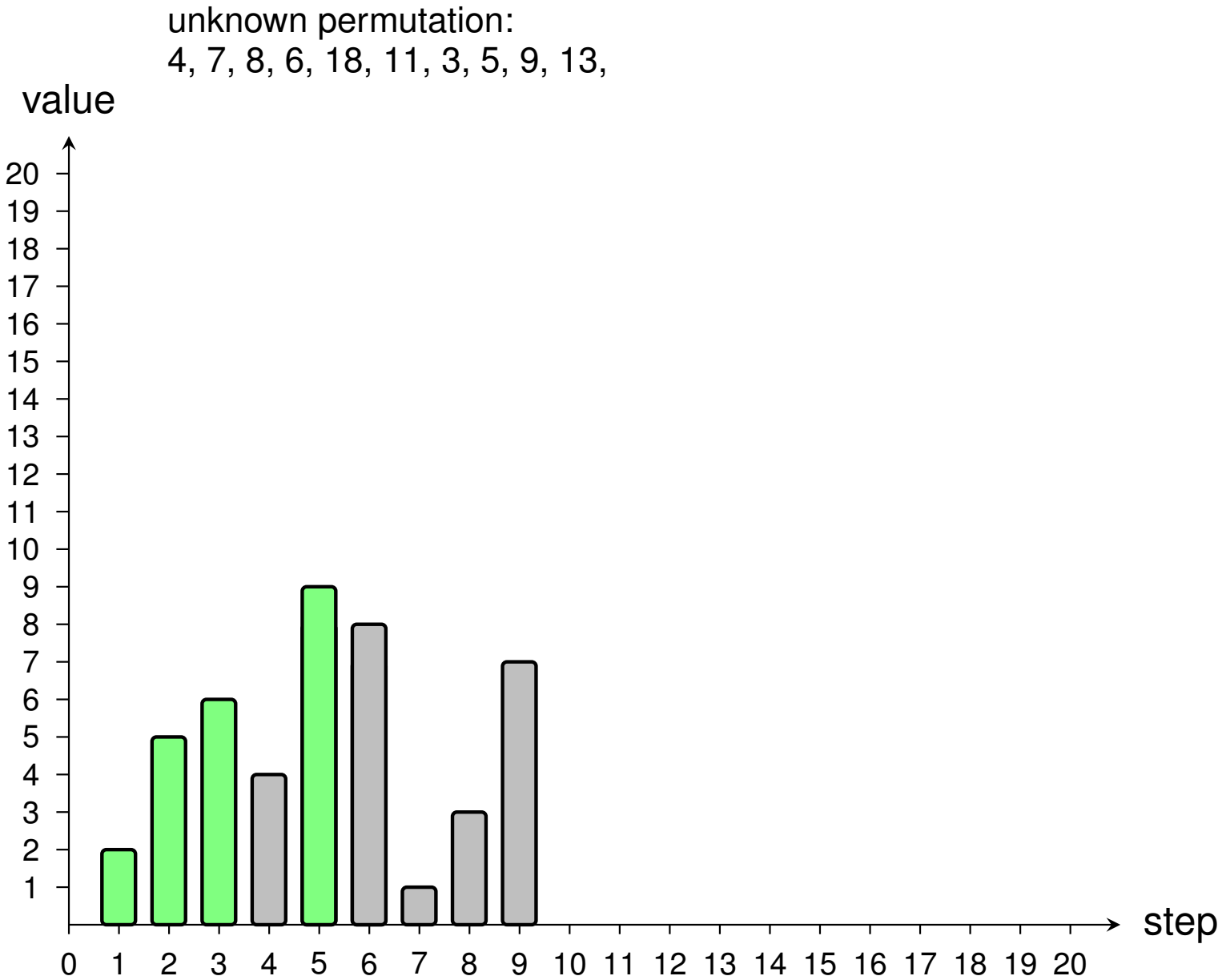

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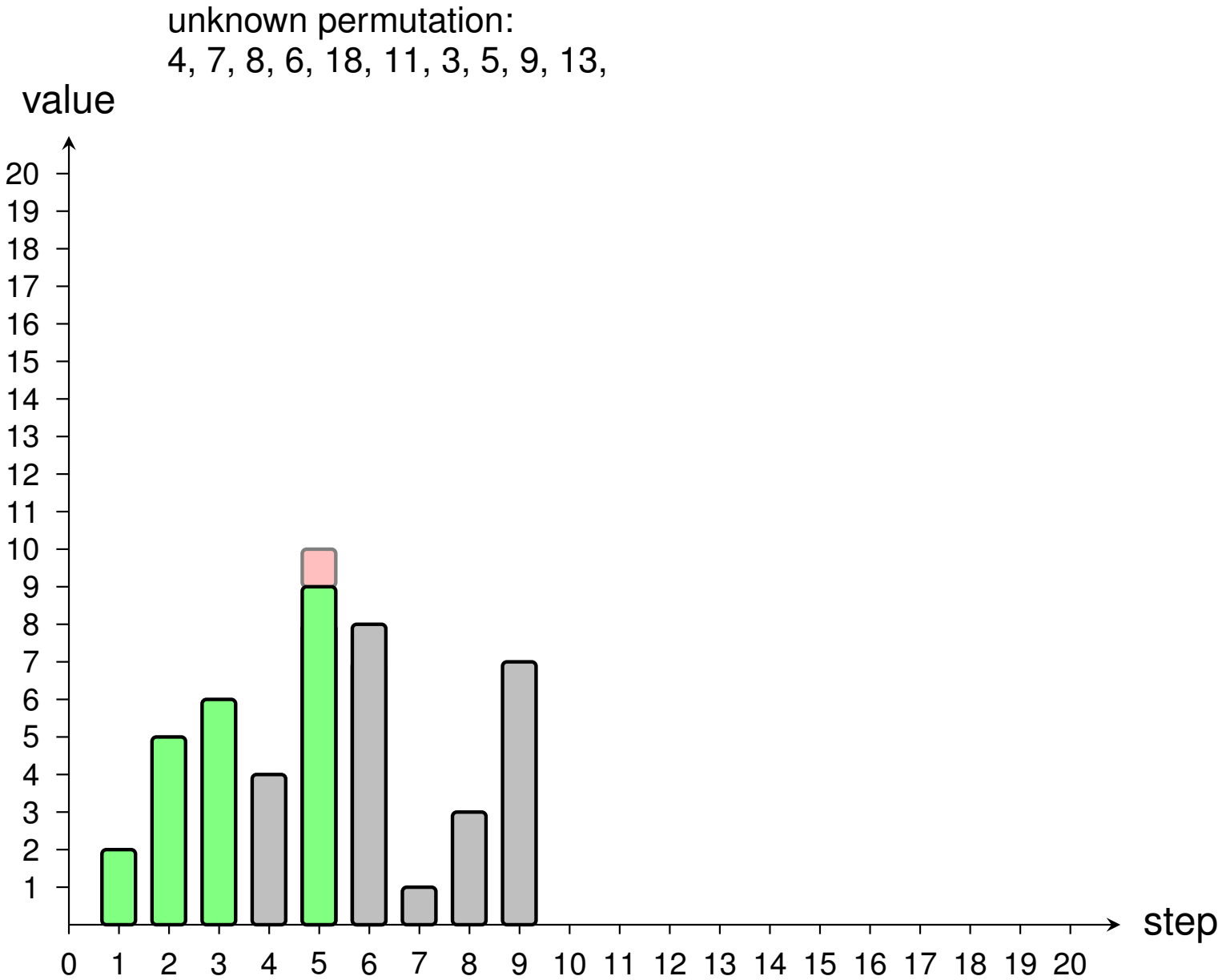

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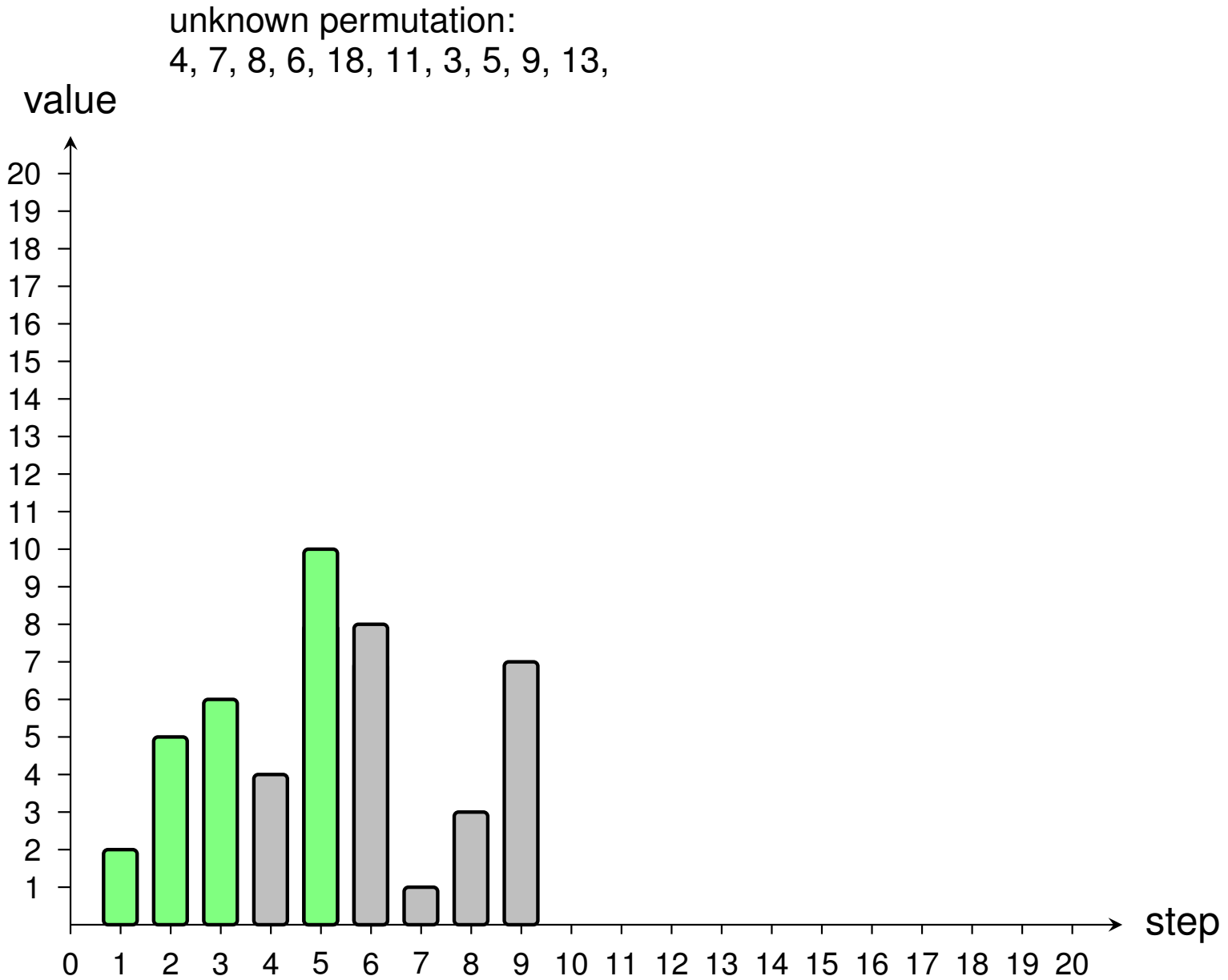

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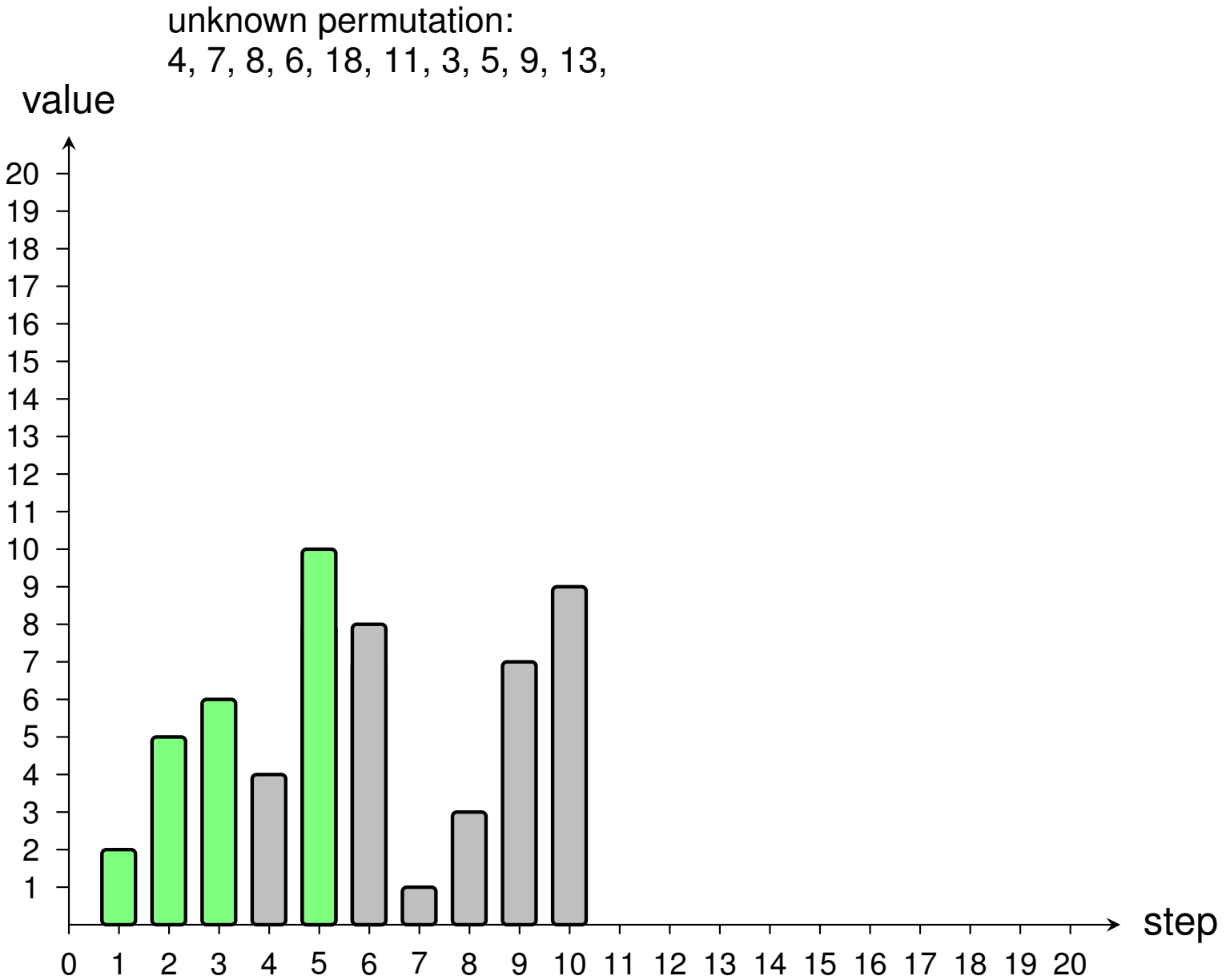

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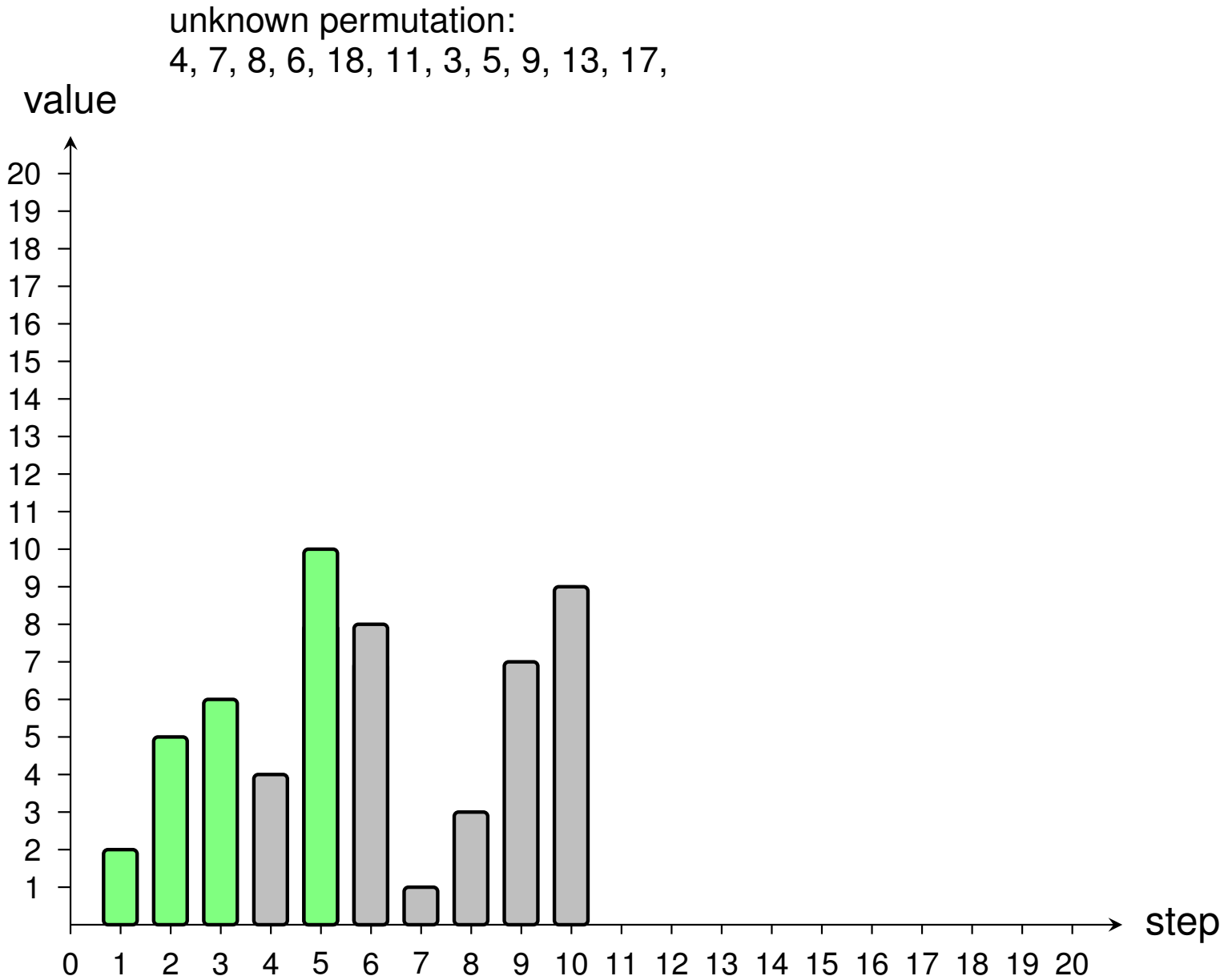

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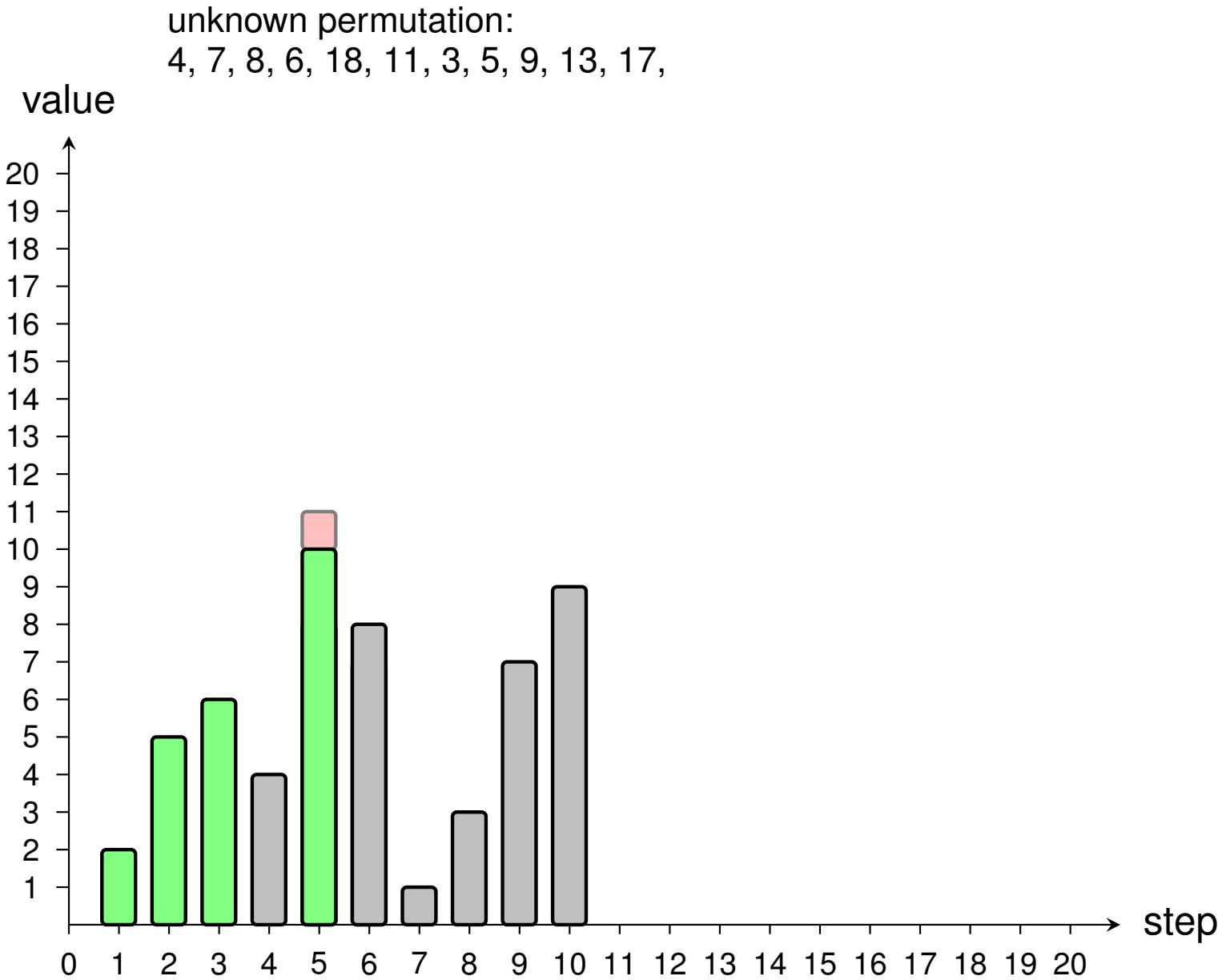

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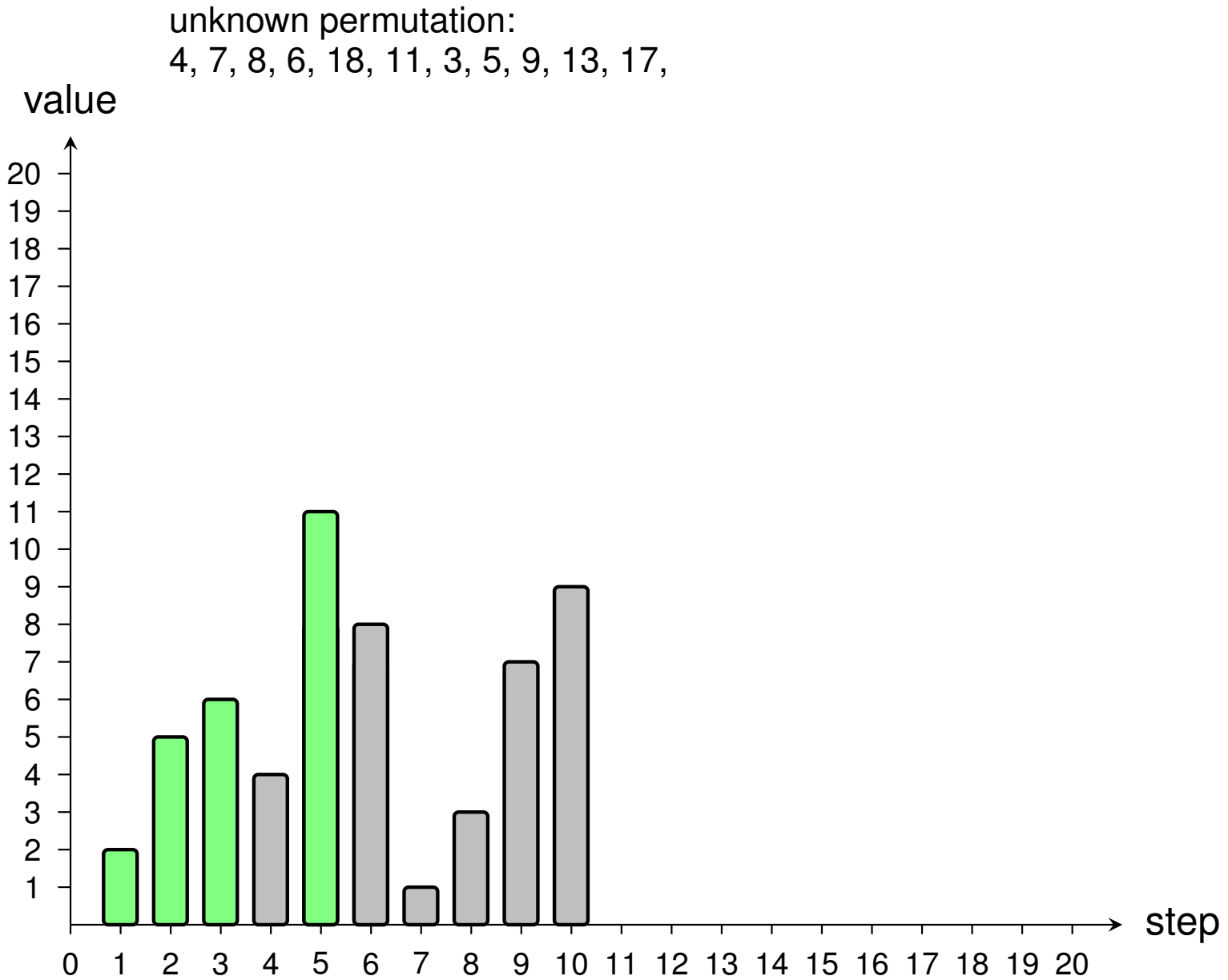

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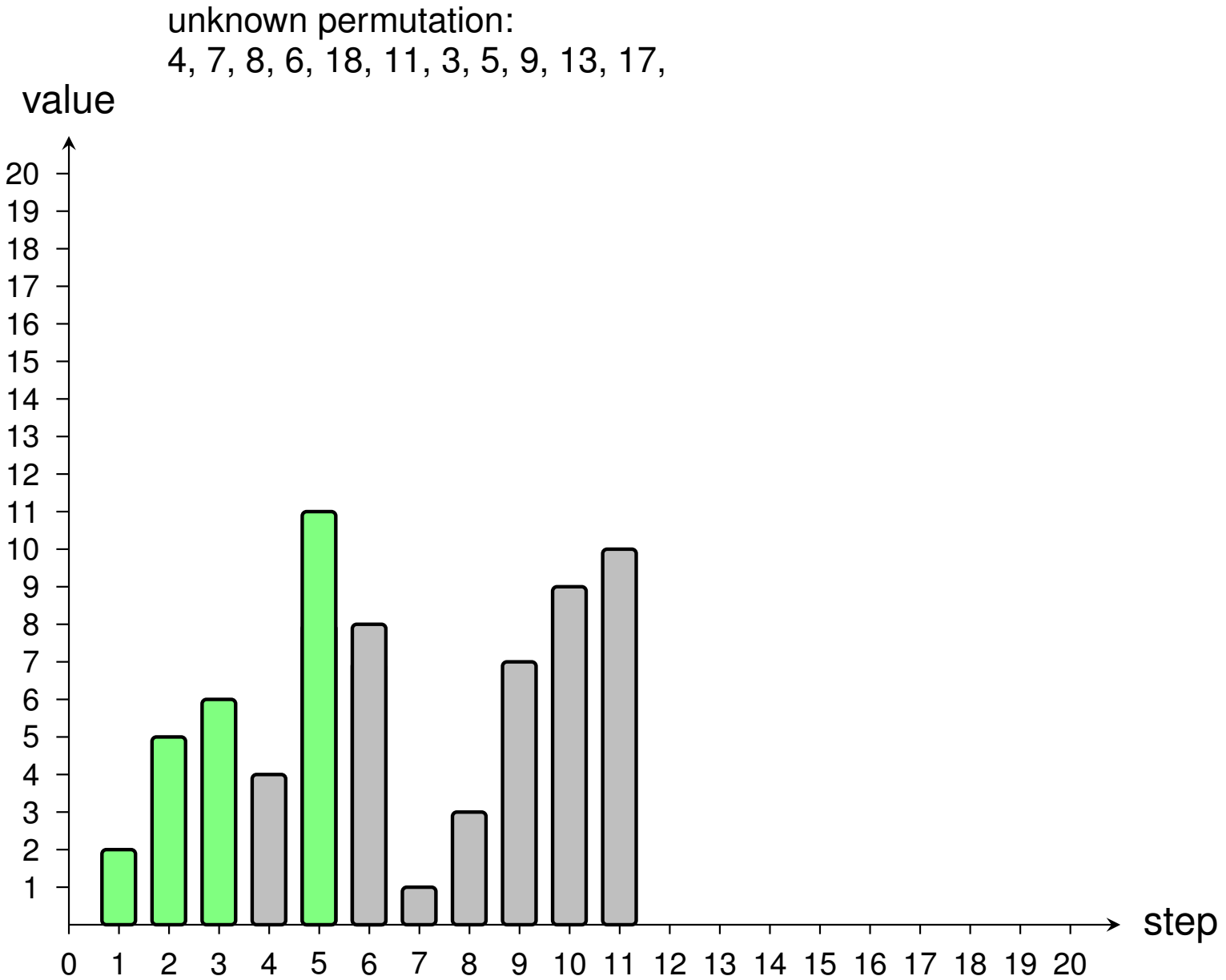

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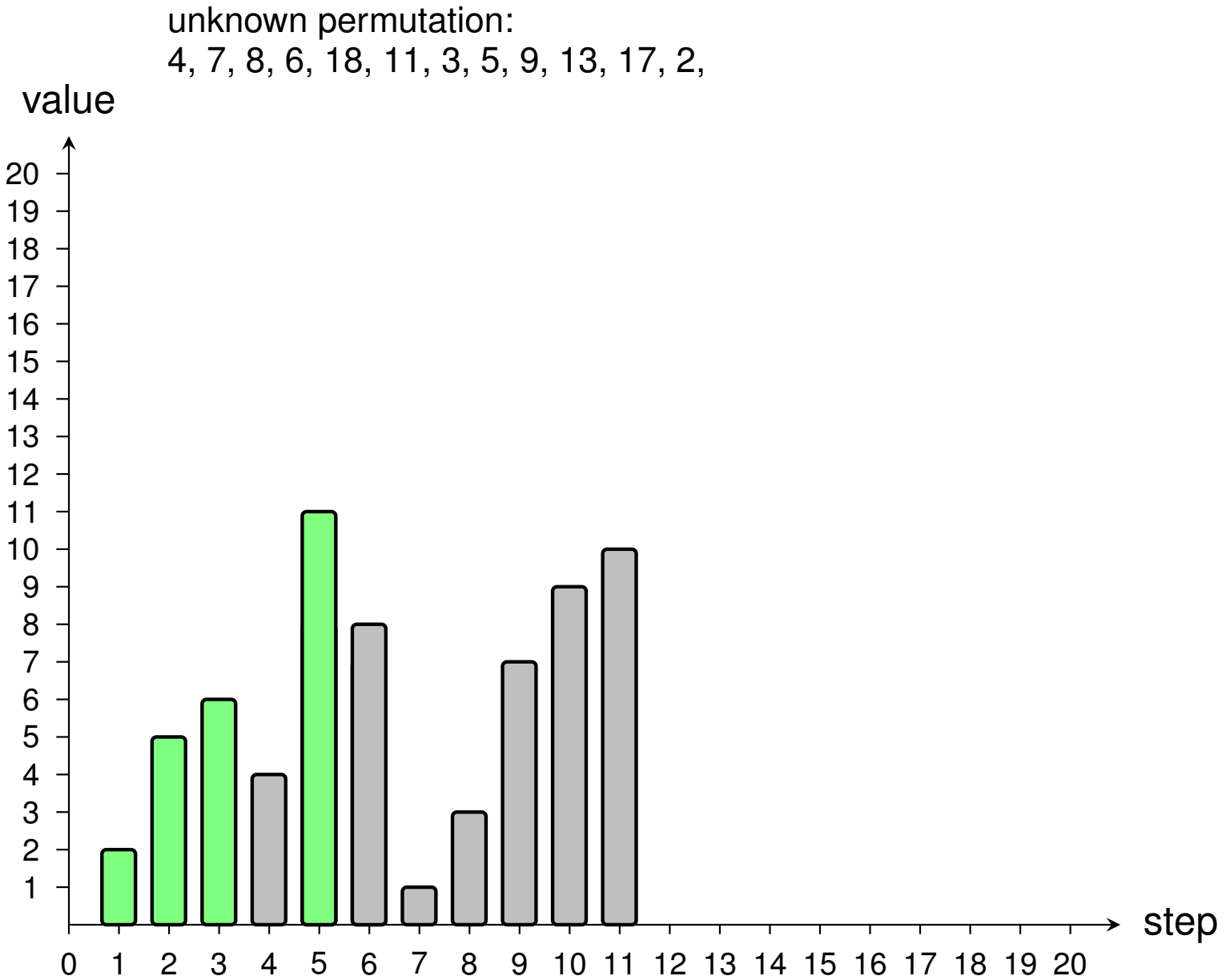

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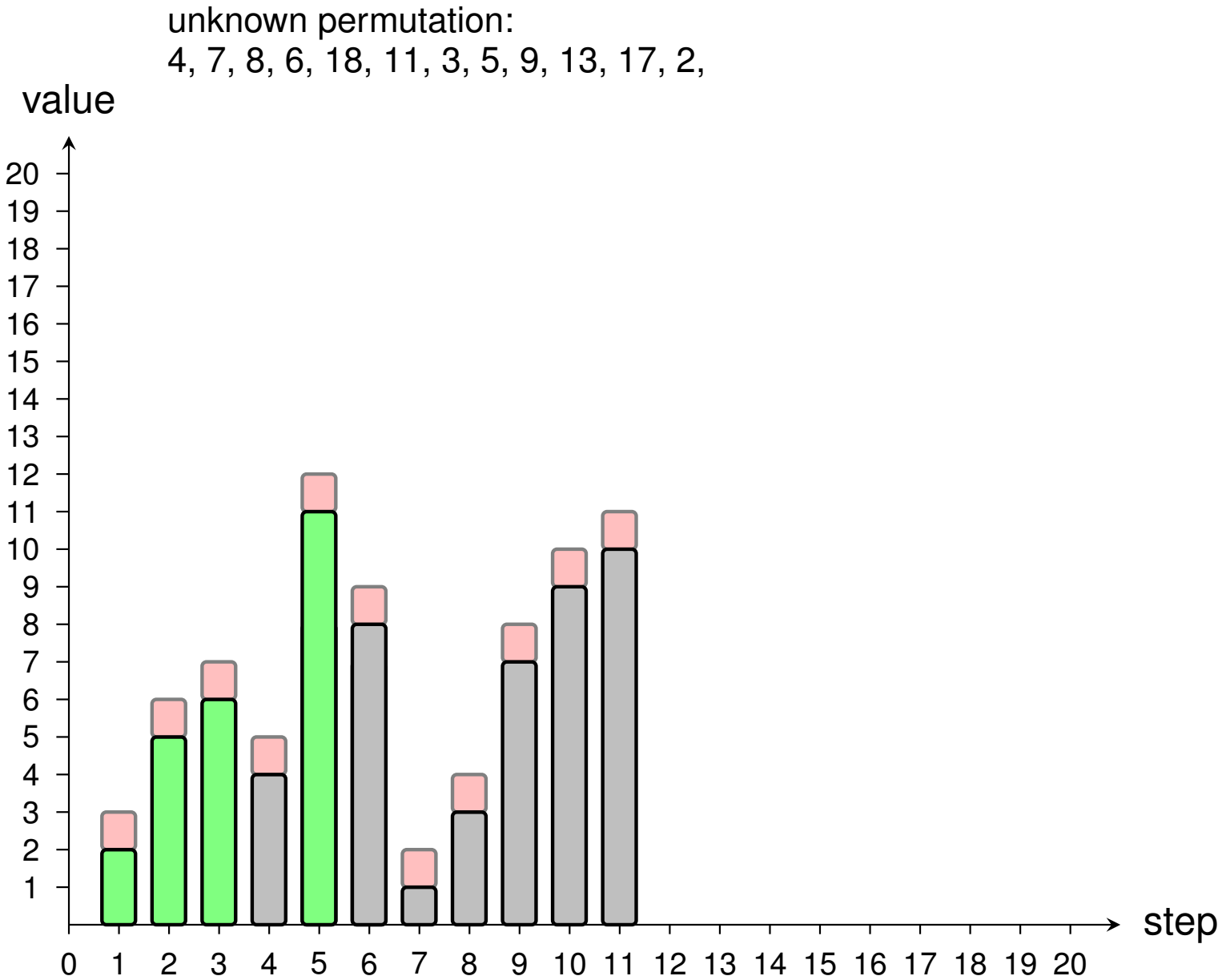

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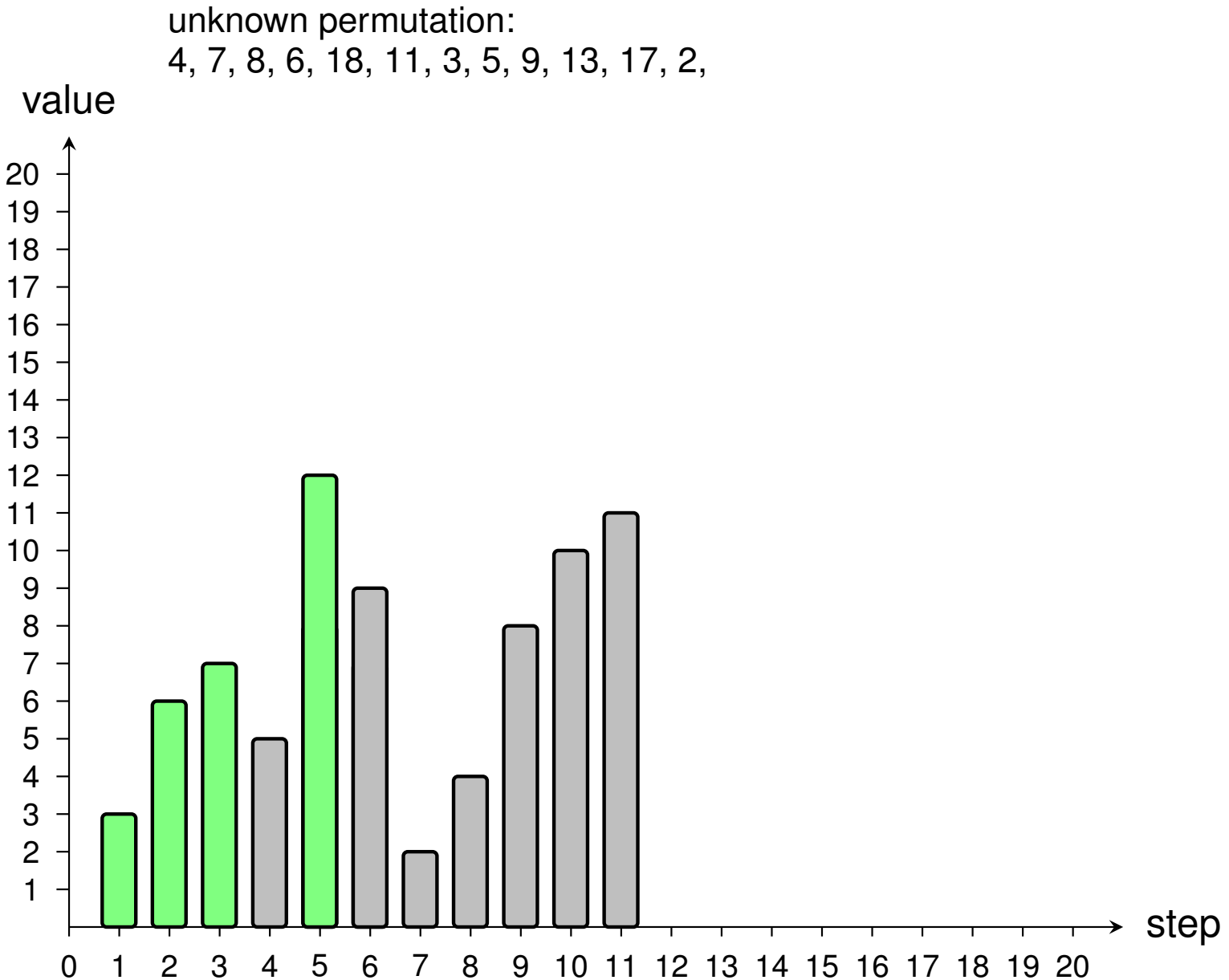

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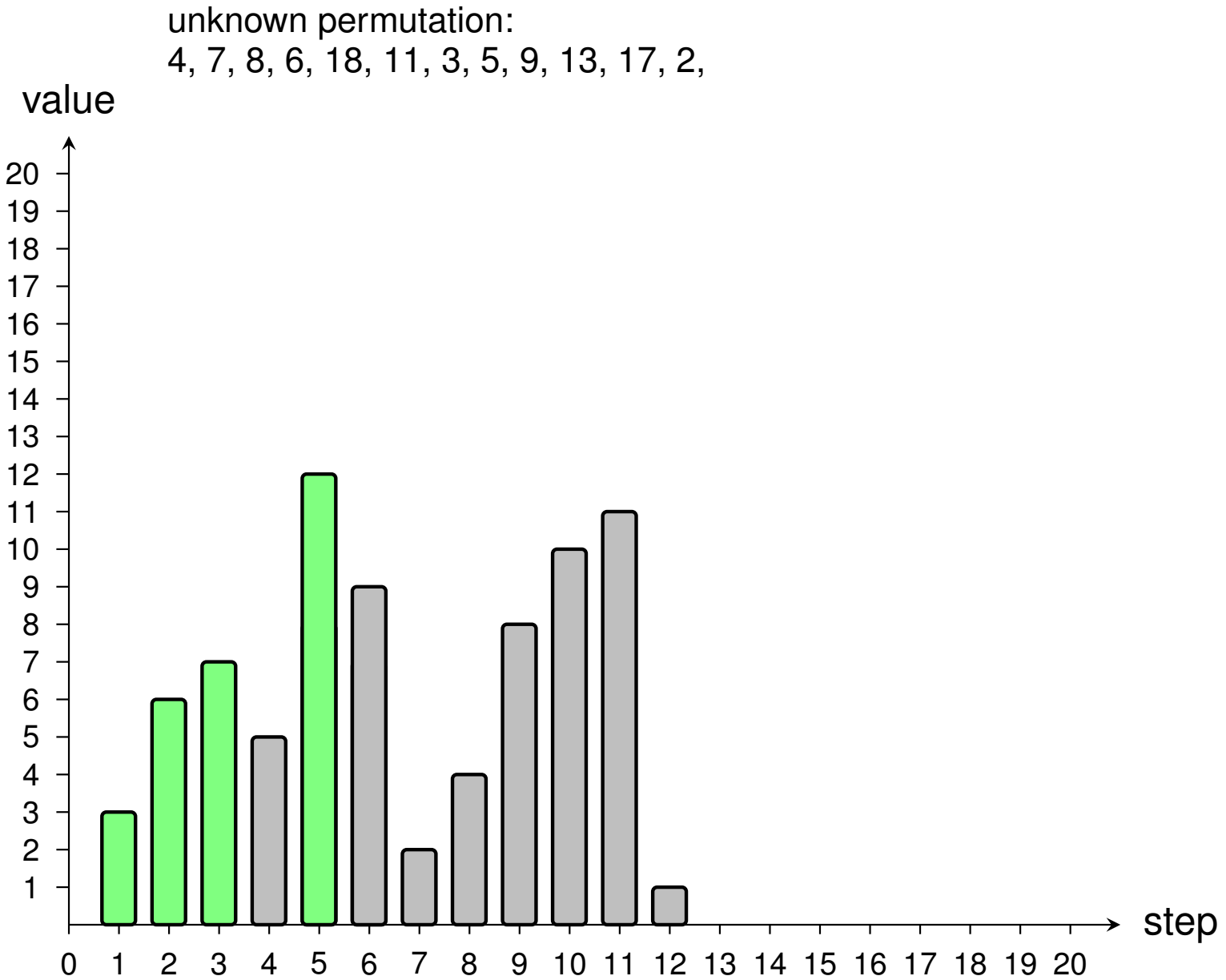

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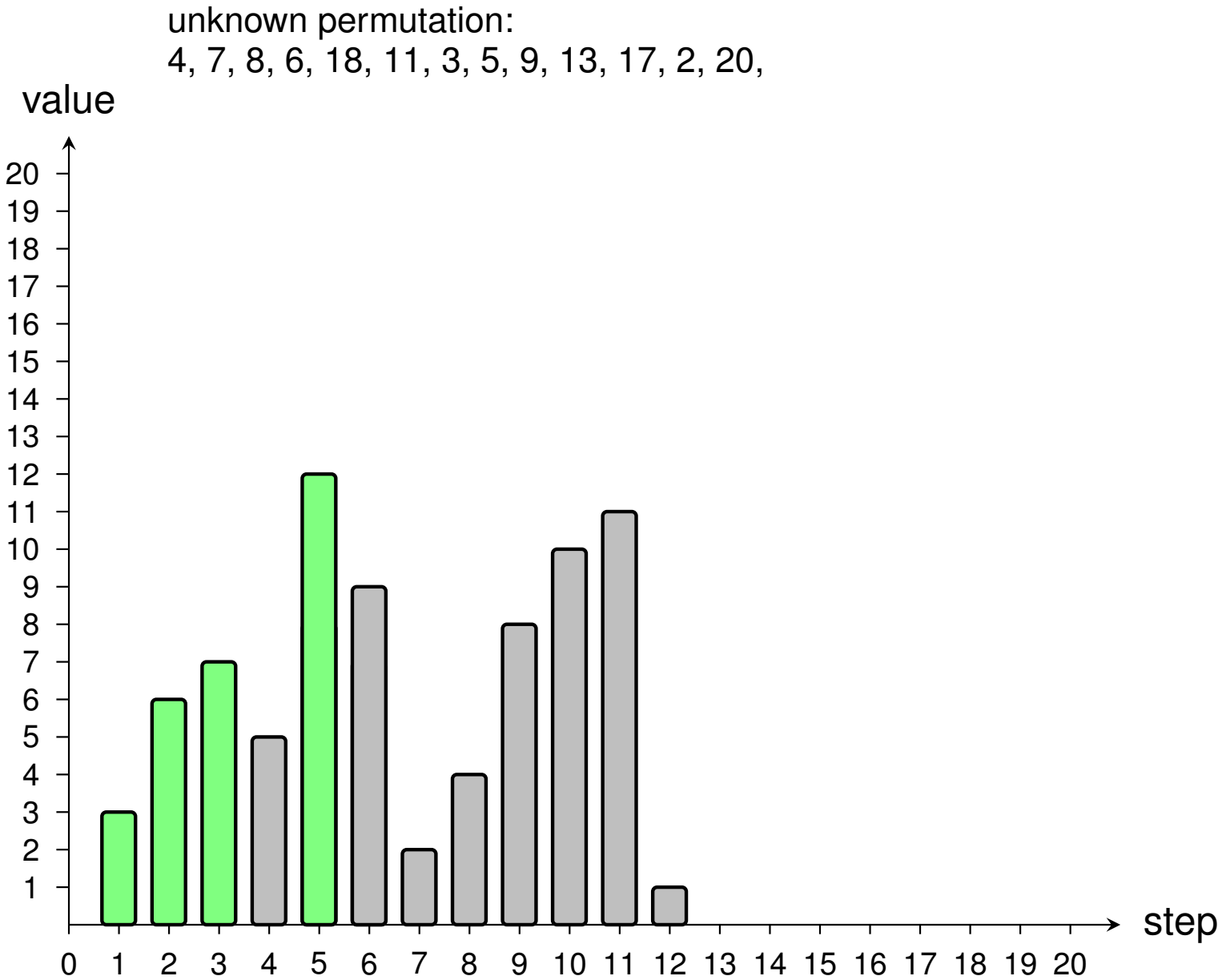

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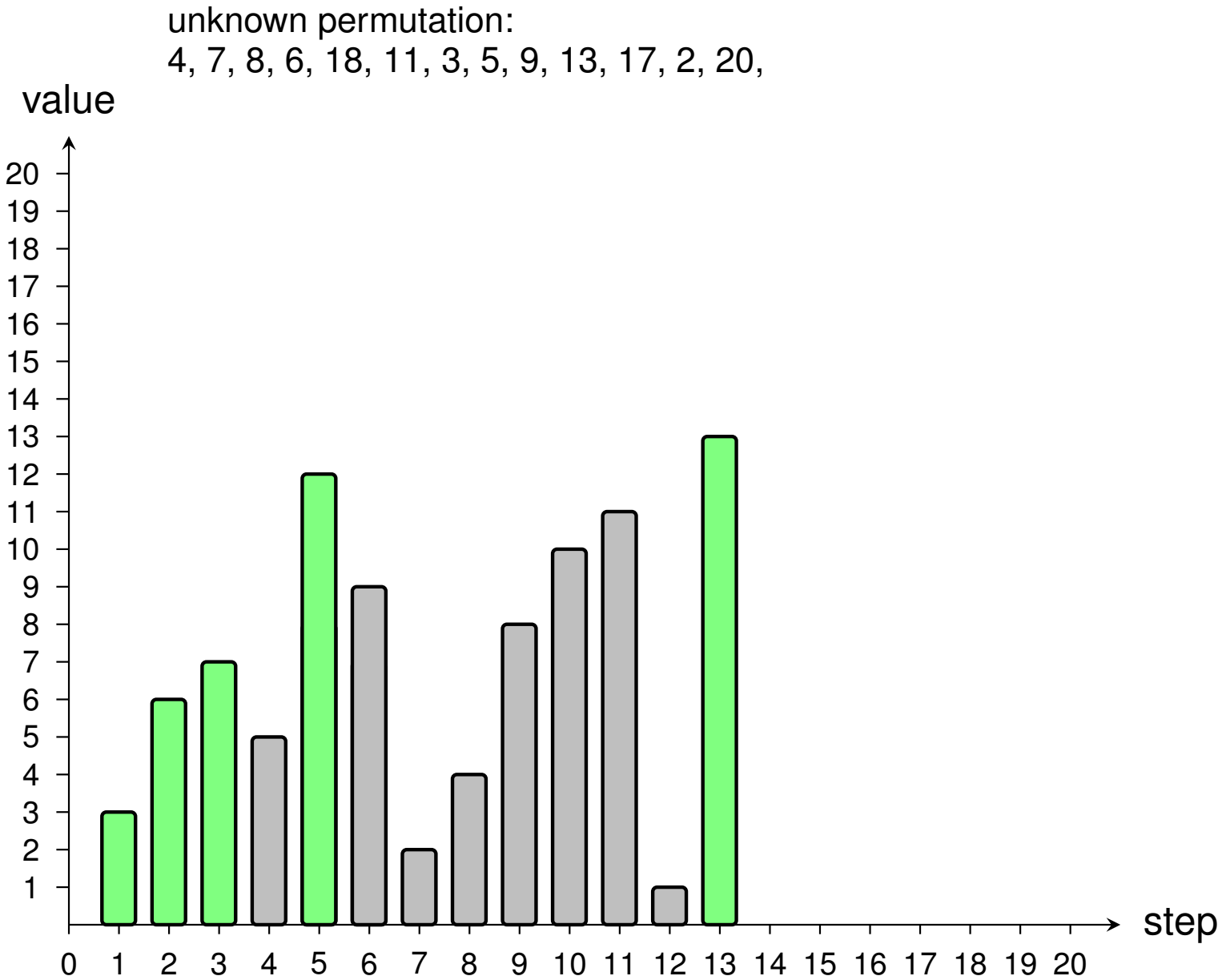

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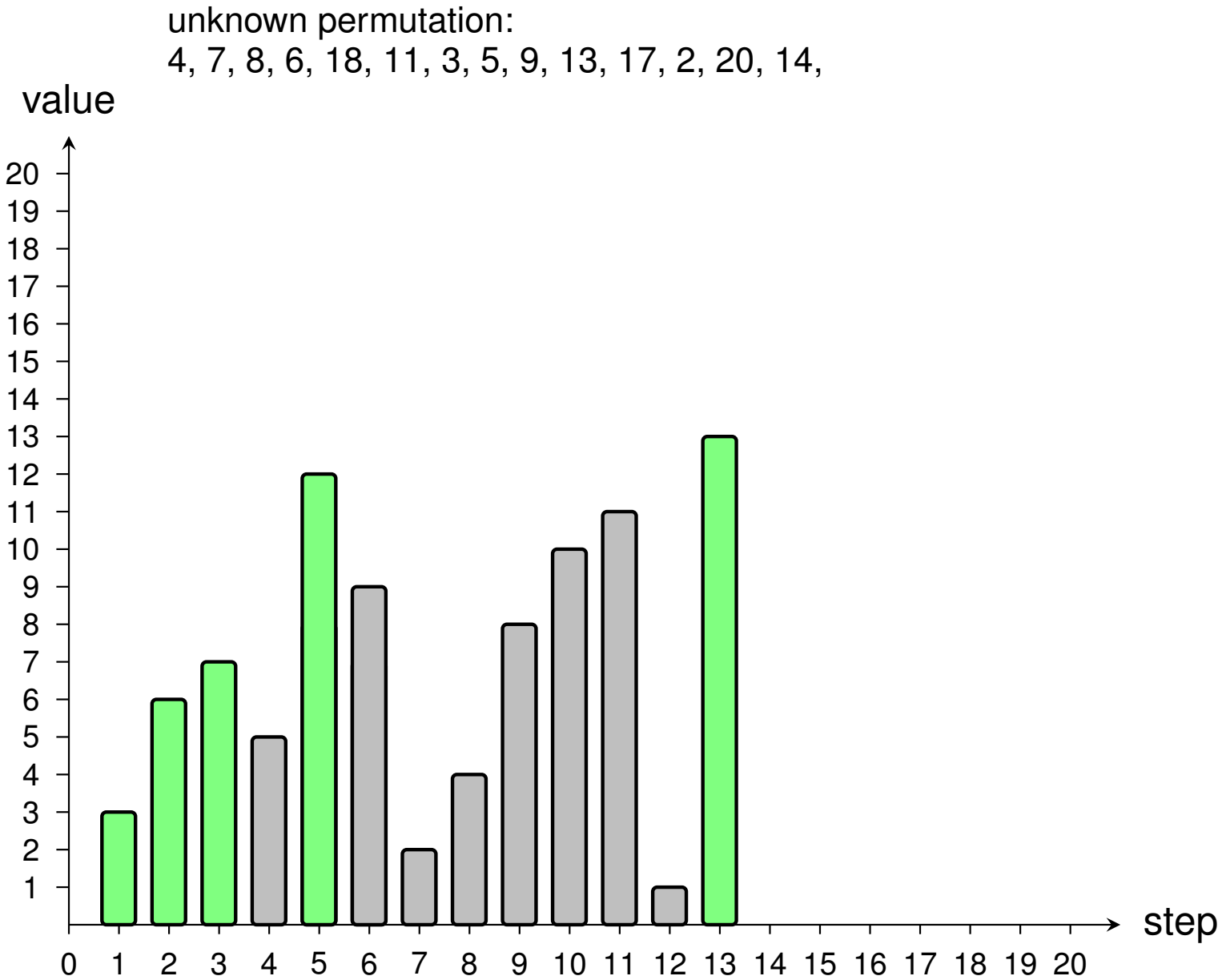

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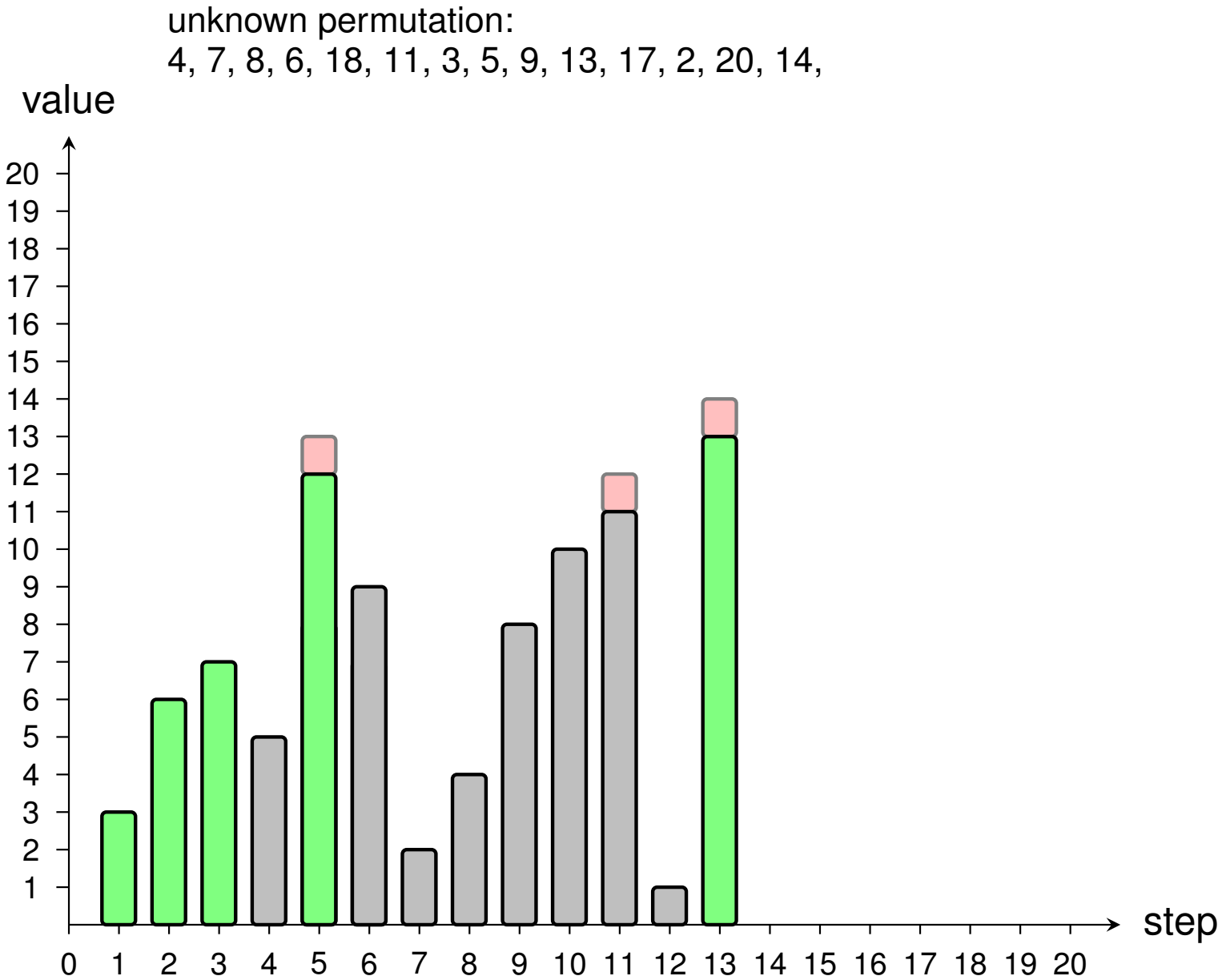

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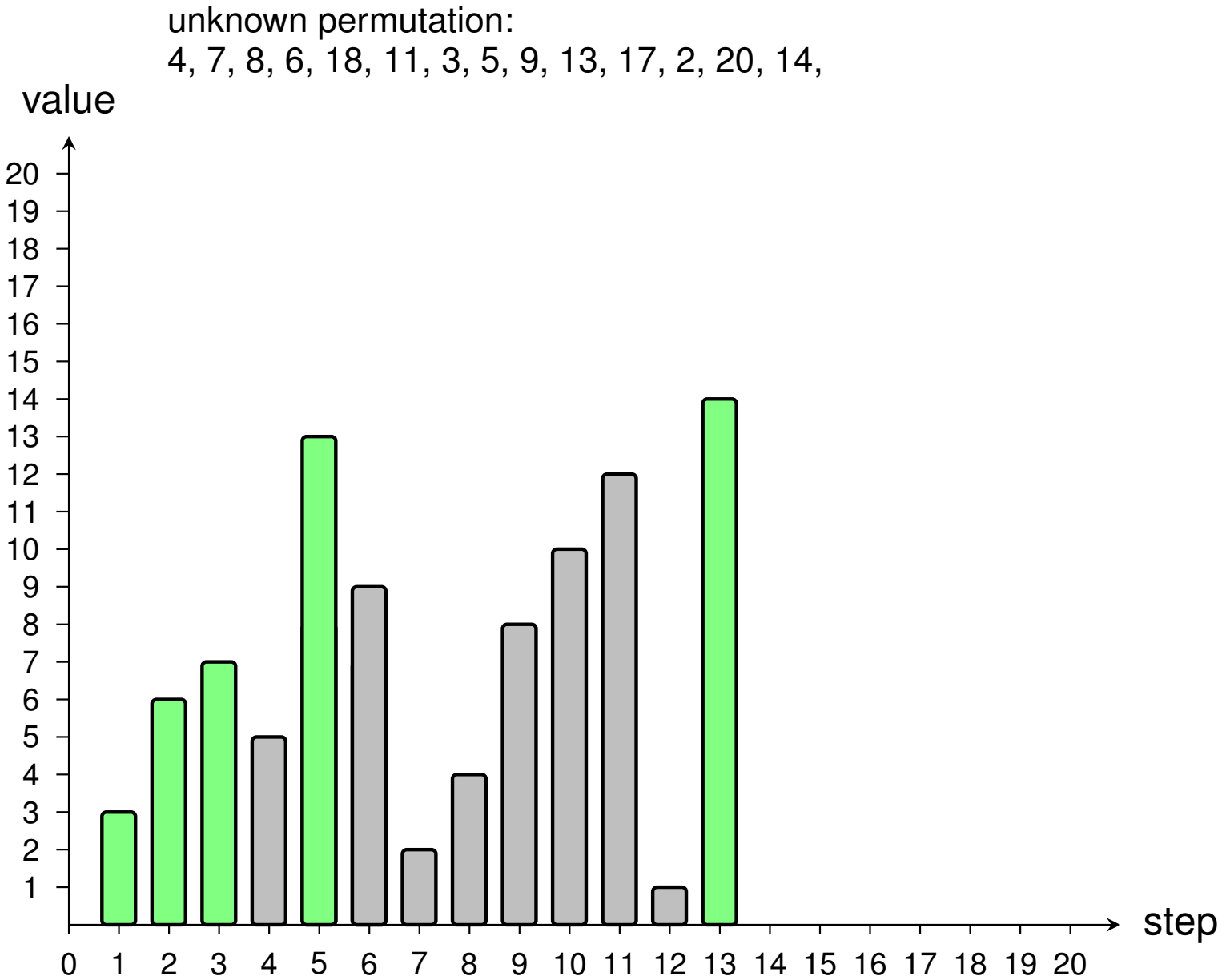

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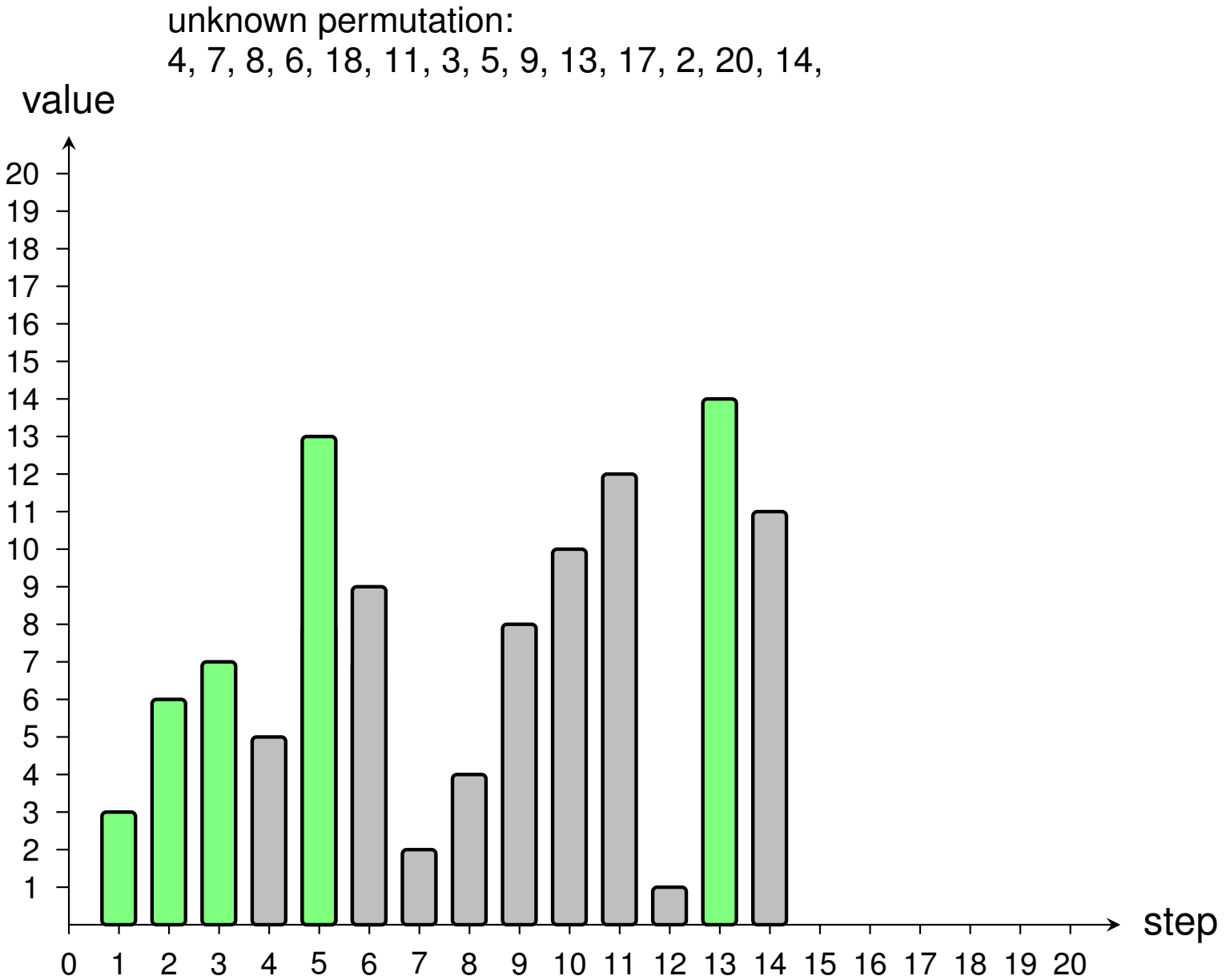

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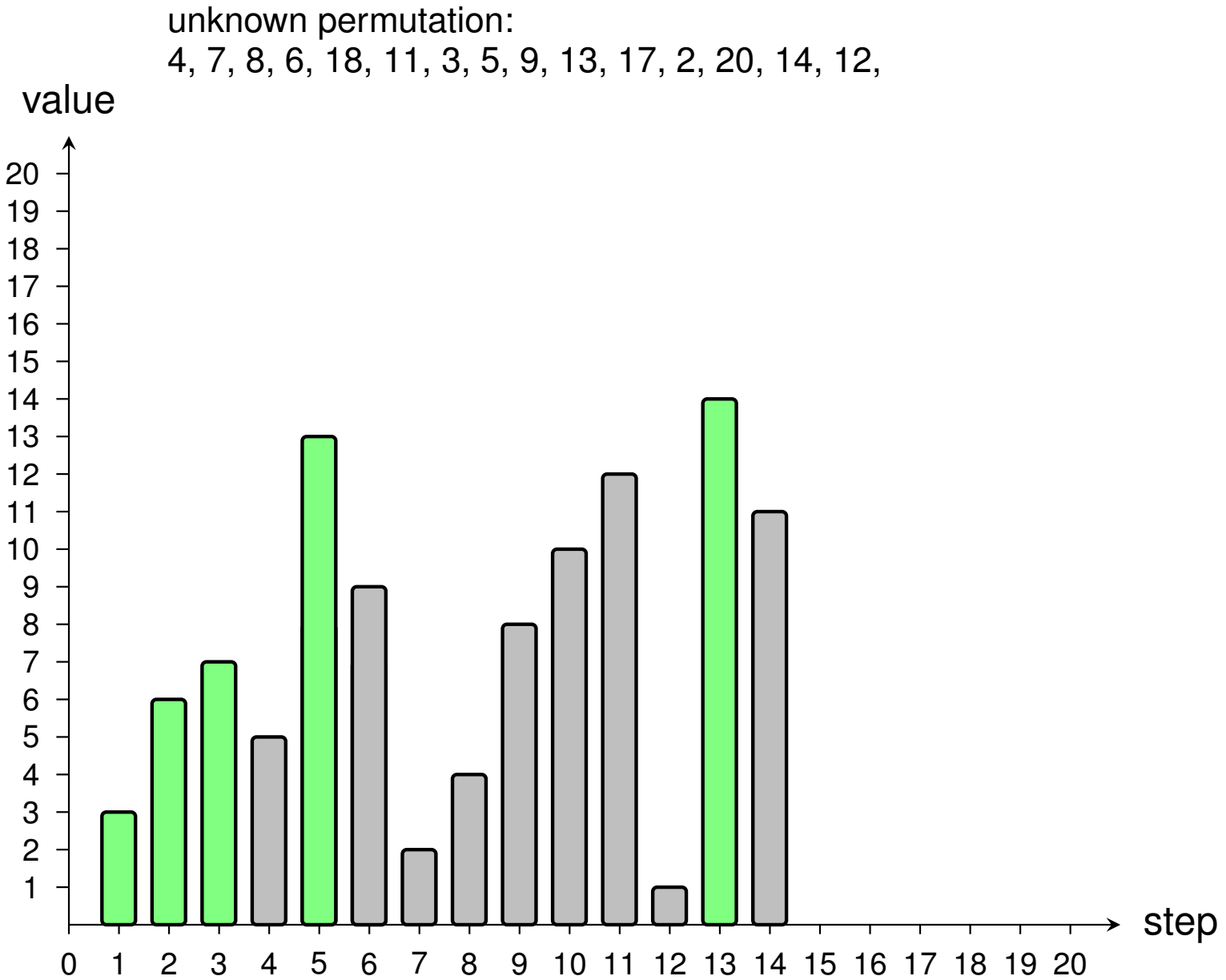

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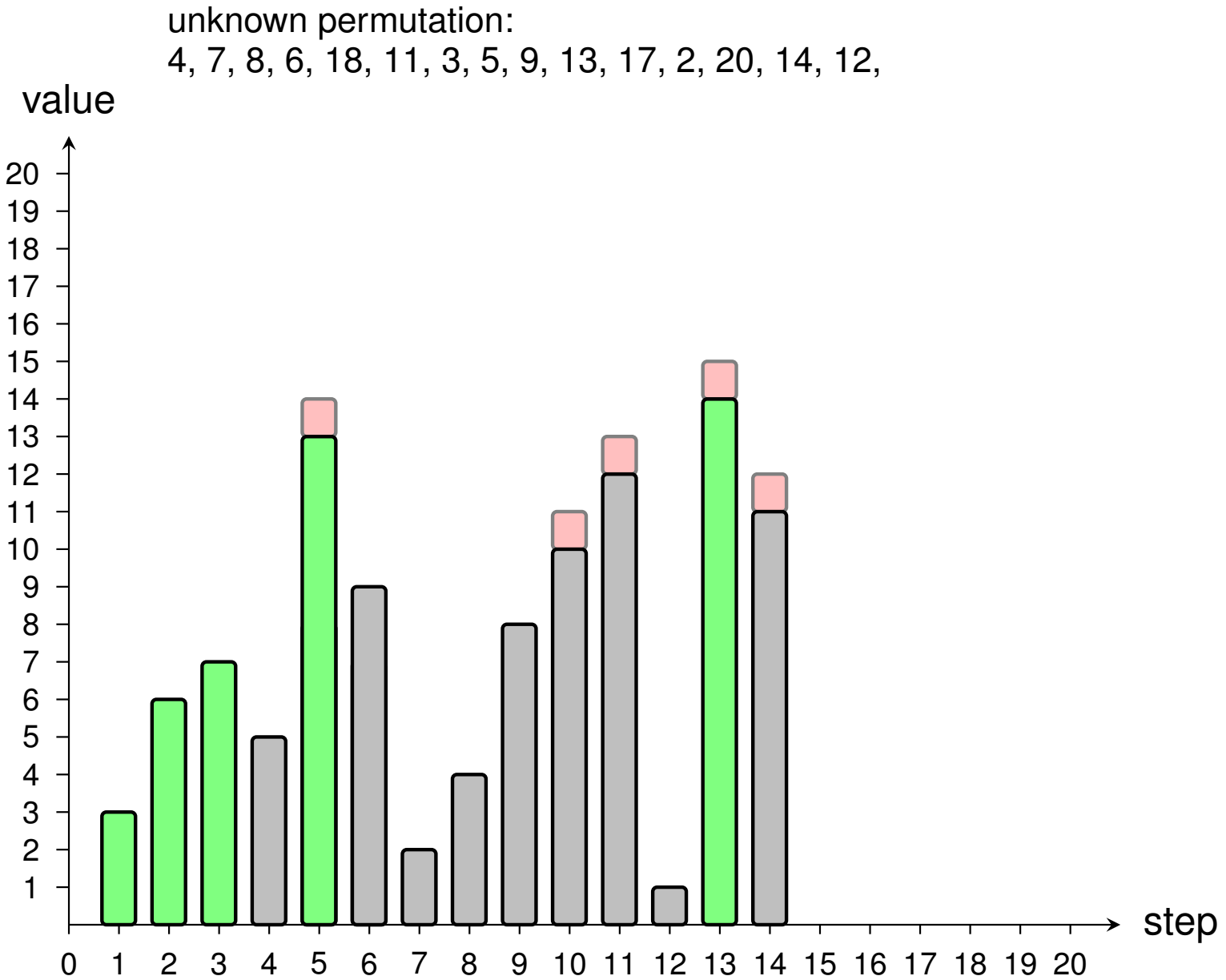

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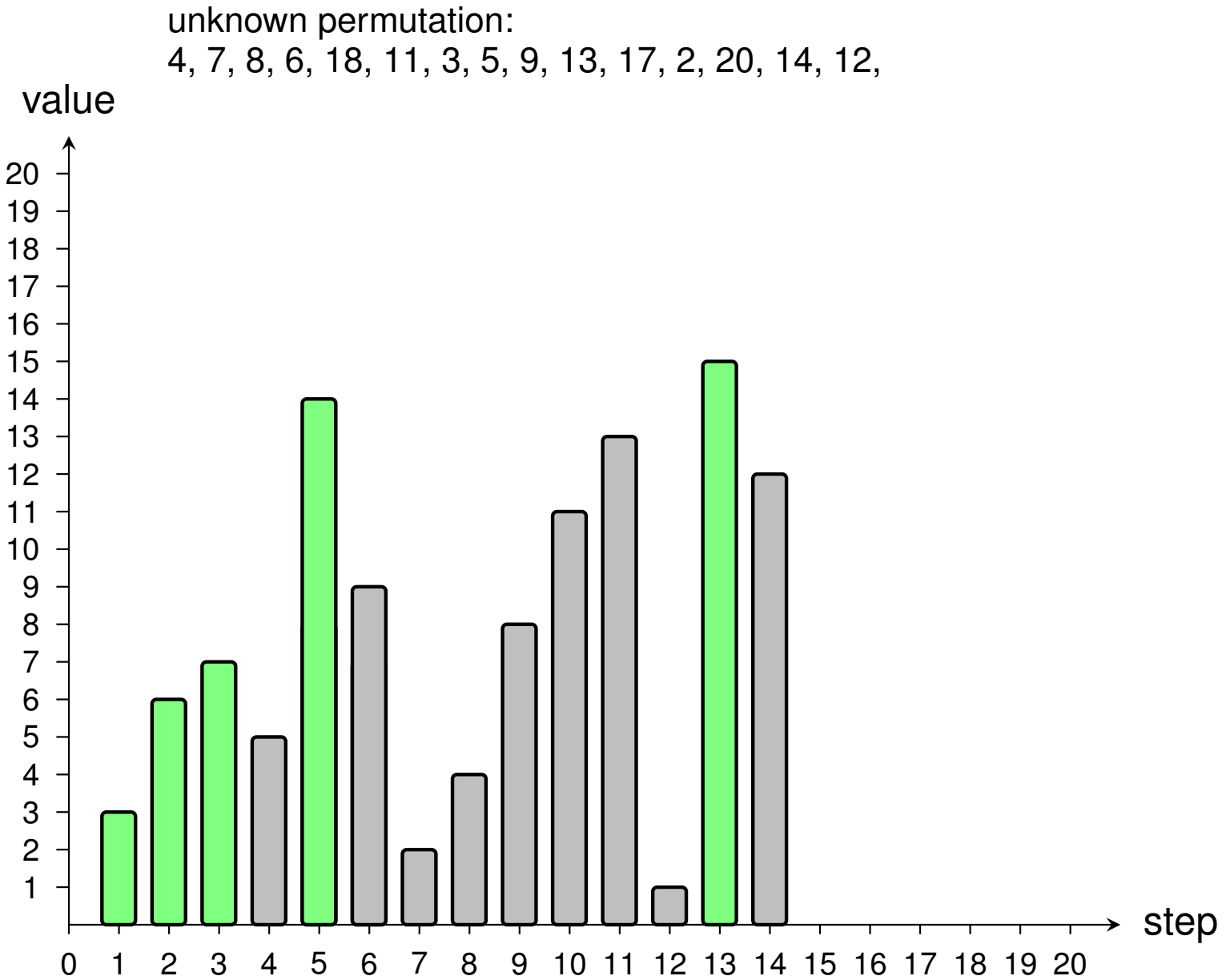

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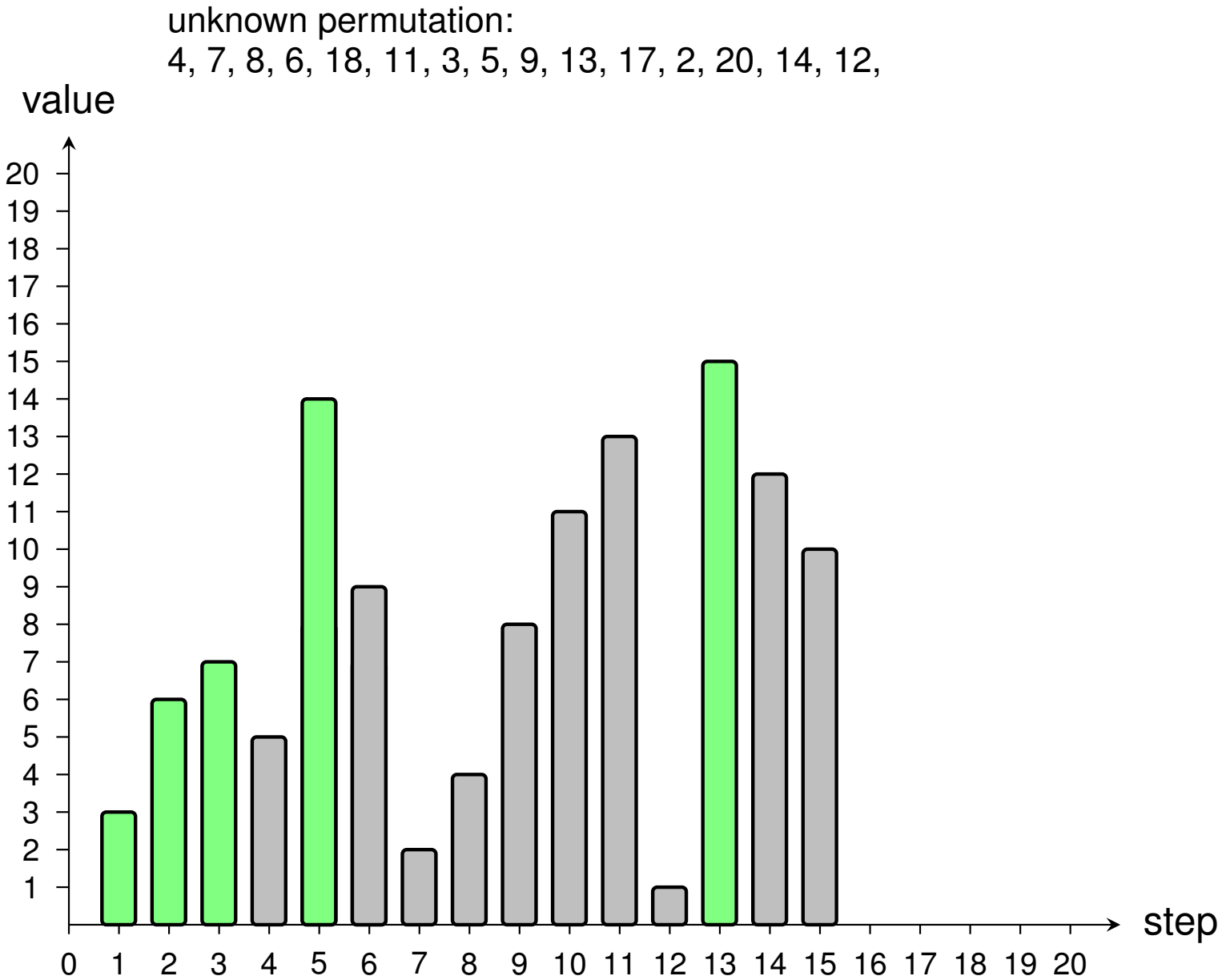

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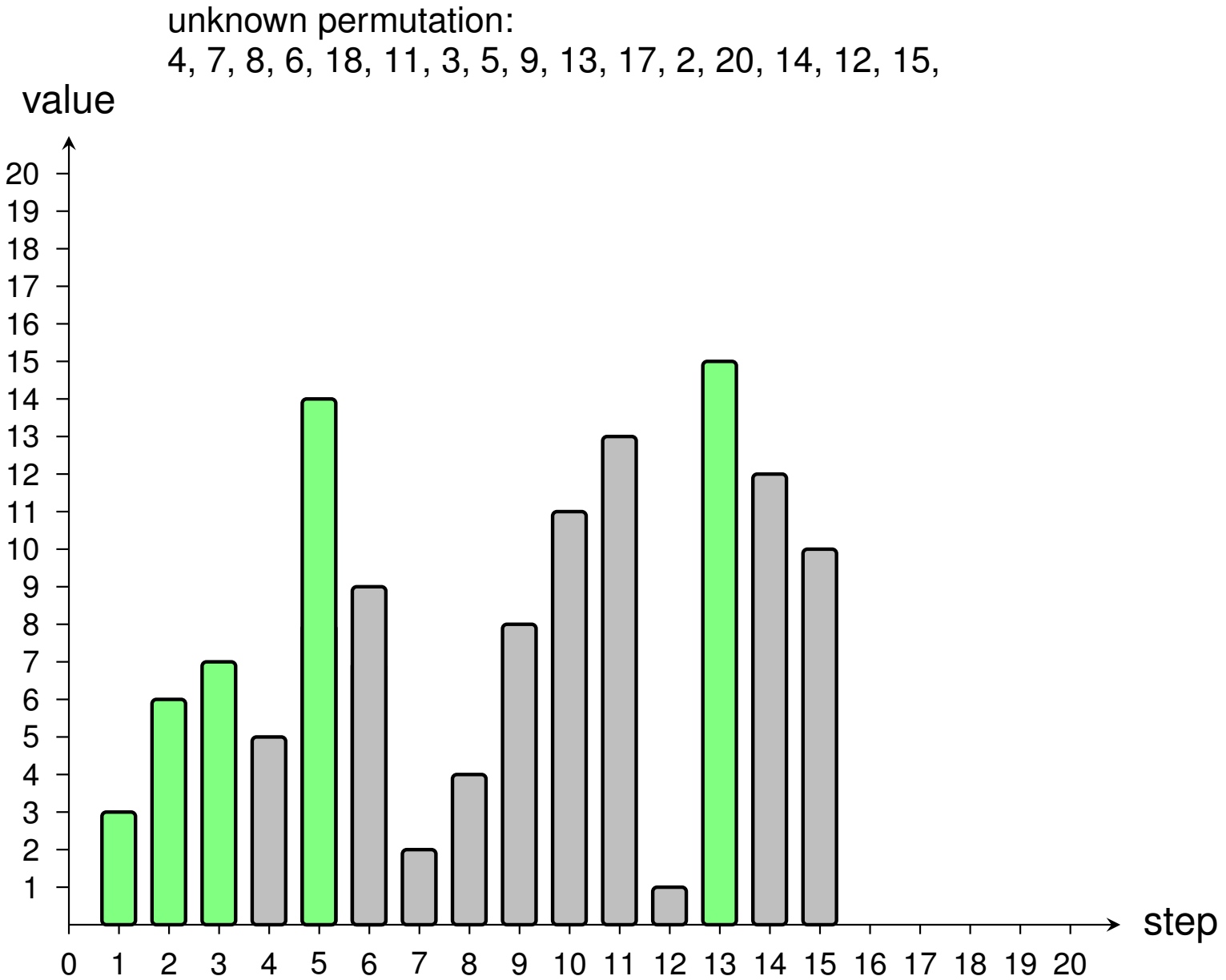

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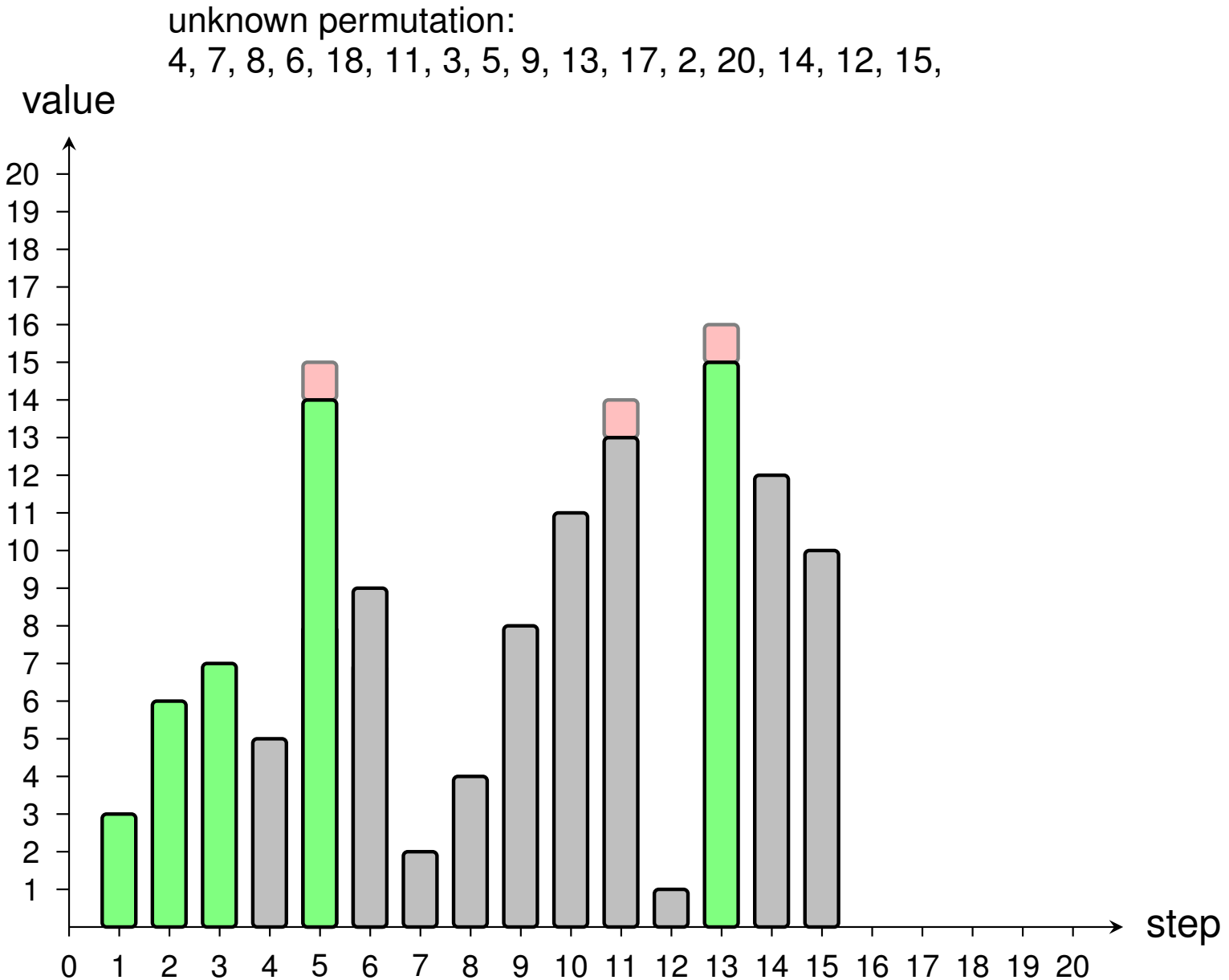

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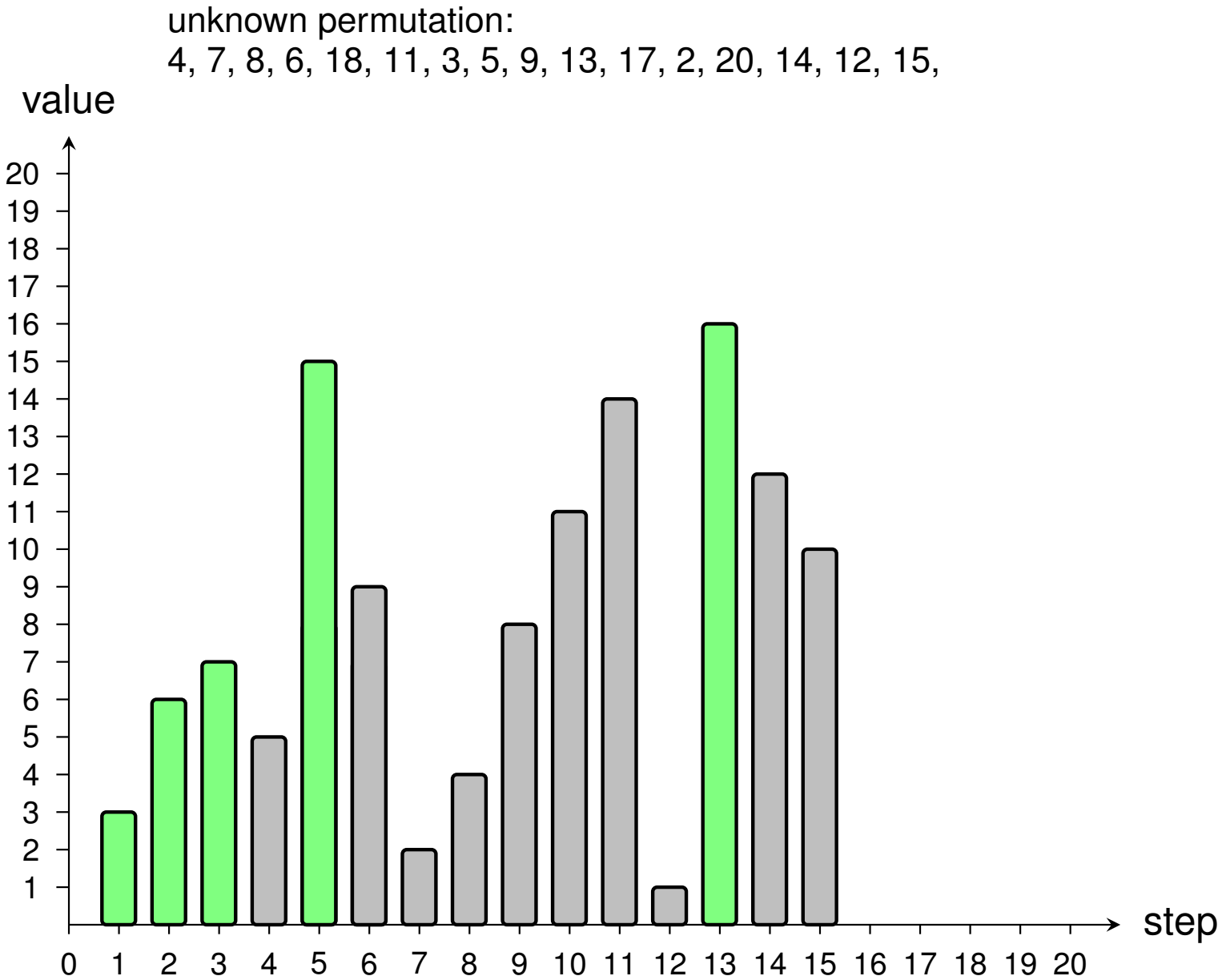

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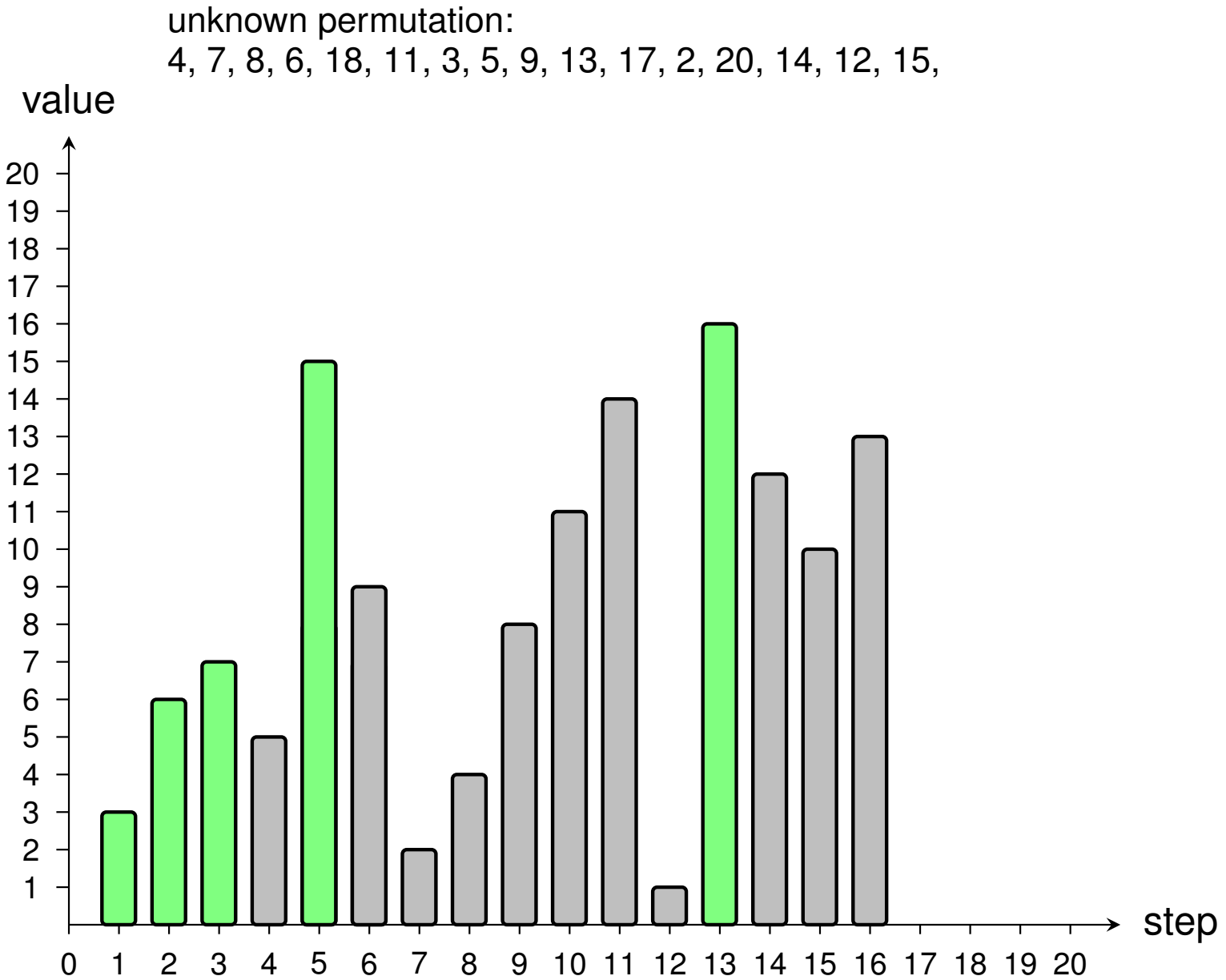

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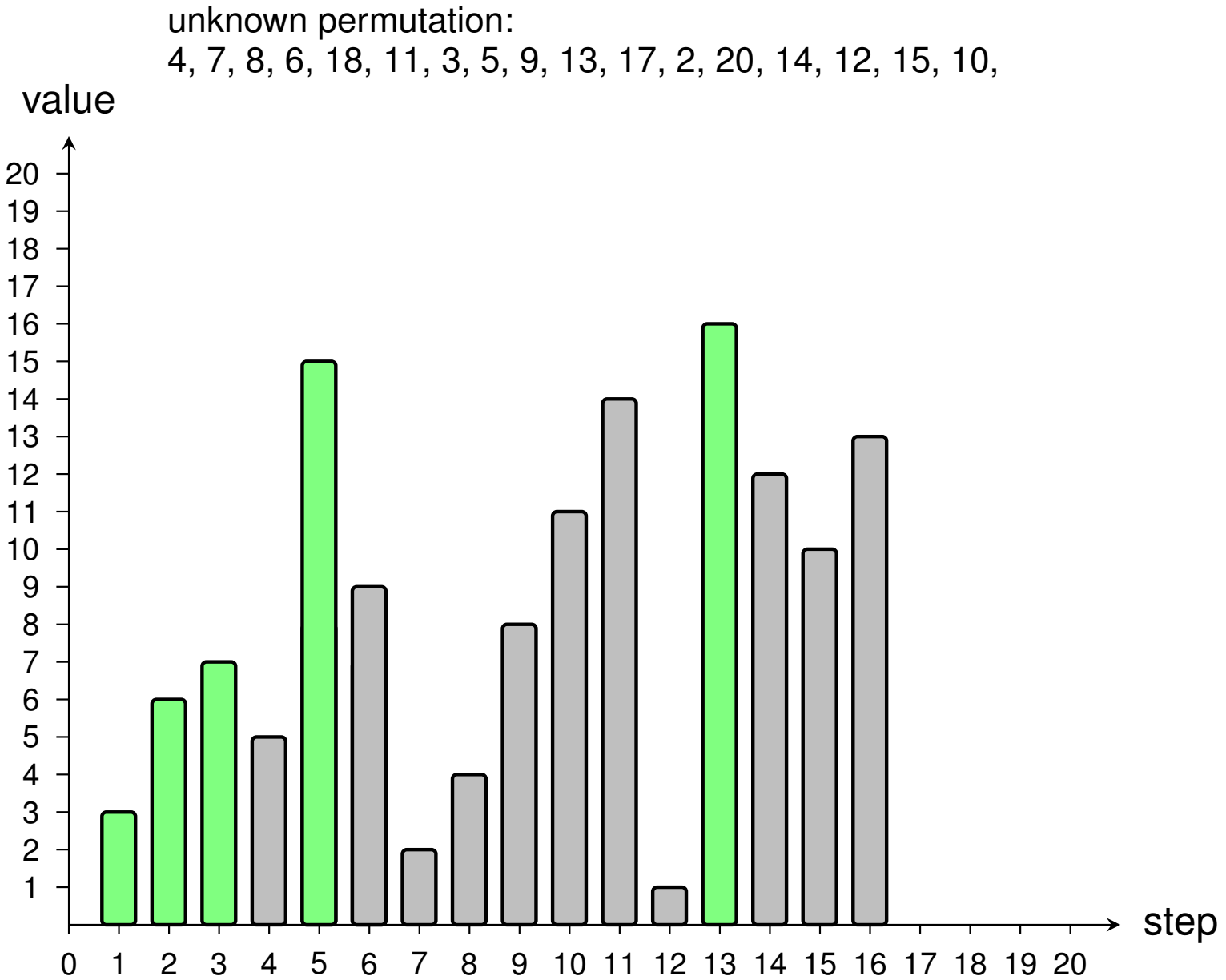

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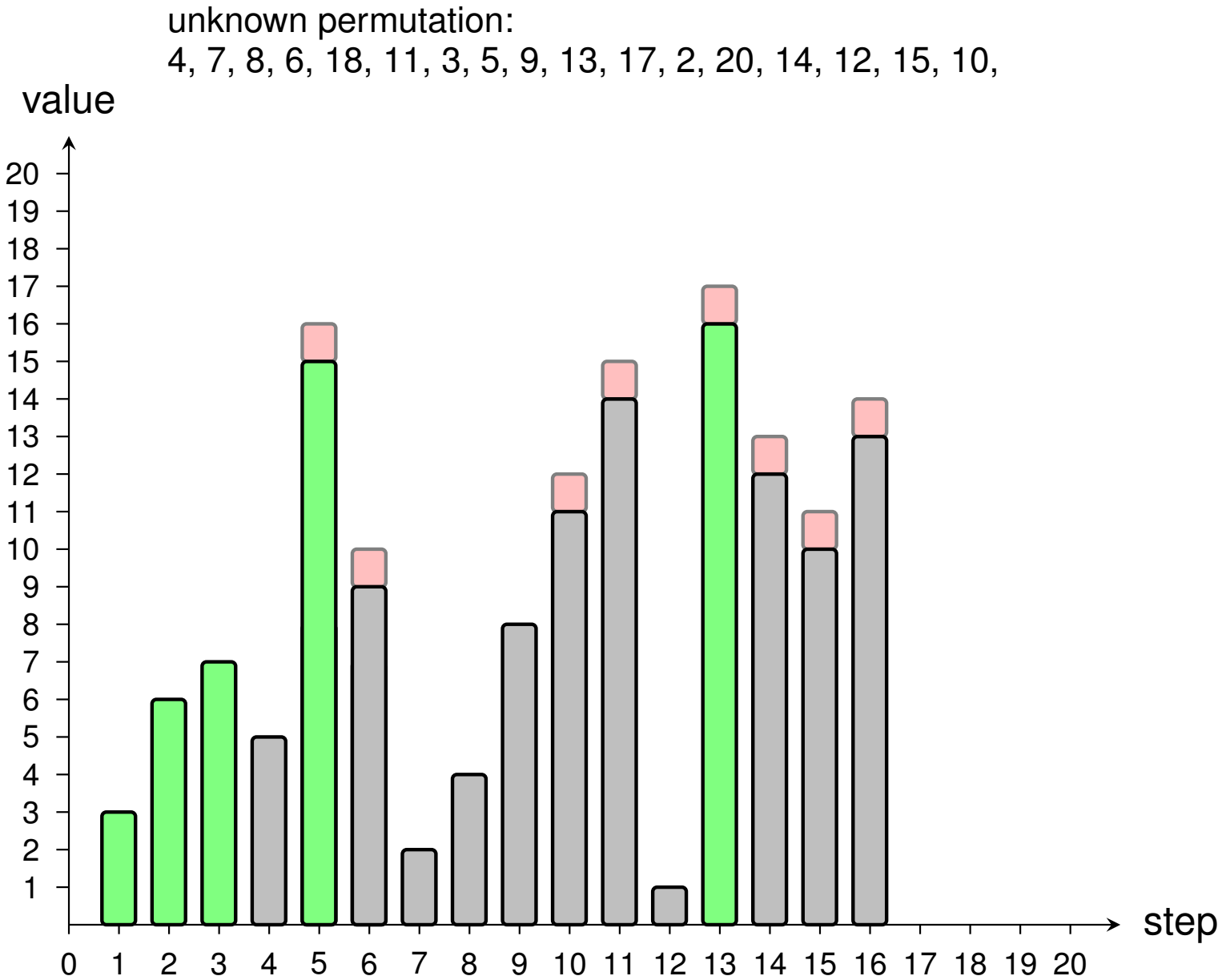

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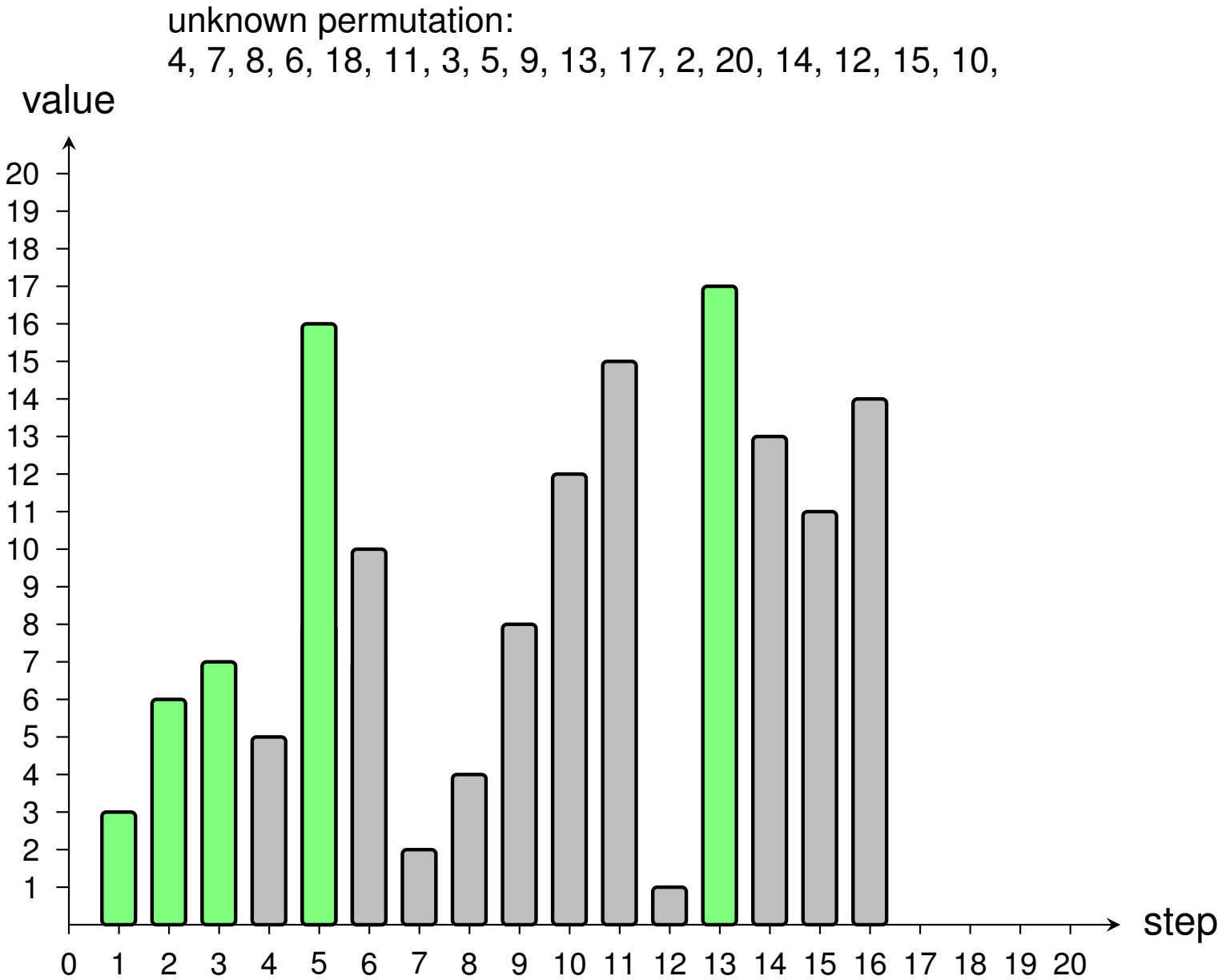

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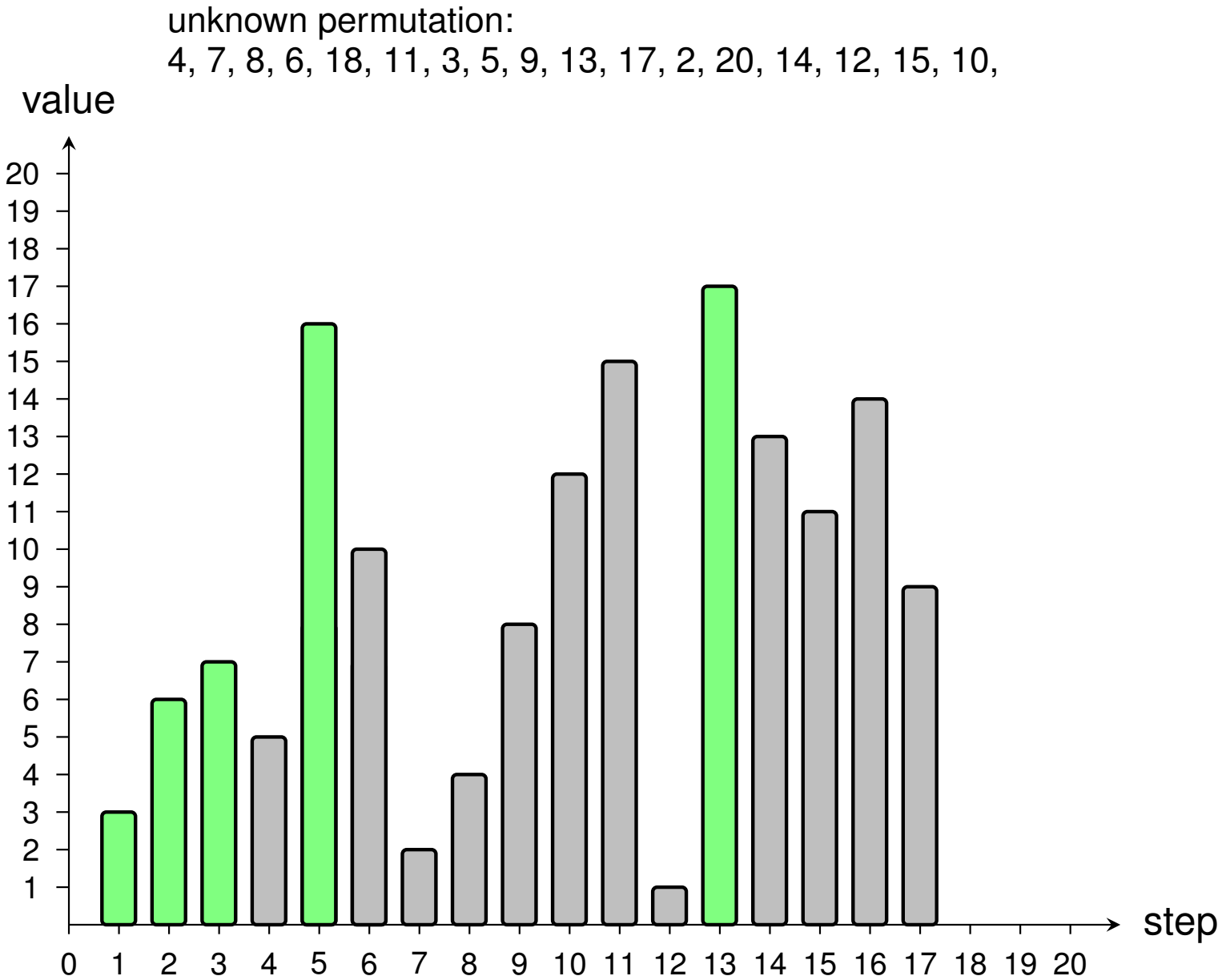

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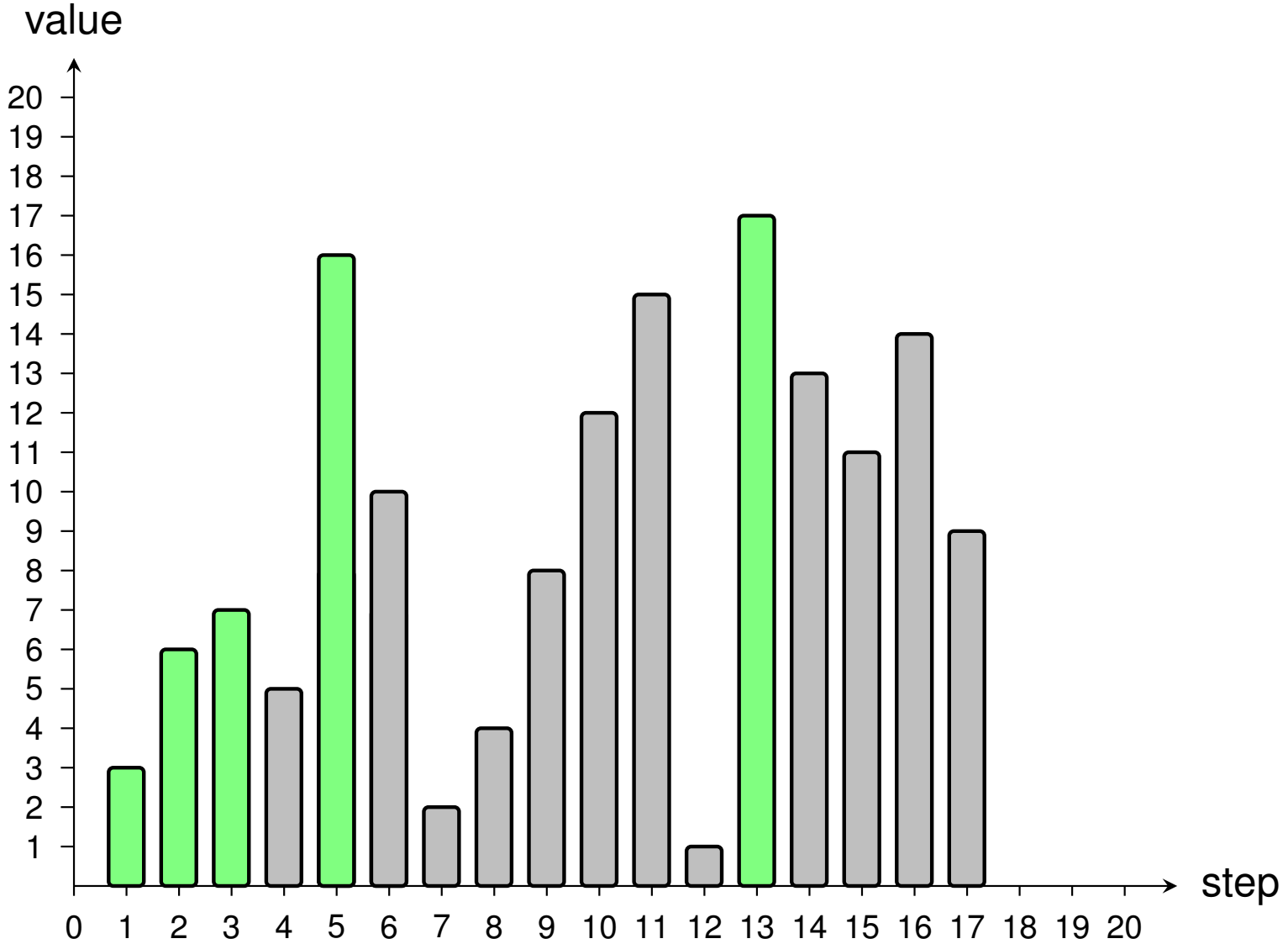

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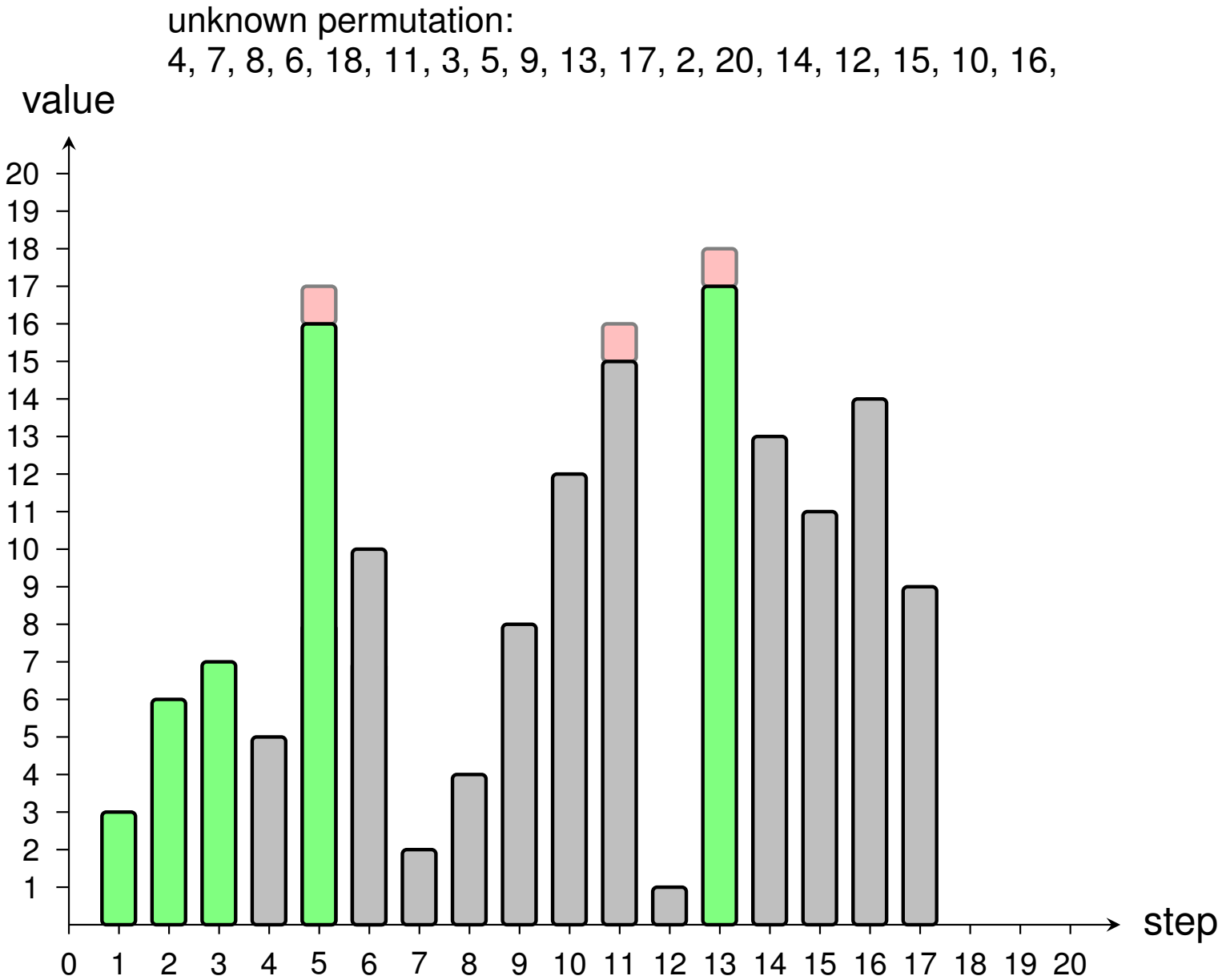

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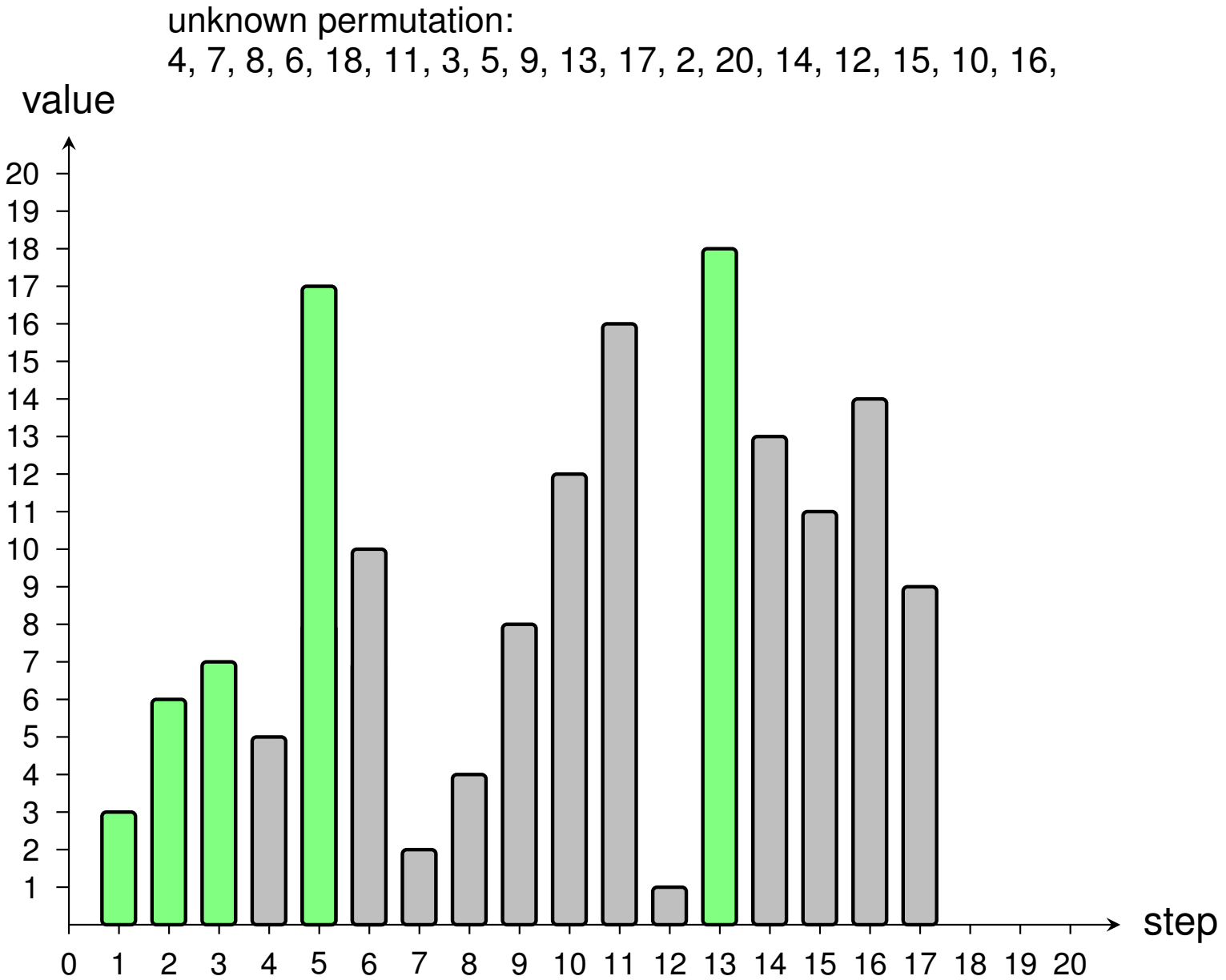

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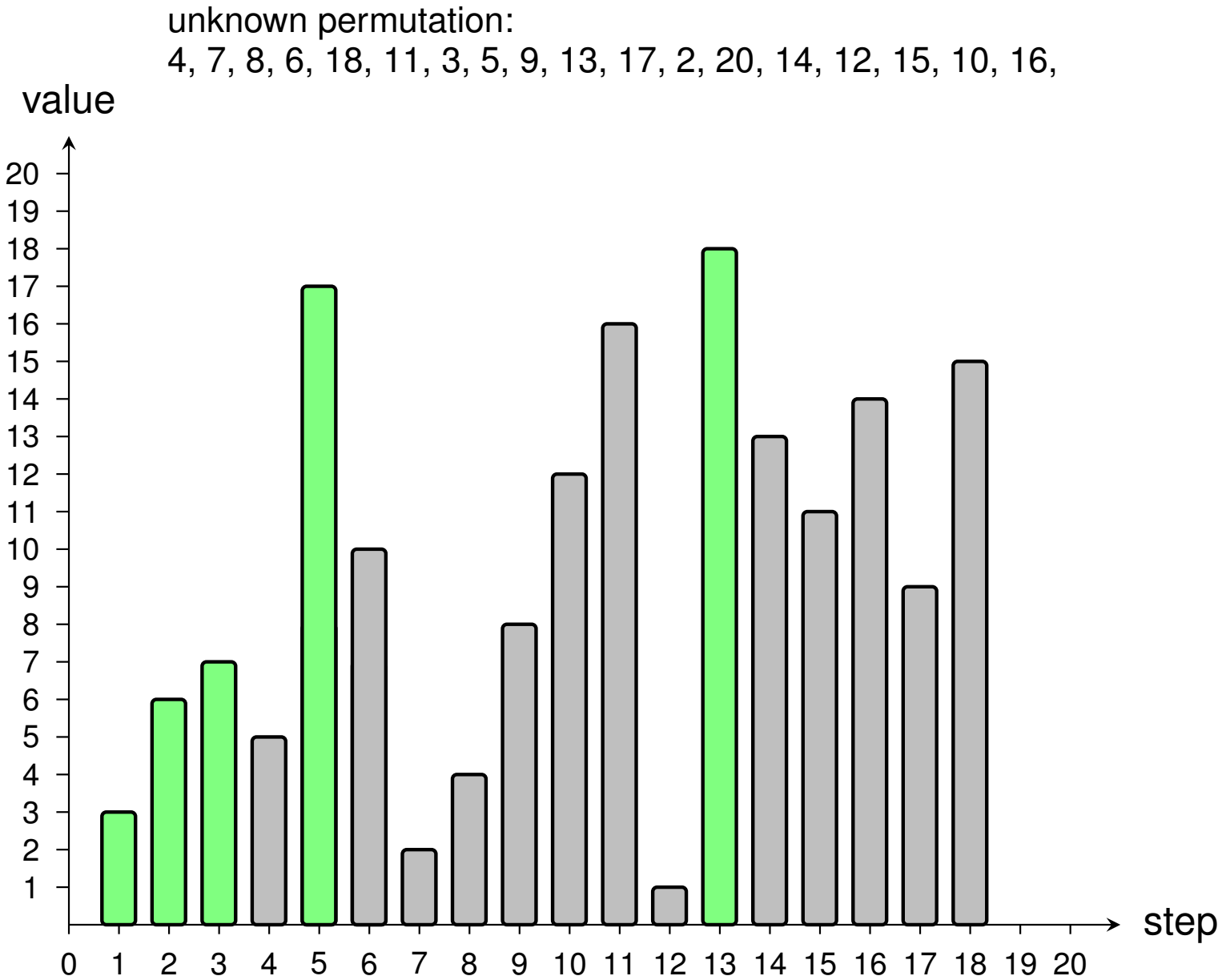

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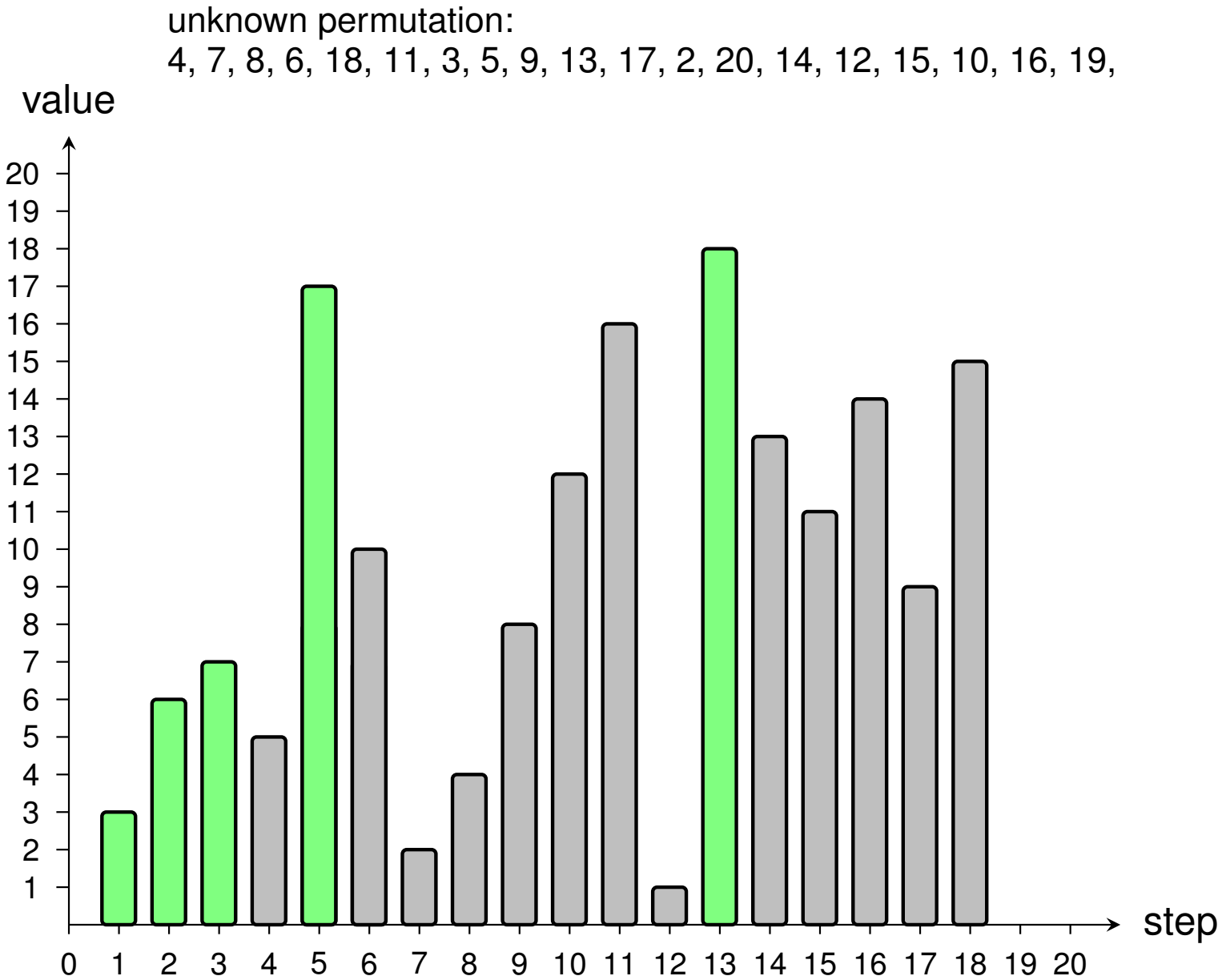

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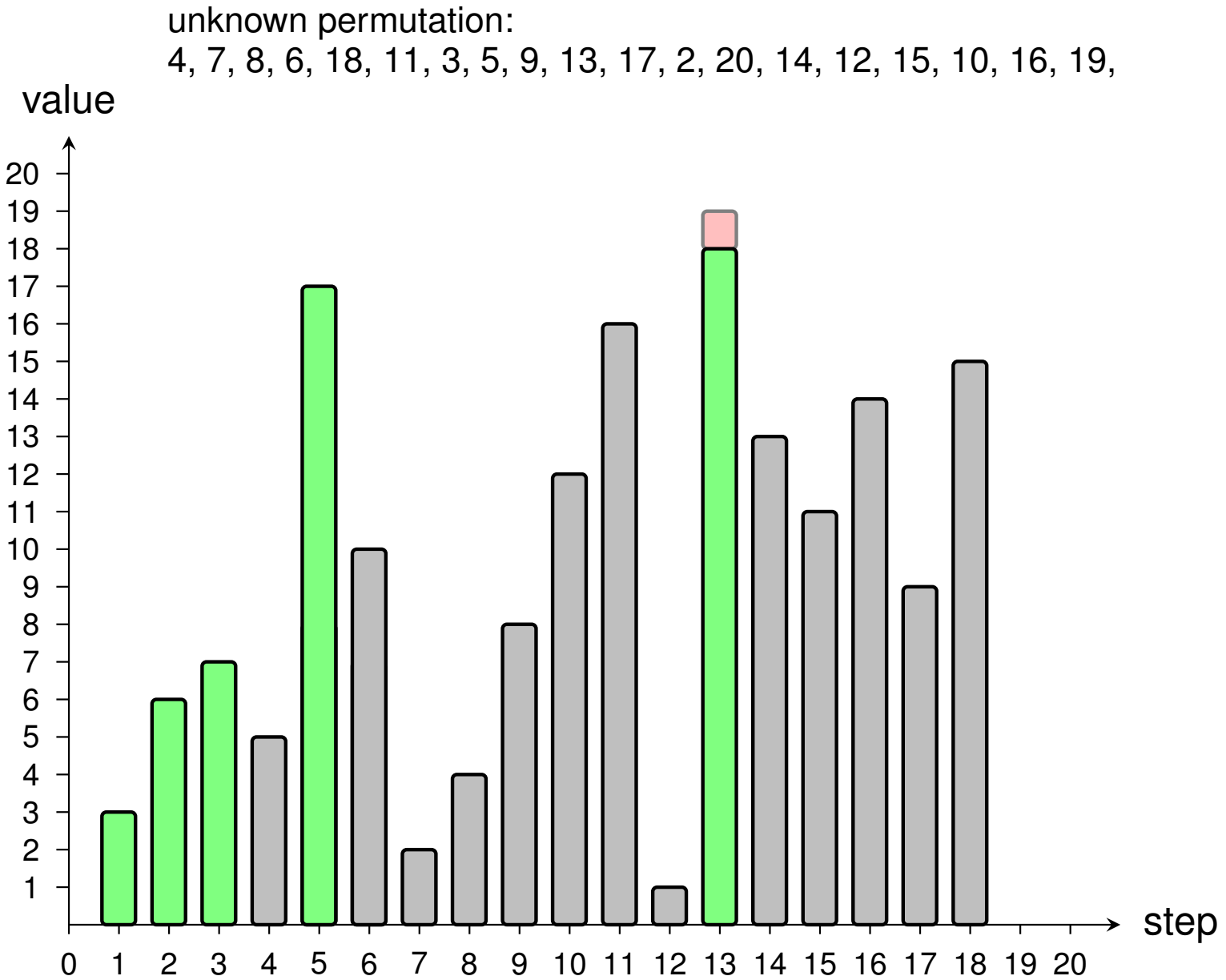

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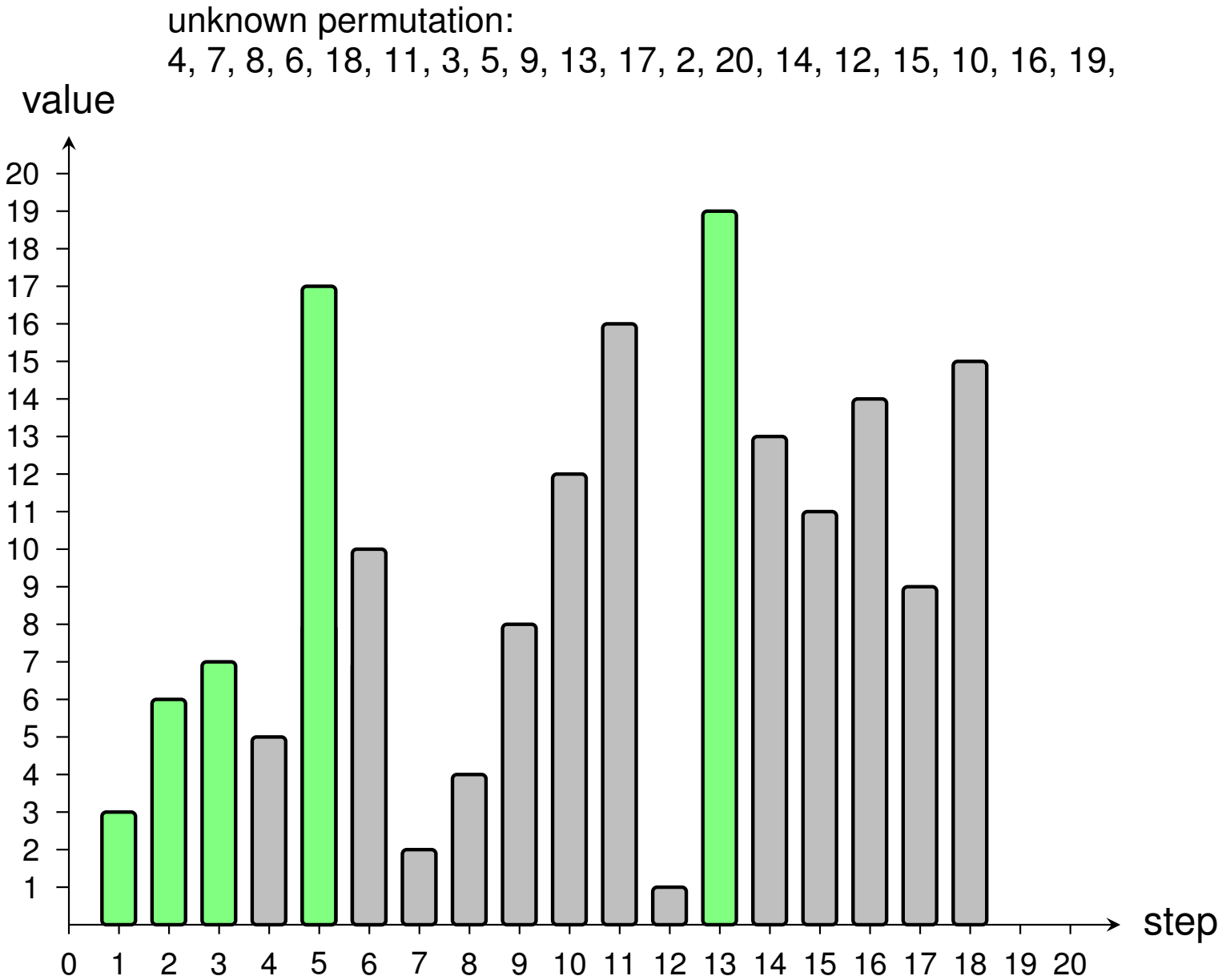

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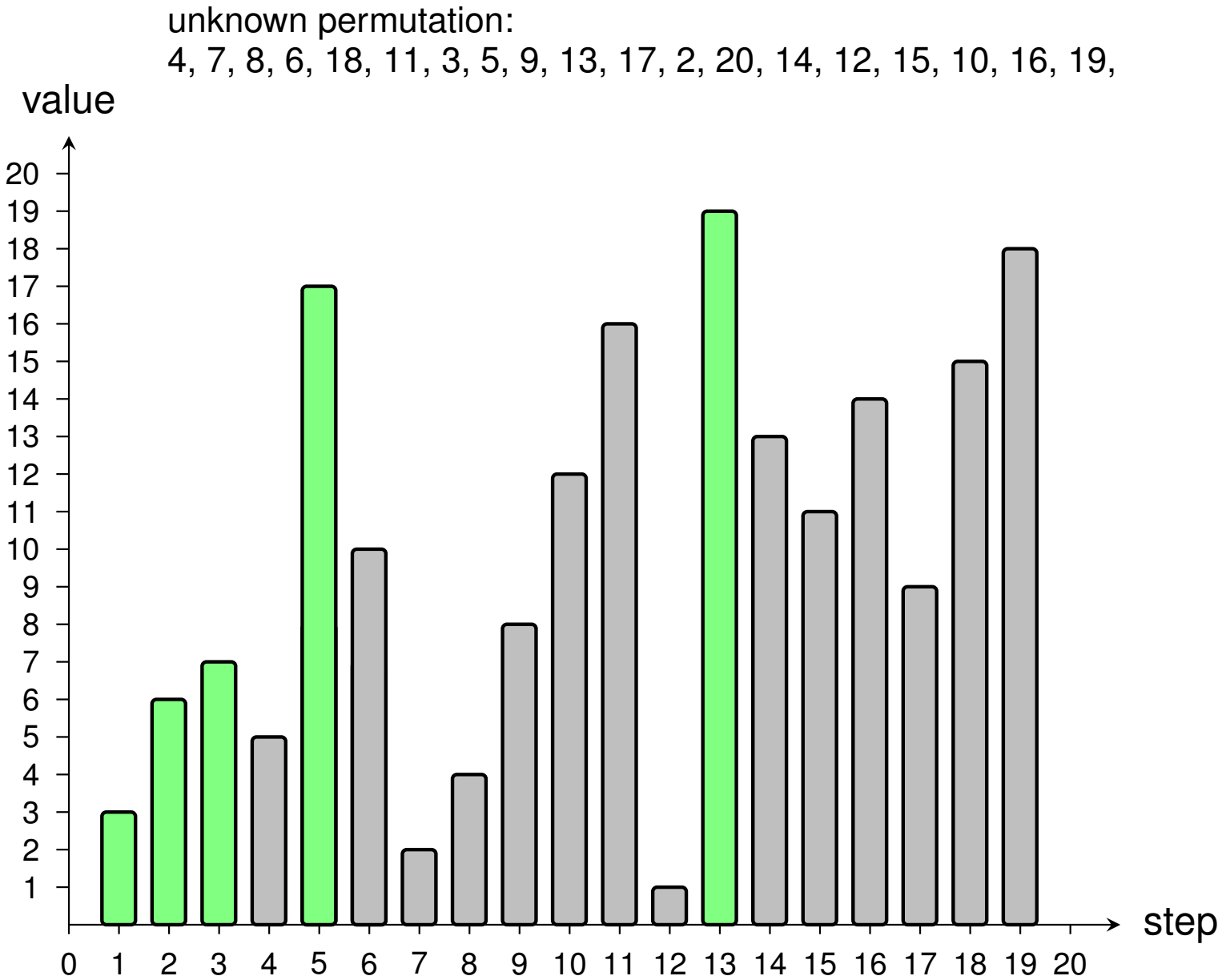

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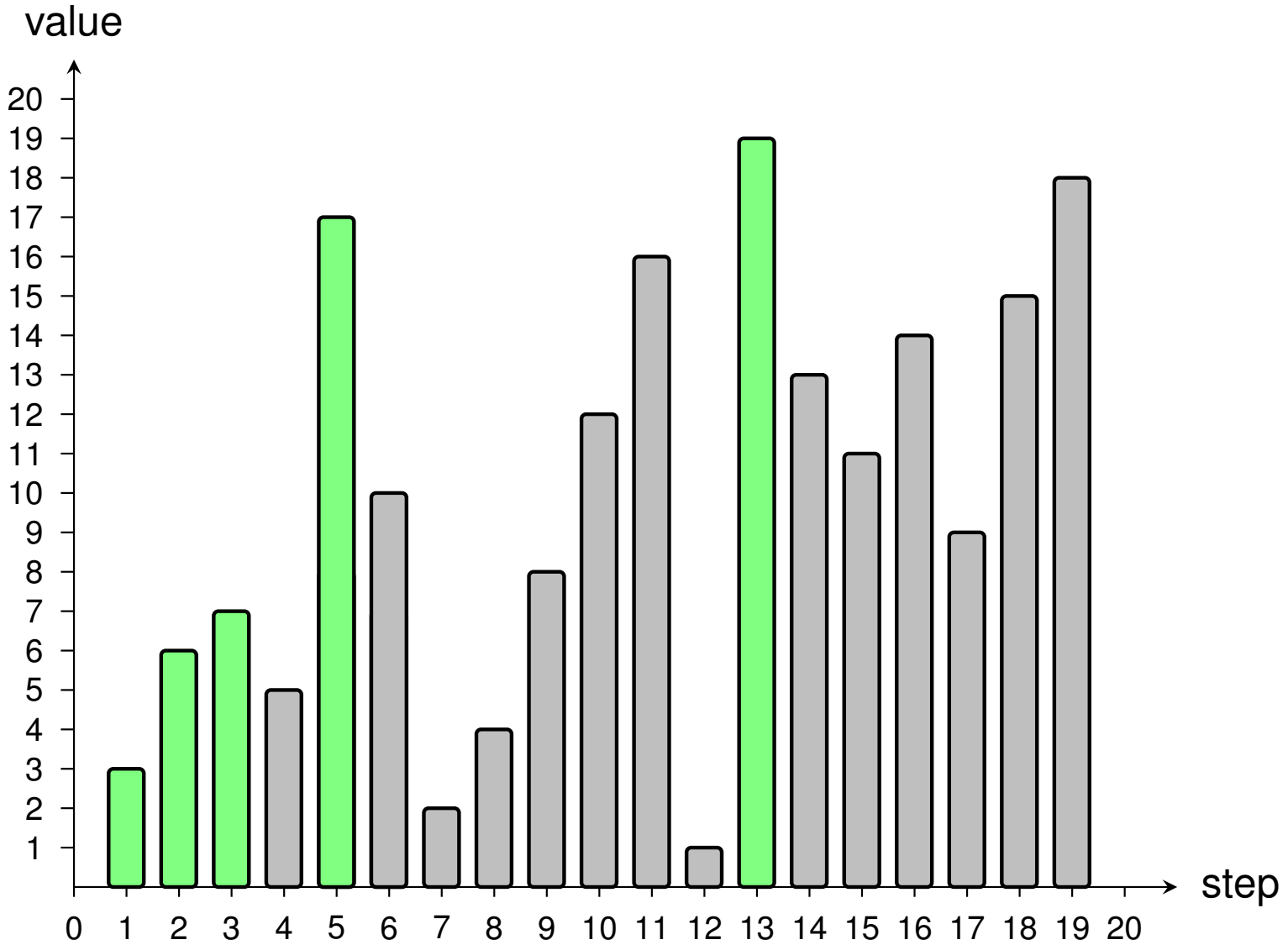

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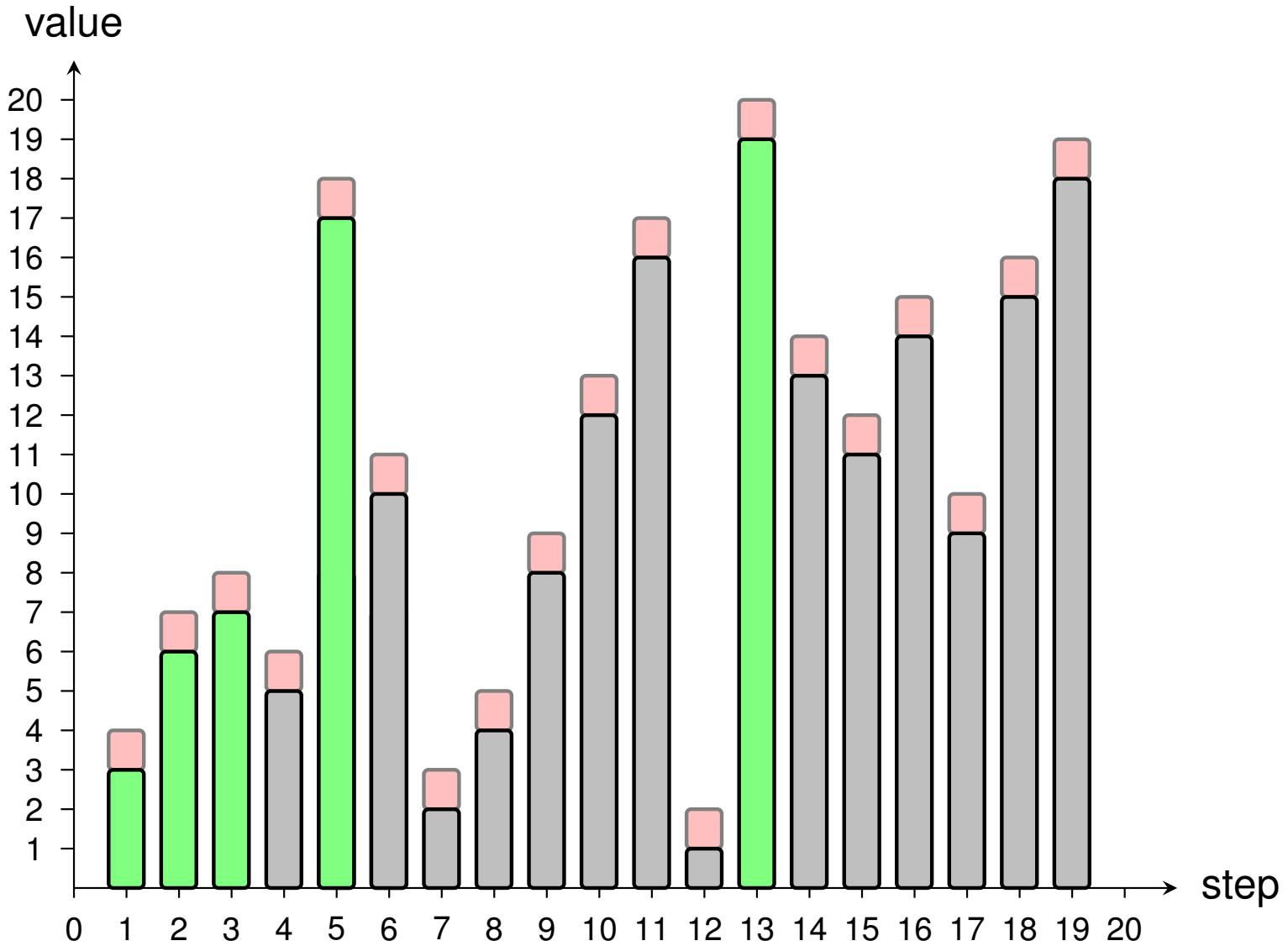

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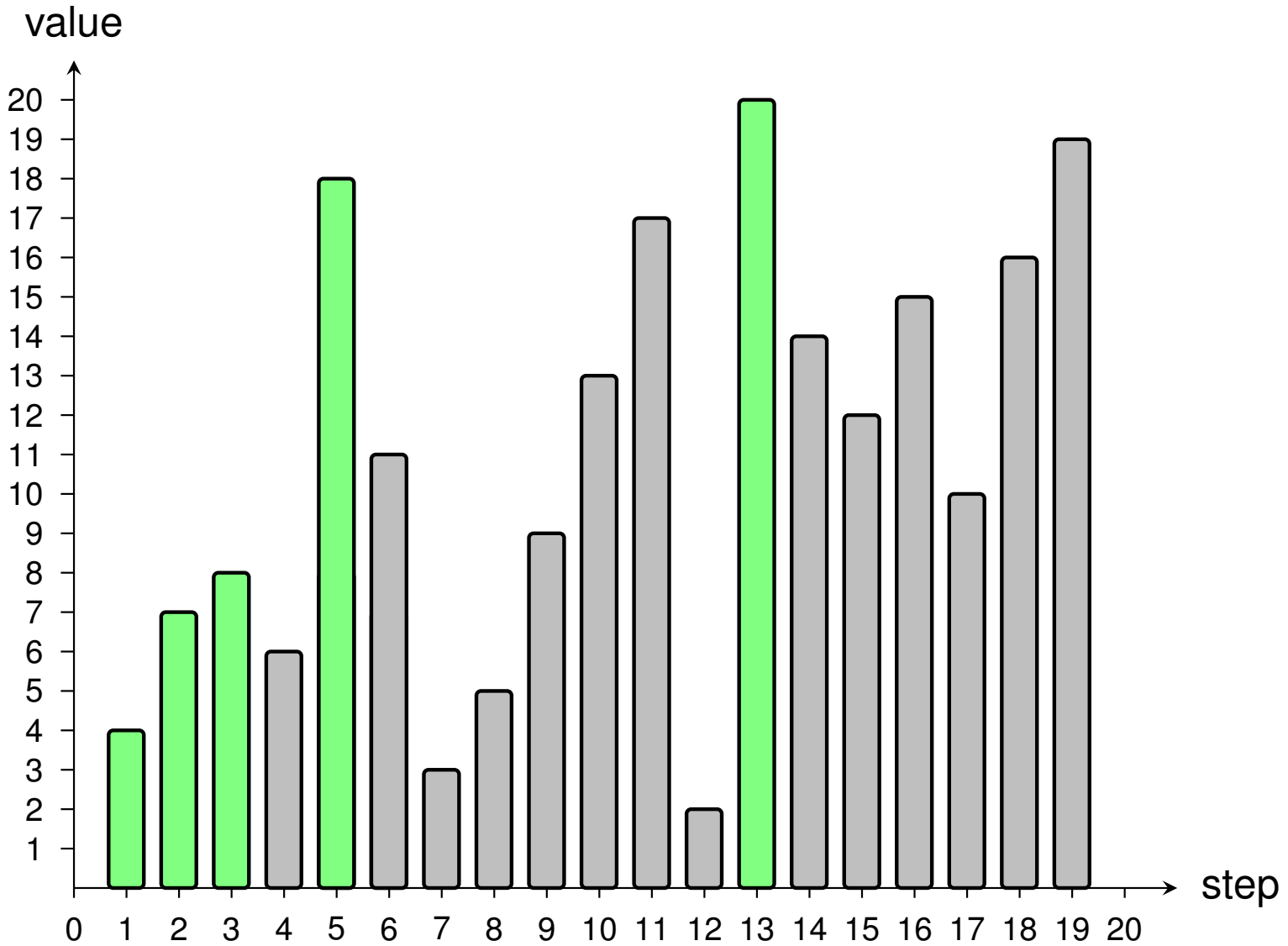

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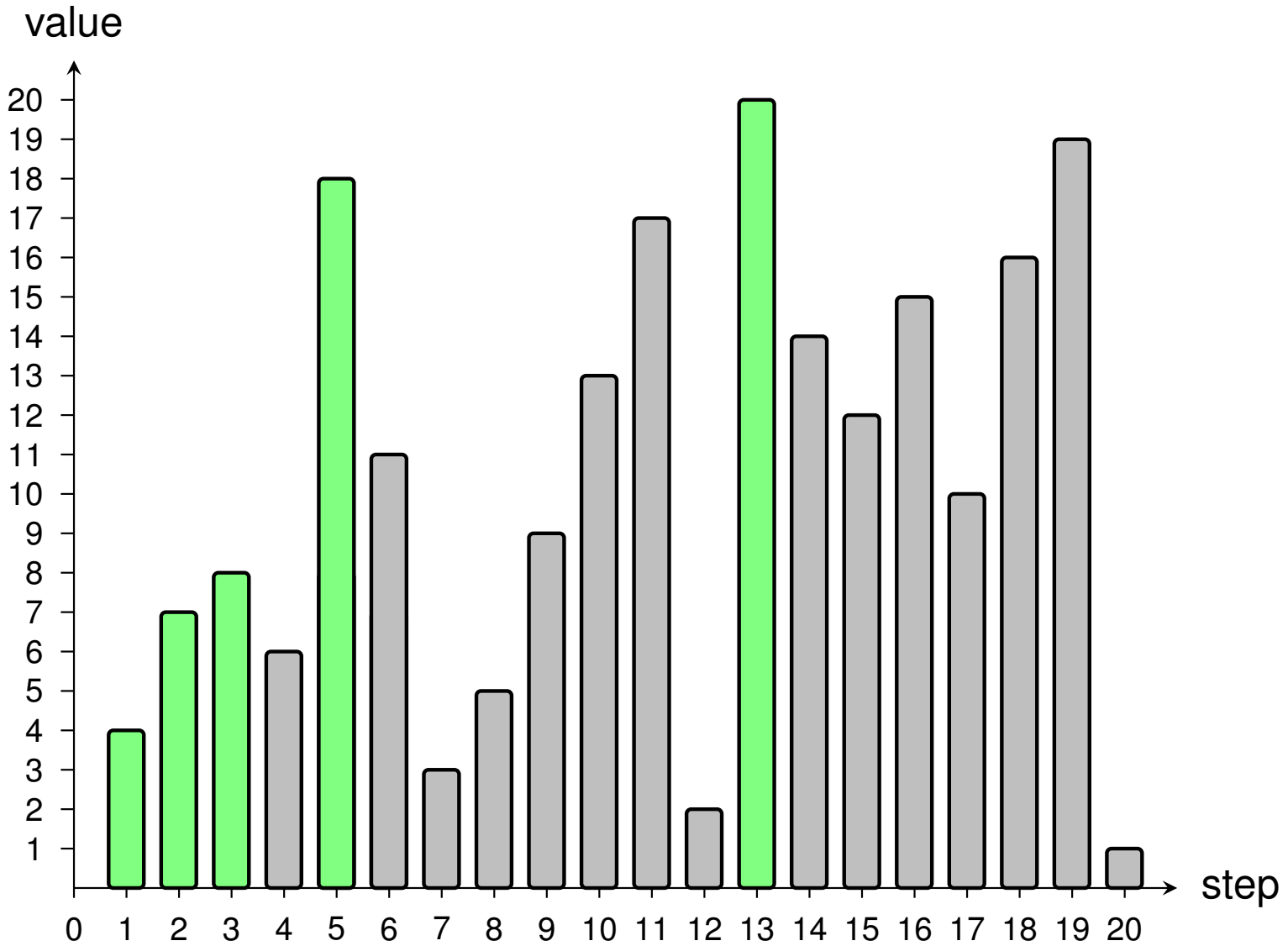

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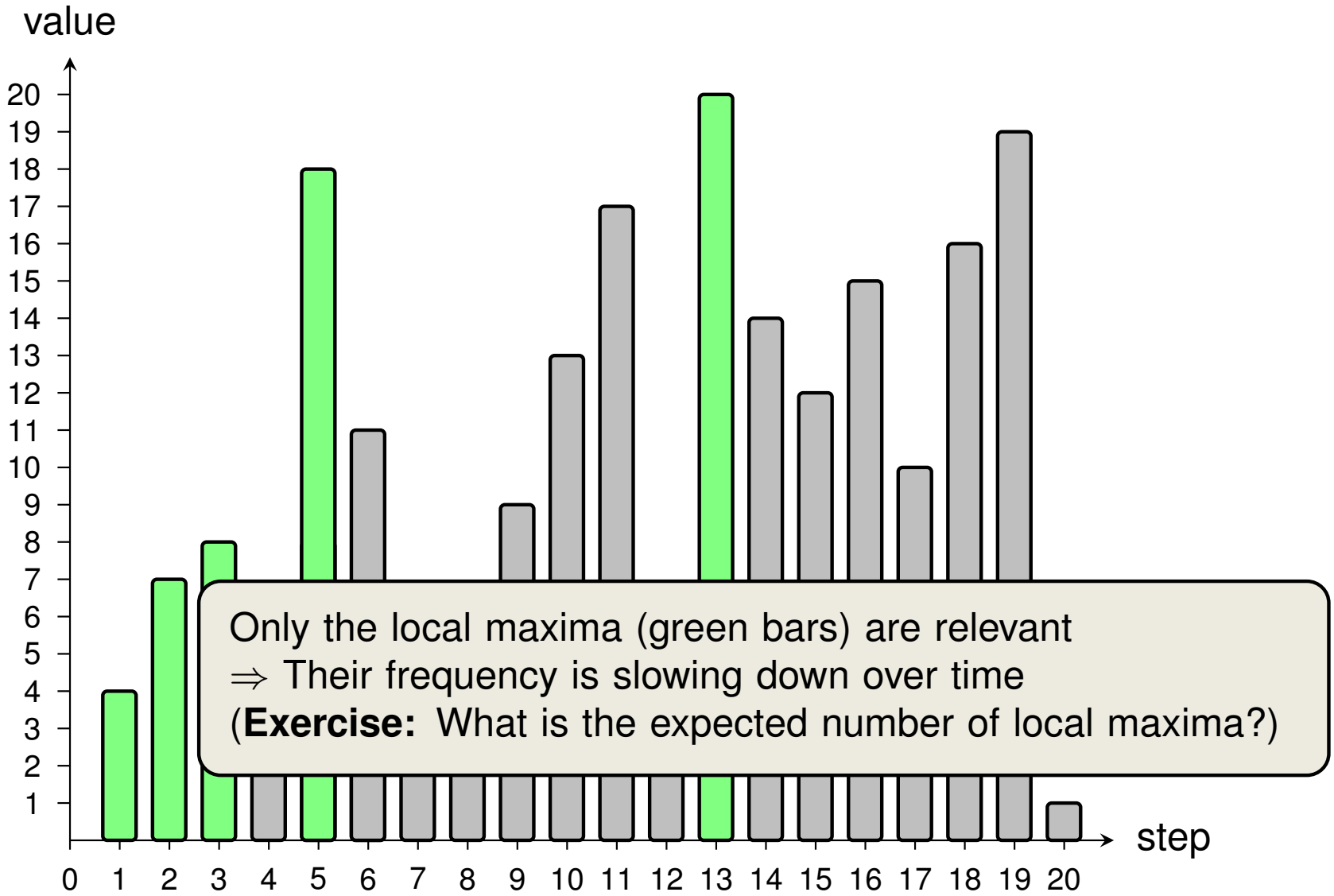

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How good is this approach?

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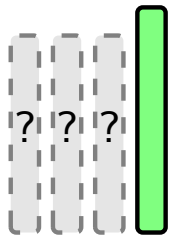

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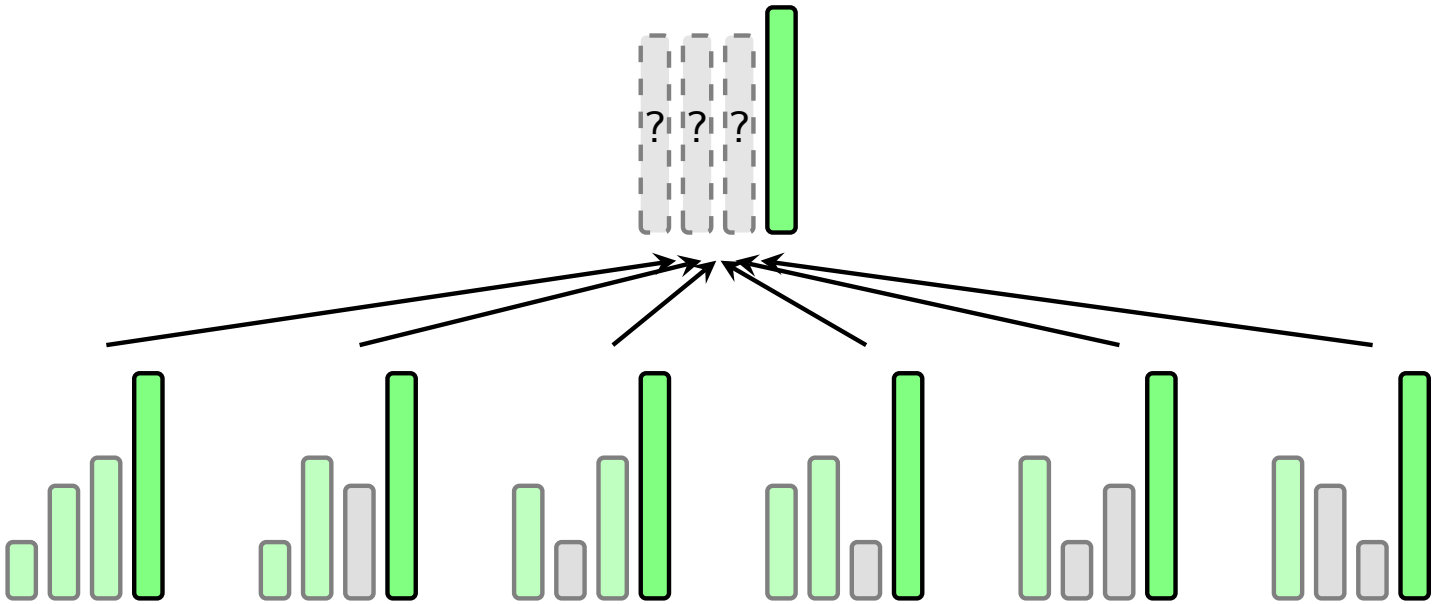

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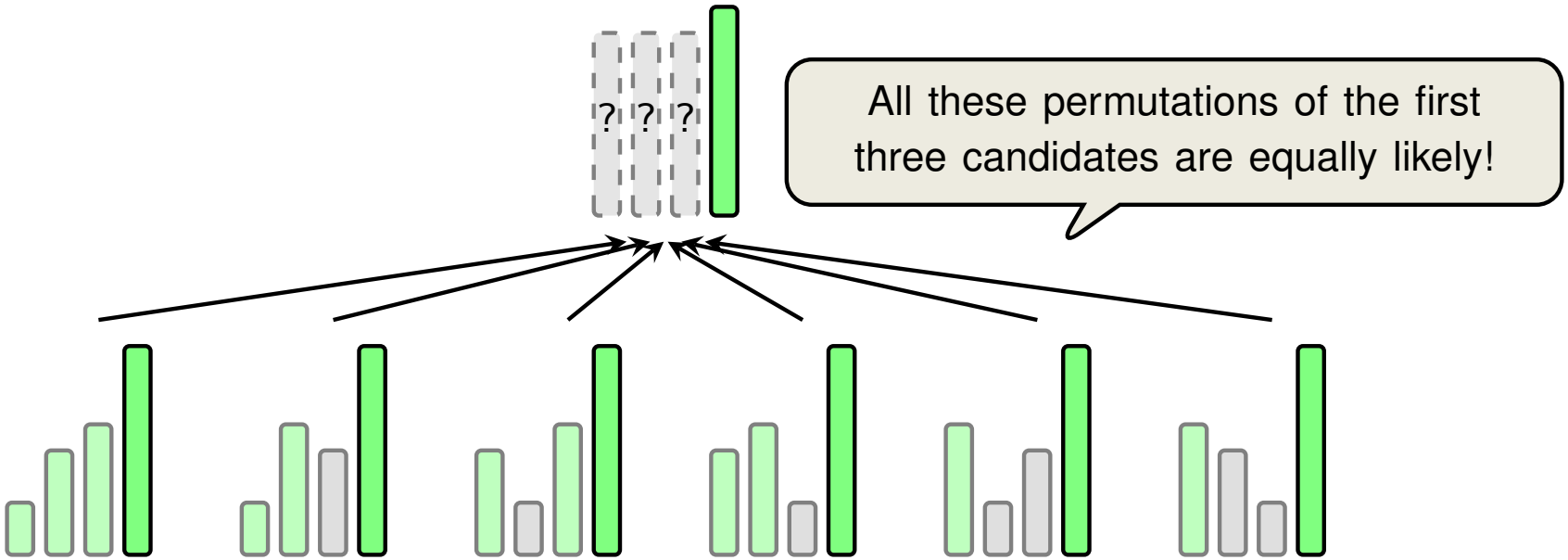

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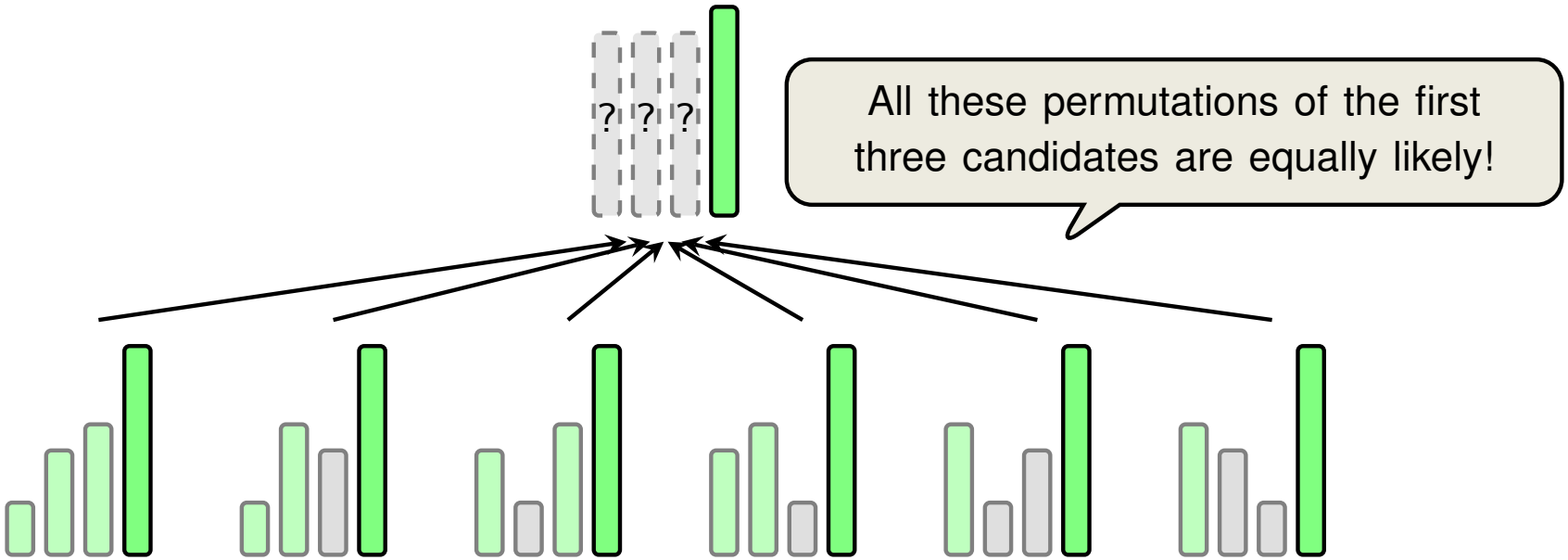

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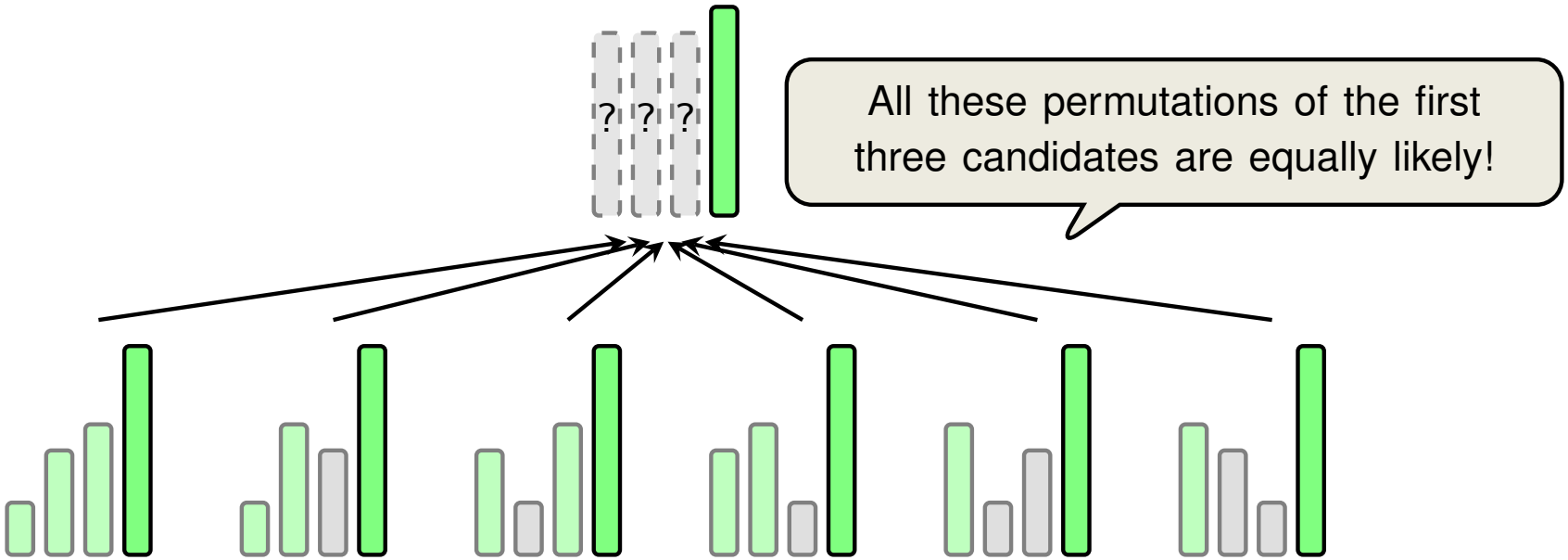

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Optimal Strategy

- Explore but reject the first $x-1$ candidates
- Accept first candidate $i \geq x$ which is better than all candidates before

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Find x which maximises the probability of hiring the best candidate.

Answer

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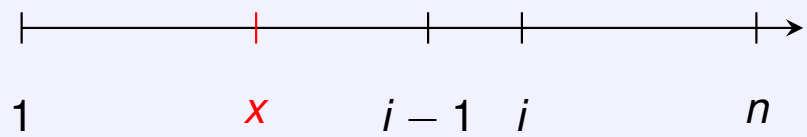

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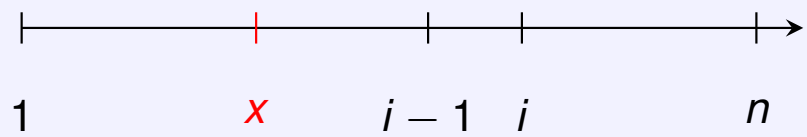

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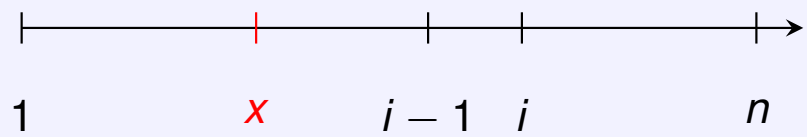

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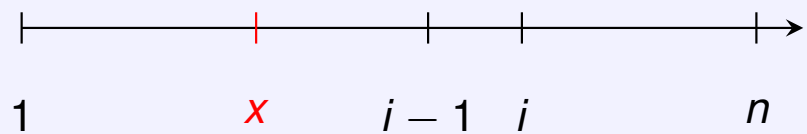

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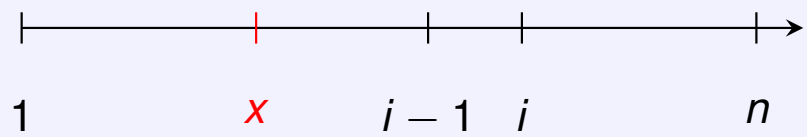

$$\Rightarrow \quad \sum_{i=x}^{n} \frac{1}{i-1} \approx \ln(n/x)$$

Example 2

Find x which maximises the probability of hiring the best candidate.

Answer

- First compute success probability for any $x \in \{1, \ldots, n\}$, and then optimise:

$$\mathbf{P}[\text{hire best candidate}]$$

$$= \sum_{i=1}^{n} \mathbf{P}[\text{hire candidate } i \cap \text{ candidate } i \text{ is best }]$$

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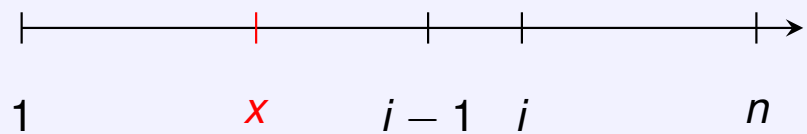
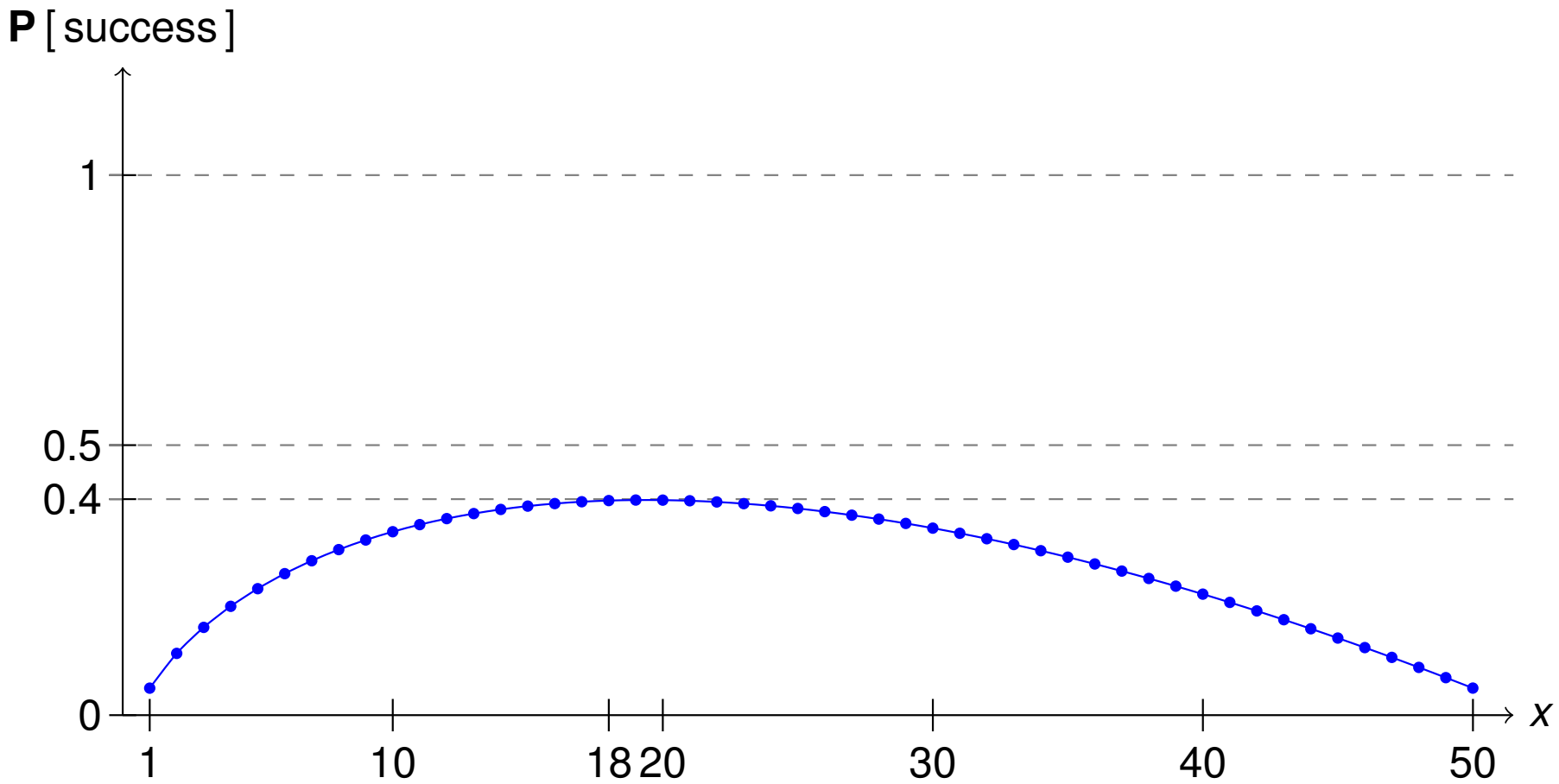


$$\Rightarrow \quad \sum_{i=x}^{n} \frac{1}{i-1} \approx \ln(n/x) \quad \Rightarrow \quad \text{maximum success probability for } x = \frac{1}{e} \cdot n.$$

Probability for Success (Illustration)

Suppose $n = 50$:



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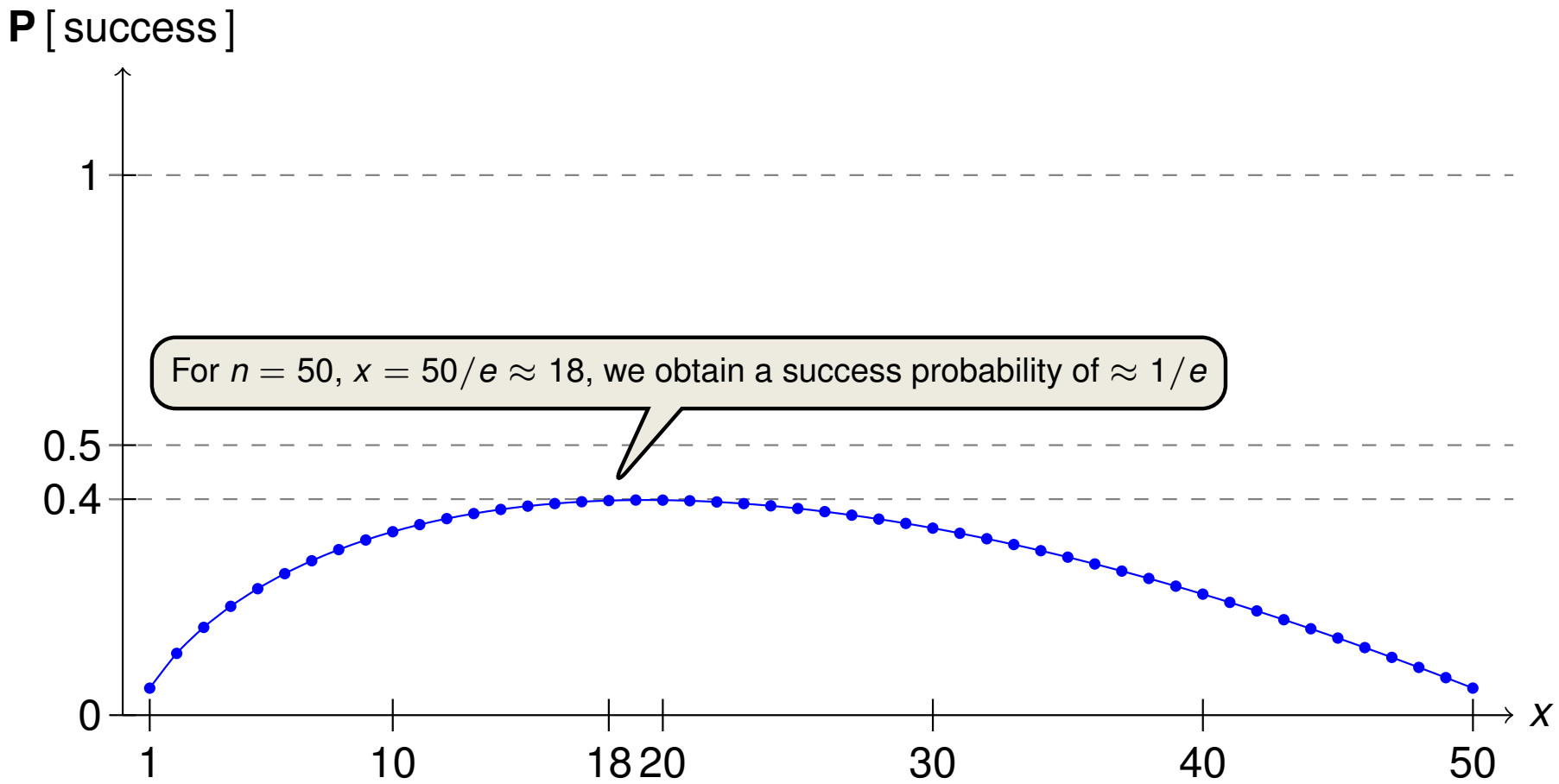

Another Variant of the Secretary Problem

"The Postdoc Variant of the Secretary Problem" (Vanderbei'80)

- same setup as in the secretary problem before
- difference: we want to pick the second-best ("the best [postdoc] is going to Harvard")
- Success probability of the optimal strategy is:

$$\frac{0.25n^2}{n(n-1)} \xrightarrow{n\to\infty} \frac{1}{4}$$

- Thus it is easier to pick the best than the second-best(!)

Outline

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- Let $I_1, I_2, \ldots, I_n$ be a sequence of independent indicators and let $p_j = \mathbf{E}[\, I_j \,]$

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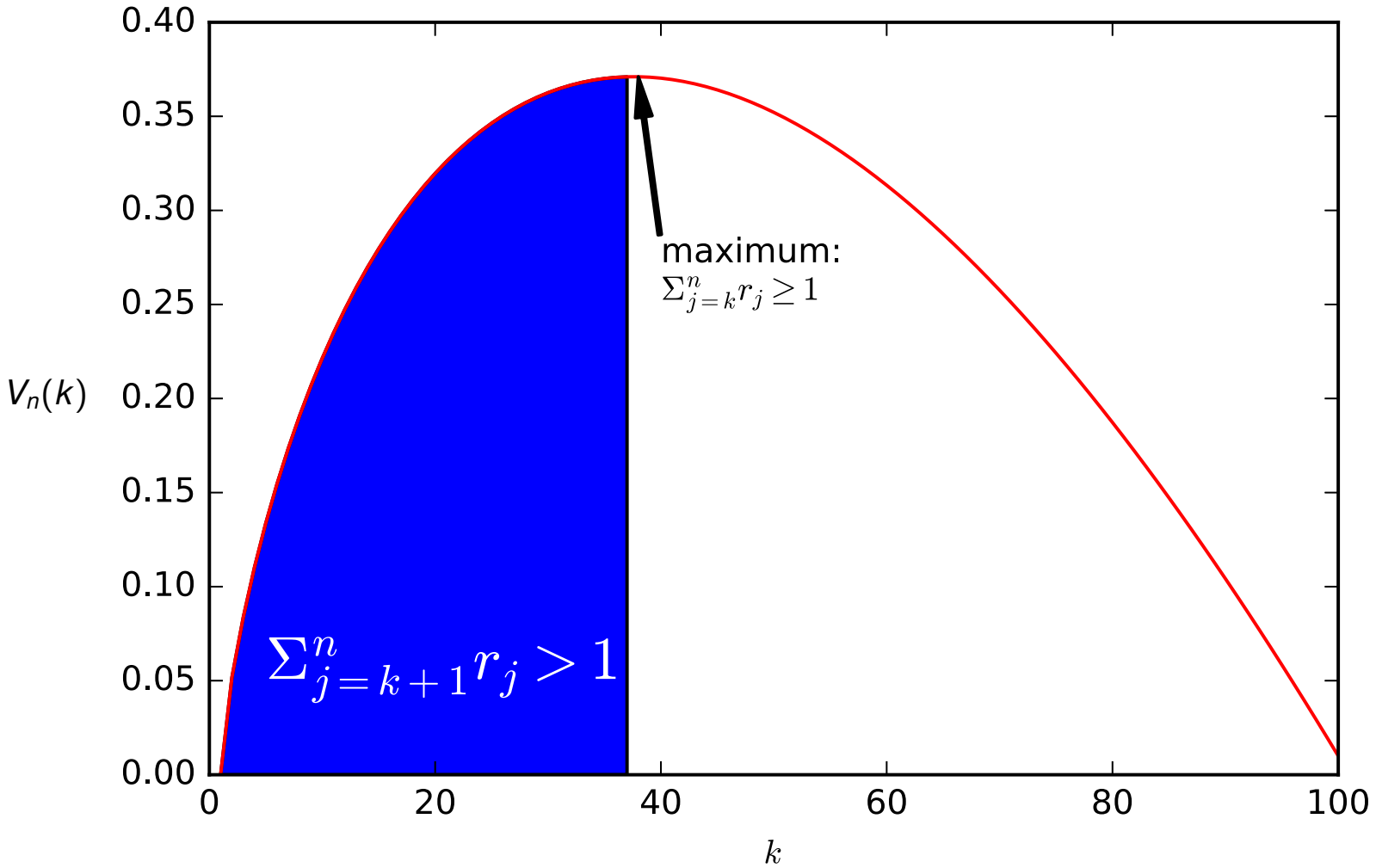
- This algorithm always executes the optimal strategy!

Illustration of the probability of having the last success ($n = 100$)



Source: Group Fibonado

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We re-derived the solution of the secretary problem as a special case!

Outline

List of Lectures

- **Part I: Introduction to Probability**
  - Lecture 1: Conditional probabilities and Bayes' theorem
- **Part II: Random Variables**
  - Lecture 2: Random variables, probability mass function, expectation
  - Lecture 3: Expectation properties, variance, discrete distributions
  - Lecture 4: More discrete distributions: Poisson, Geometric, Negative Binomial
  - Lecture 5: Continuous random variables
  - Lecture 6: Marginals and Joint Distributions
  - Lecture 7: Independence, Covariance and Correlation
- **Part III: Moments and Limit Theorems**
  - Lecture 8: Basic Inequalities and Law of Large Numbers
  - Lecture 9: Central Limit Theorem
- **Part IV: Applications and Statistics**
  - Lecture 10: Estimators (Part I)
  - Lecture 11: Estimators (Part II)
  - Lecture 12: Online Algorithms

List of Distributions

Very Important:

- Bernoulli, Binomial, Poisson
- (Continuous) Uniform, Normal, Exponential

(Somewhat Less) Important:

- Geometric, Negative Binomial, Hypergeometric, Discrete Uniform

Not used or not defined in this course (and thus not examinable):

- Cauchy, Gamma, bivariate Normal
- Beta

Thank you and Best Wishes for the Exam!