Introduction to Probability

Lecture 8: Basic Inequalities and Law of Large Numbers

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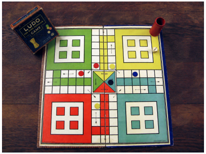
Outline

Board Games Involving Dice

- Games with One Dice: ⚃



Board Games Involving Dice

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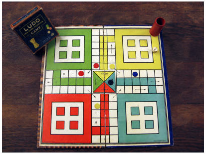
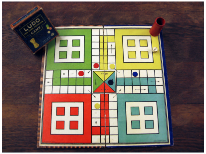


- Games with Two Dice: ⚂ ⚅

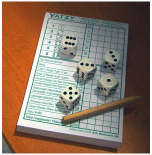
Board Games Involving Dice

- Games with One Dice: ⚃



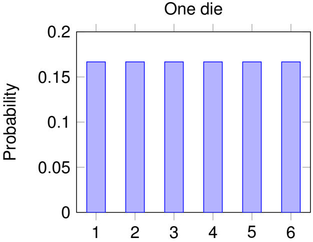
- Games with Two Dice: ⚂ ⚅

- Games with Five Dice: ⚂ ⚅ ⚄ ⚀ ⚄



Source: All images from Wikipedia.

Joint Distributions of Sums



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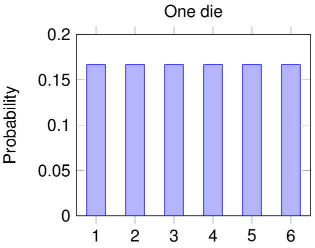

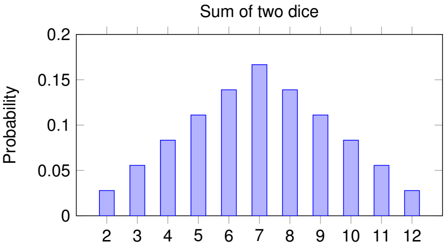

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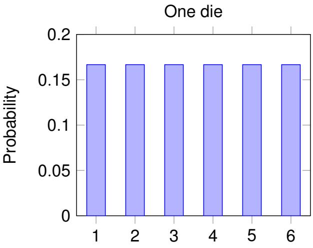
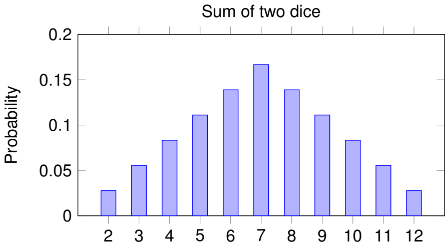
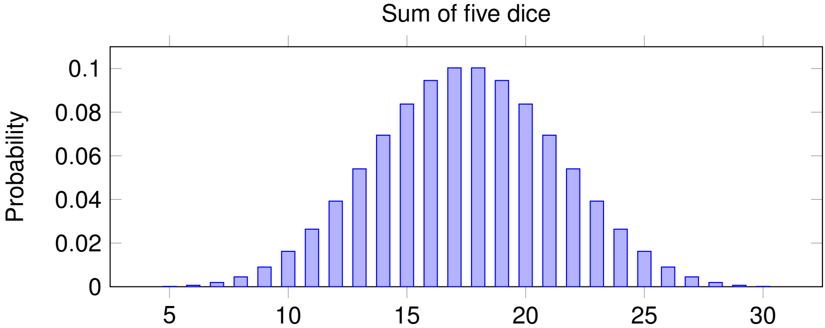
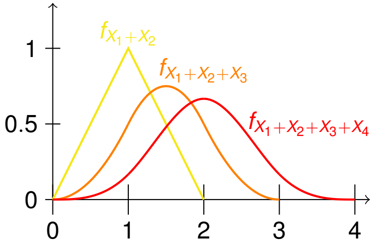


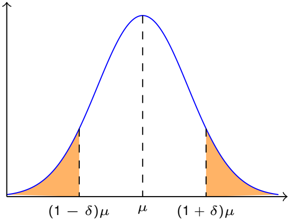




Motivation

We will study sums of independent and identically distributed variables. How does their distribution look like, and how well do they concentrate around the expectation?





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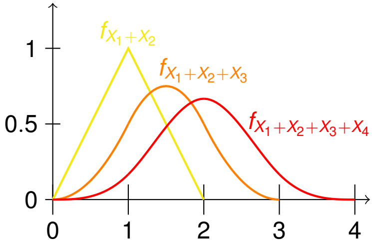

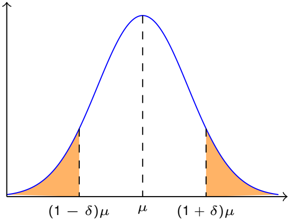

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2. Chebyshev's inequality
3. Law of Large Numbers
4. Central Limit Theorem

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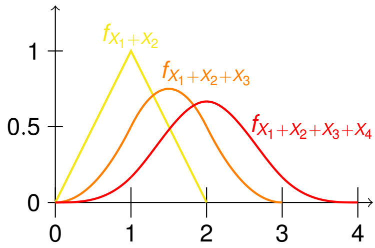

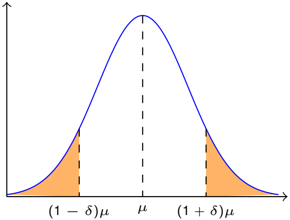

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2. Chebyshev's inequality
3. Law of Large Numbers
4. Central Limit Theorem

Re-use concepts from previous lectures:

1. Independence (Random Var.) (Lec. 1, 7)
2. Expectation and Variance (Lec. 2, 3)
3. Normal Distribution (Lec. 5)
4. Sums of Random Variables (Lec. 6)

Outline

Introduction

Markov's Inequality and Chebyshev's Inequality

Weak Law of Large Numbers

Markov's Inequality

Markov's Inequality

For any non-negative random variable X with finite $\mathbf{E}[X]$, it holds for any $a > 0$,

$$\mathbf{P}[X \geq a] \leq \frac{\mathbf{E}[X]}{a}.$$

A. Markov (1856-1922)

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- Advantage: Very basic inequality, we only need to know $\mathbf{E}[X]$
- Downside: For many distributions, the tail bound might be quite loose
- Proof is similar to the proof of Chebyshev's inequality (Exercise!)

Applying Markov's Inequality

Example 2

Consider throwing an unbiased, six-sided dice 120 times and let X denote the number of times we obtain a six.

1. Derive an upper bound on $\mathbf{P}[\,X \geq 30\,]$.
2. Can you also derive an upper bound on $\mathbf{P}[\,X \leq 10\,]$?

Answer

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1. First compute $\mathbf{E}[X]$

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   - Define a new random variable $Y := 120 - X$.

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$$\leq \frac{100}{} = \frac{10}{}.$$

Both bounds, especially the second, are quite loose!

Chebyshev's Inequality

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For any random variable X with finite $\mathbf{E}[X]$ and $\mathbf{V}[X]$, for any $a > 0$,

$$\mathbf{P}[|X - \mathbf{E}[X]| \geq a] \leq \mathbf{V}[X]/a^2.$$

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Comments:

- can be rewritten as:

$$\mathbf{P}\left[|X - \mathbf{E}[X]| \geq \sqrt{\delta \cdot \mathbf{V}[X]}\right] \leq 1/\delta.$$

The "$\mu \pm$ a few σ" rule. Most of the probability mass is within a few standard deviations from μ.

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- Unlike Markov, Chebyshev's inequality is two-sided and also holds for random variables with negative values
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- Chebyshev's inequality is also known as Second Moment Method

Derivation of Chebychev's inequality

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- We will give a self-contained proof for a continuous random variable X (the case for discrete X is analogous).

Exercise: Can you find a proof that uses Markov's inequality?

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- We will give a self-contained proof for a continuous random variable X (the case for discrete X is analogous).
- Write down the definition of $\mathbf{V}[X]$ and then lower bound:

$$\mathbf{V}[X] = \mathbf{E}\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)\, dx$$

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- Dividing both sides by a^2 yields the result.

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Example: Chebychev is (usually) much stronger than Markov

Example 3

Throw an unbiased coin n times and let X be the total number of heads. In an experiment, with n large, we would usually expect a number of heads that is close to the expectation. Can we justify that?

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$\Rightarrow$ We have $\mathbf{V}[X] = np(1-p) = n \cdot 1/2 \cdot 1/2$. For any $\delta > 0$,

Example: Chebychev is (usually) much stronger than Markov

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Throw an unbiased coin n times and let X be the total number of heads. In an experiment, with n large, we would usually expect a number of heads that is close to the expectation. Can we justify that?

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$X \sim Bin(n, 1/2)$ so $\mathbf{E}[X] = n \cdot \frac{1}{2}$.

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\end{aligned}$$

Much better! (Inversely) Linear in n

Outline

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Markov's Inequality and Chebyshev's Inequality

Weak Law of Large Numbers

Law of Large Numbers

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= independent and identically distributed

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"For even the most stupid of men, by some instinct of nature, by himself and without any instruction (which is a remarkable thing), is convinced that the more observations have been made, the less danger there is of wandering from one's goal."

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- A similar statement holds even if the X_i's are not identically distributed

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- "Power of Averaging": repeated samples allow us to estimate μ
- A similar statement holds even if the X_i's are not identically distributed
- There is also a strong law of large numbers:

$$\mathbf{P}\left[\lim_{n \to \infty} \overline{X}_n = \mu \right] = 1.$$

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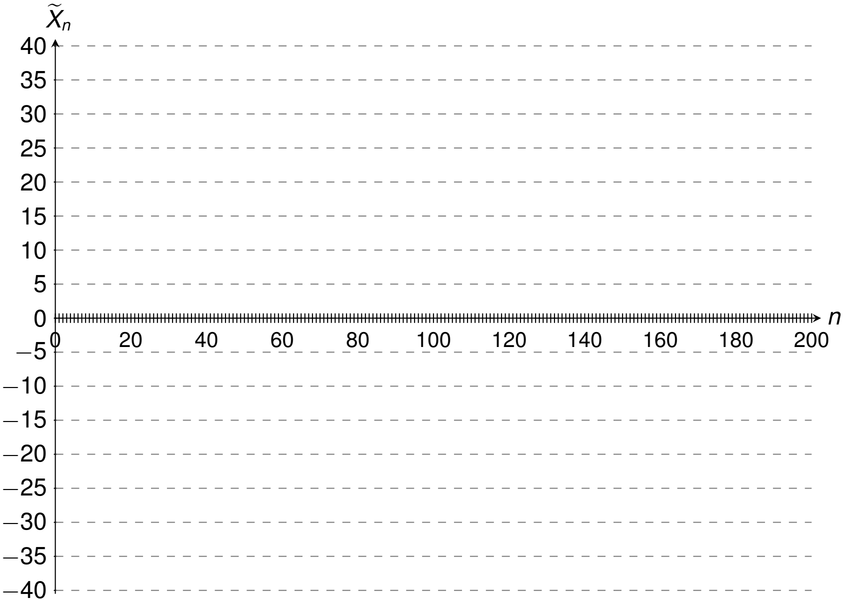
How does a "typical" realisation look like?

Illustration of Weak Law of Large Numbers (2/4)



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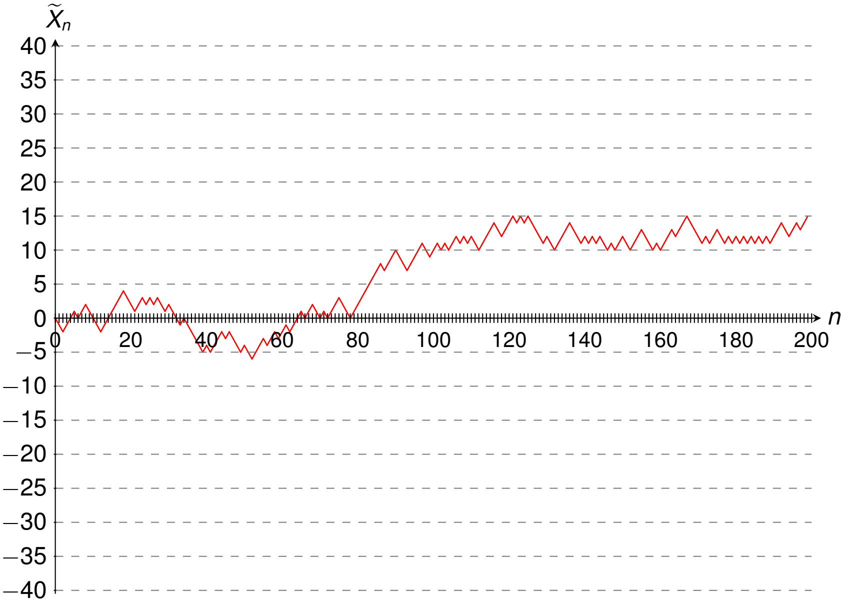

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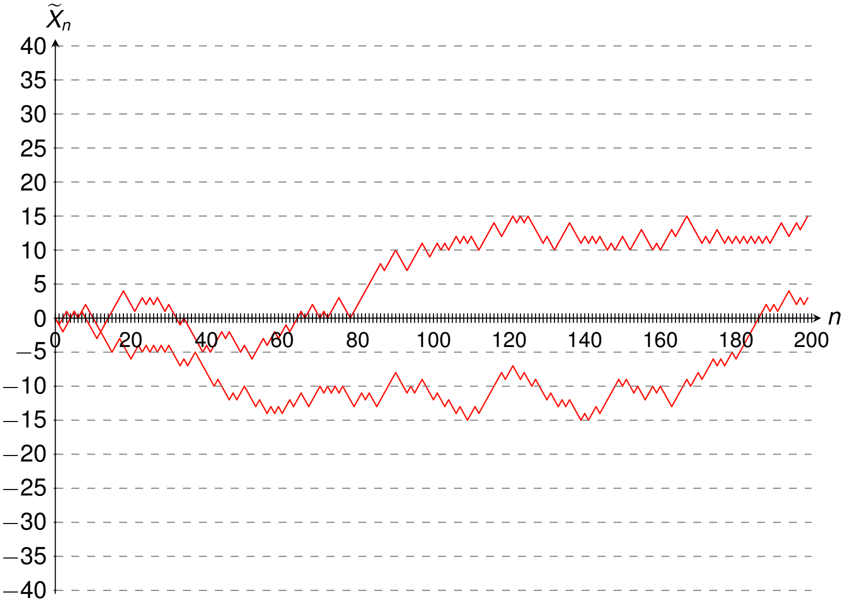

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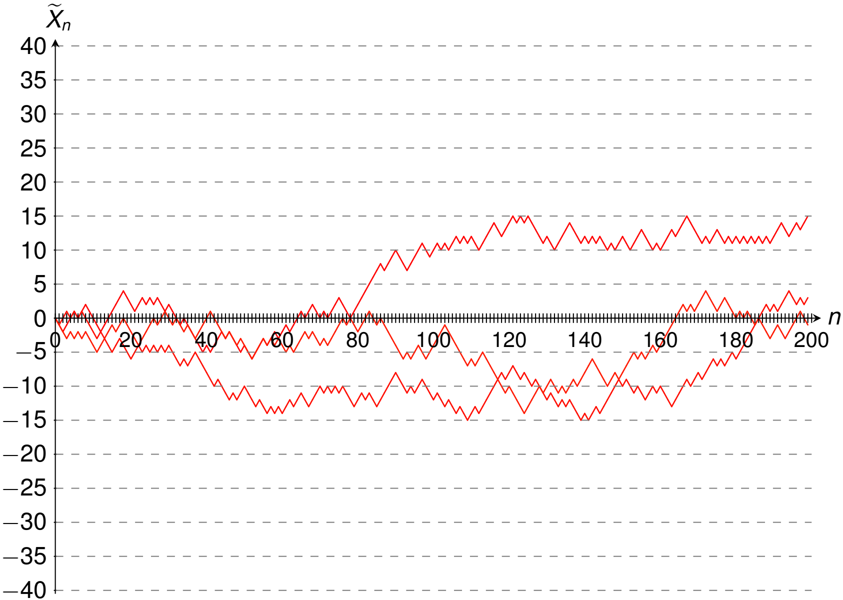

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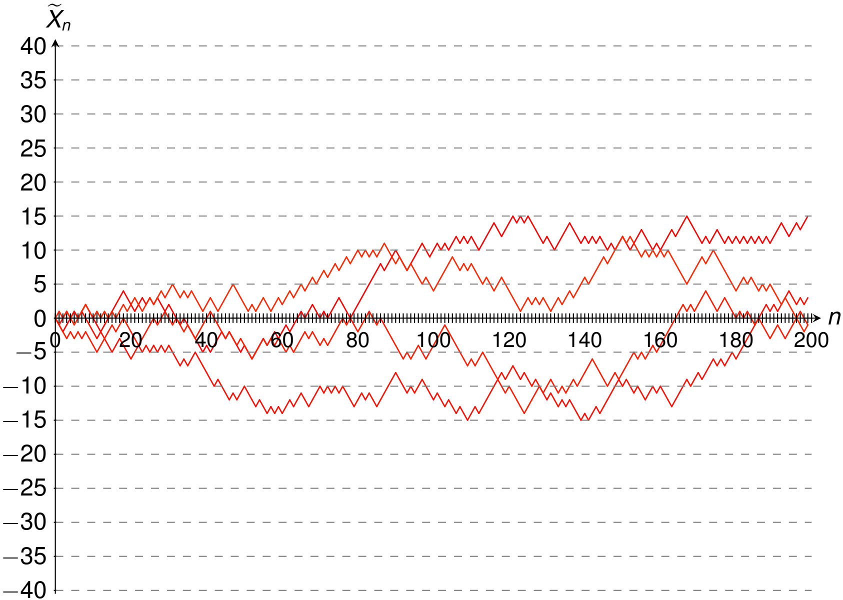

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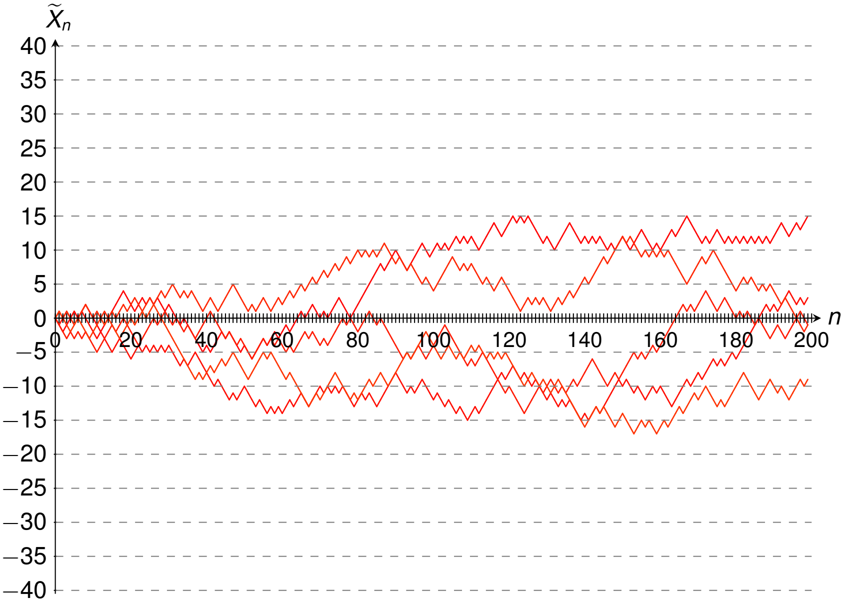

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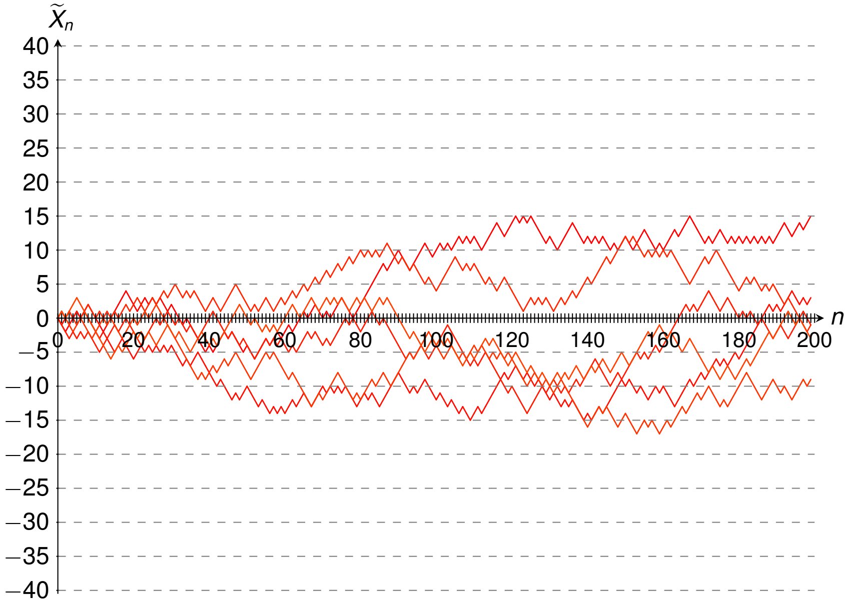

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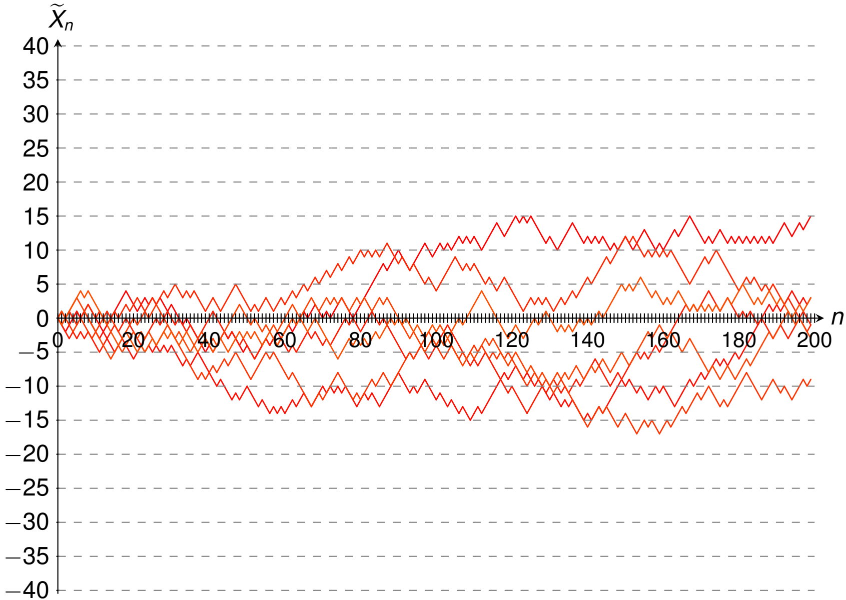

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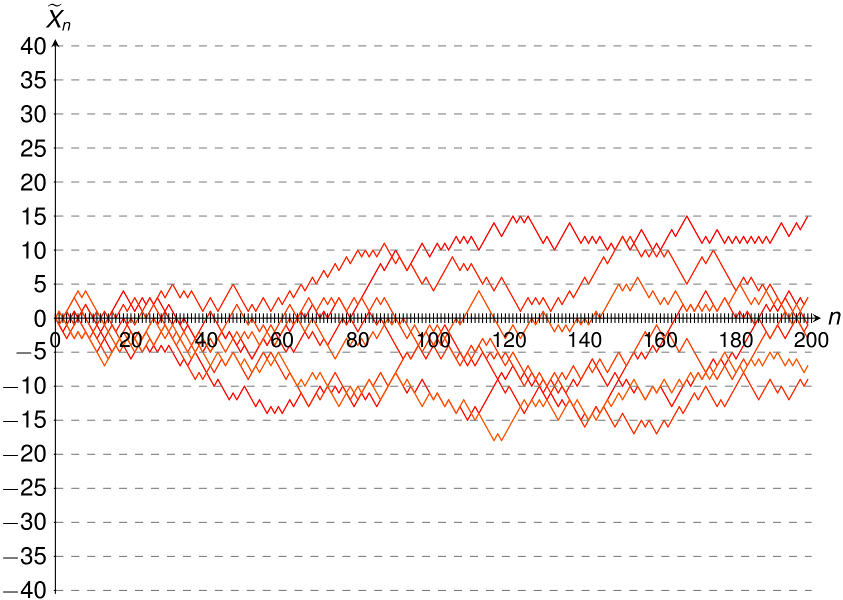

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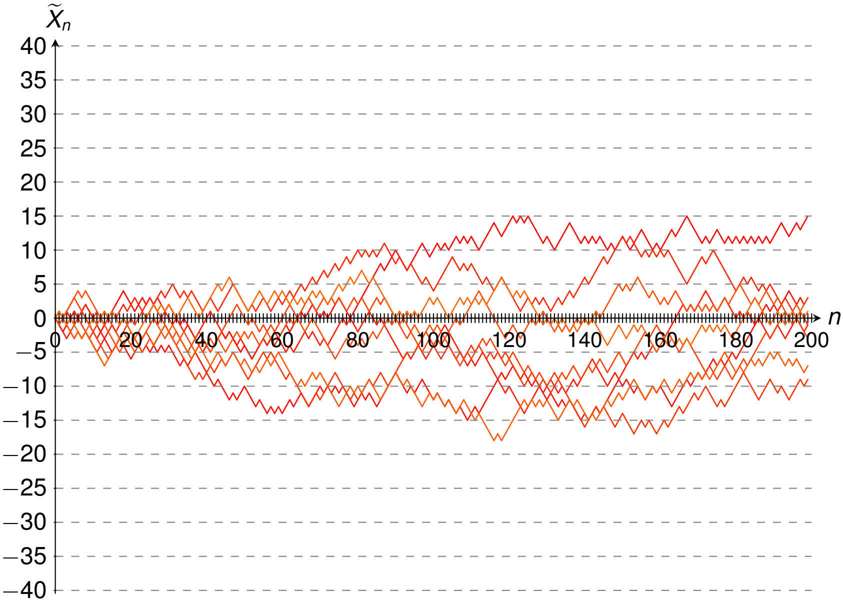

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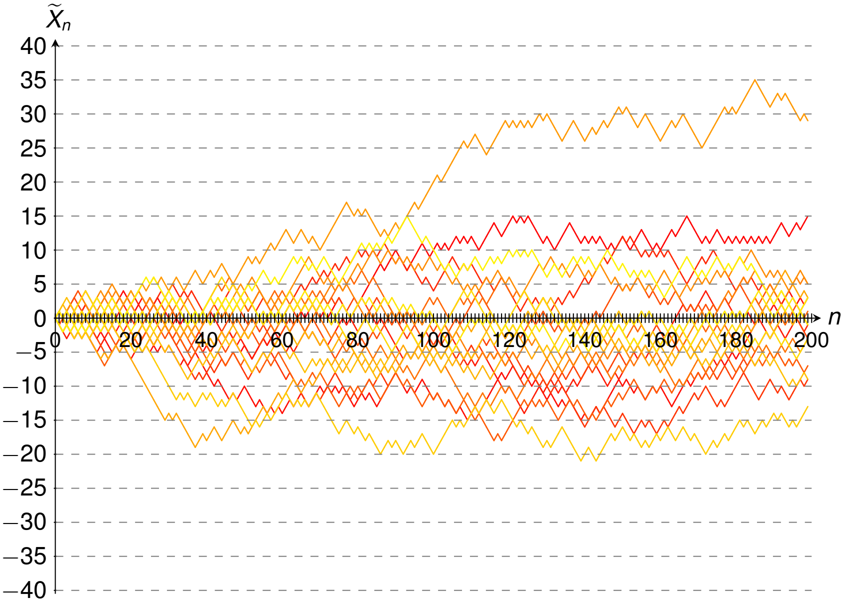

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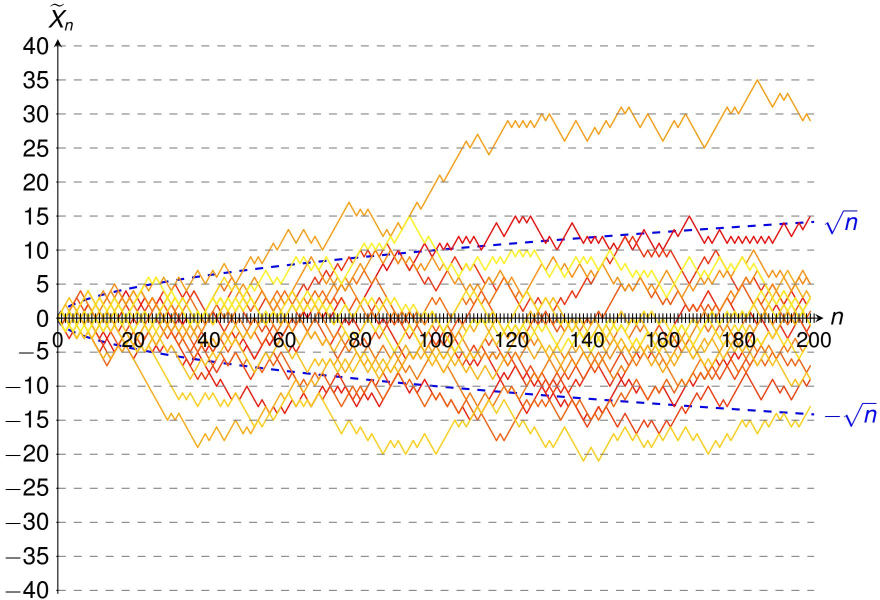
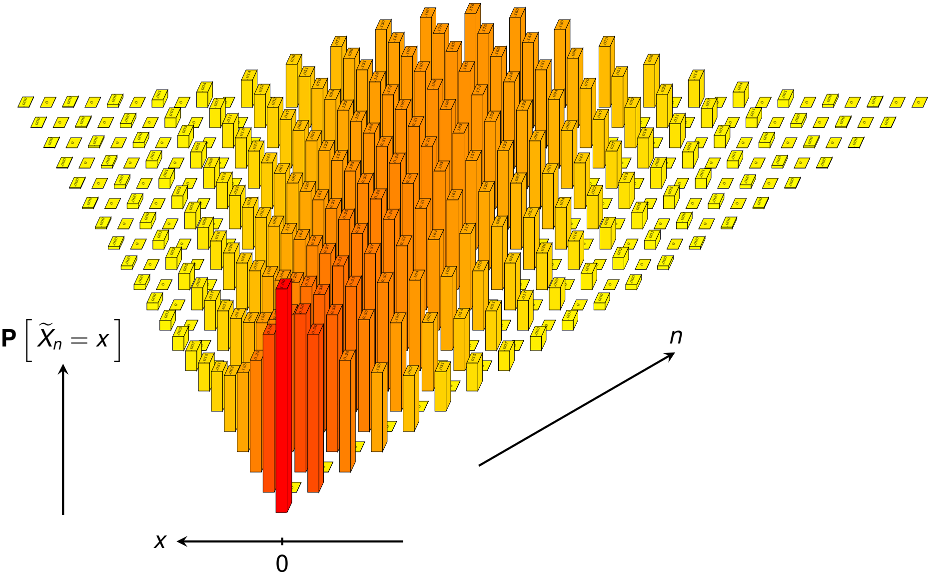
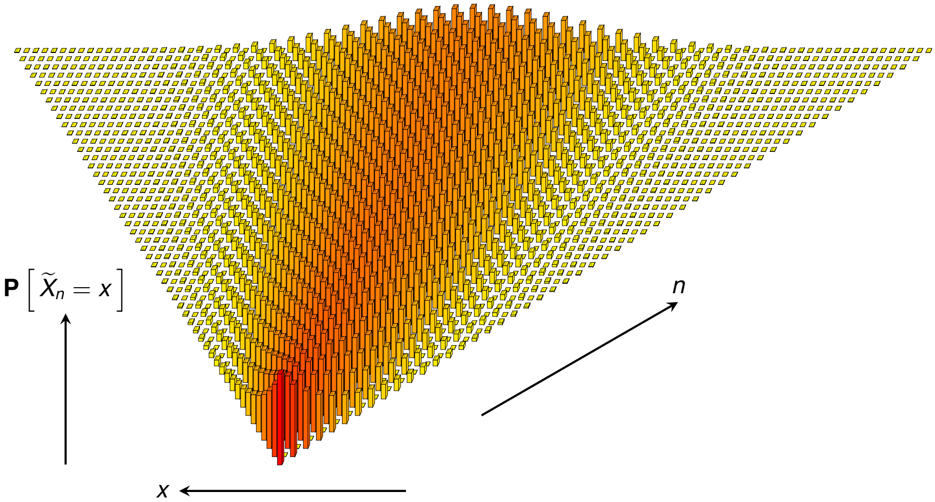


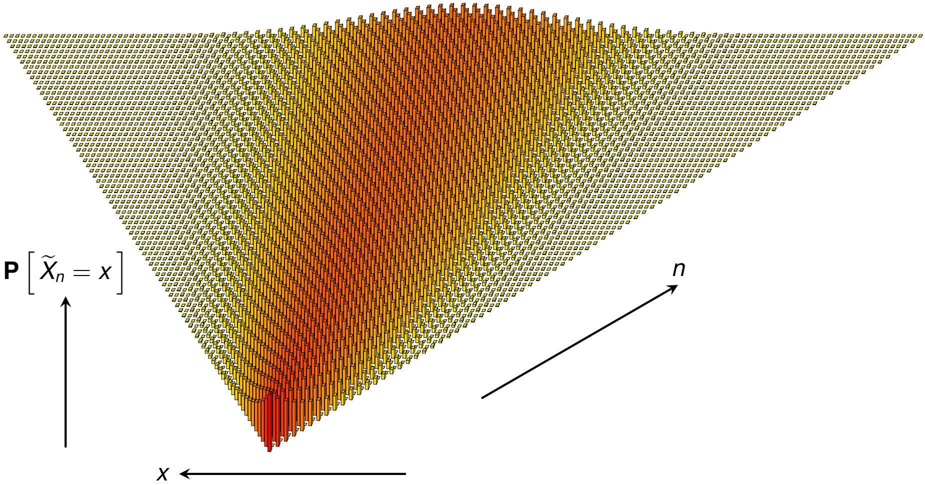
Plot of the Distributions for $n = 0, 1, \ldots, 20$



Plot of the Distributions for $n = 0, 1, \ldots, 50$



Plot of the Distributions for $n = 0, 1, \ldots, 80$



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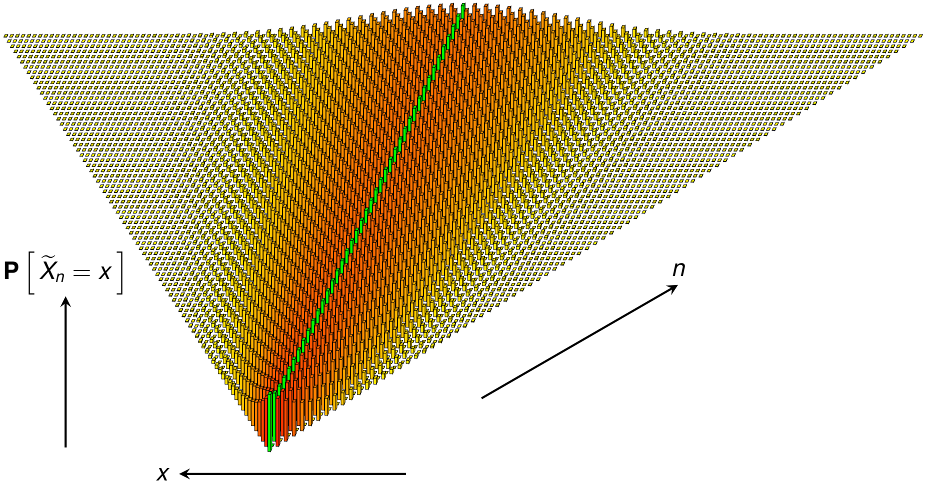
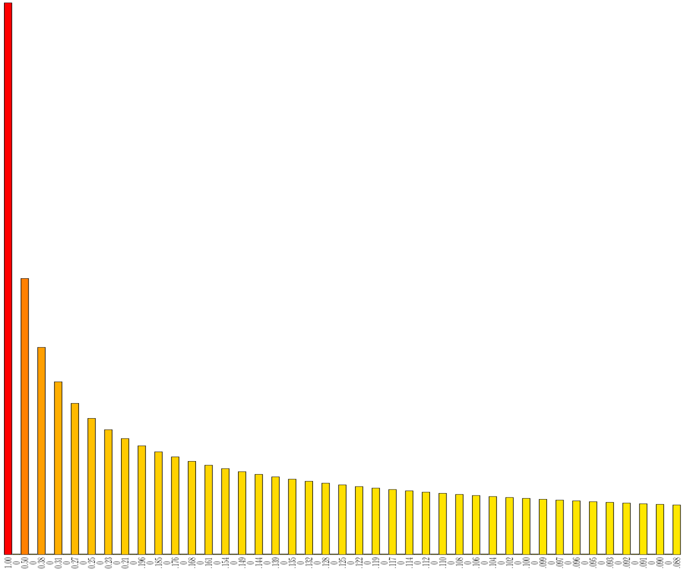


Interlude: Approximation of $\mathbf{P}[\,\widetilde{X}_n = 0\,]$



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Exercise

Try to find an expression for $\mathbf{P}\left[\widetilde{X}_n = 0\right]$. Using Stirling's approximation for $n!$, conclude that $\mathbf{P}\left[\widetilde{X}_n = 0\right] = \Theta(1/\sqrt{n})$ for even integers n.

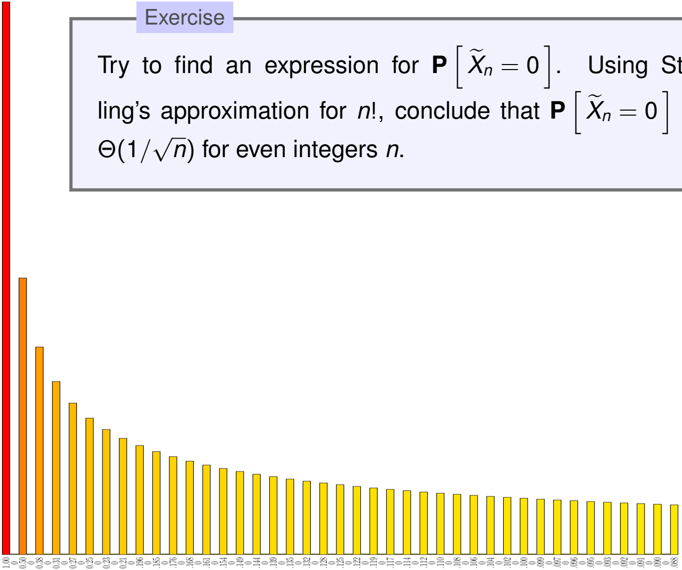

Illustration of Weak Law of Large Numbers (3/4)

- Let X_i be independent random variables taking values $\in \{-1, +1\}$ with probability $1/2$ each
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This does **not** converge!

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Consider now the average (sample mean): $\overline{X}_n := 1/n \cdot \sum_{i=1}^{n} X_i$.

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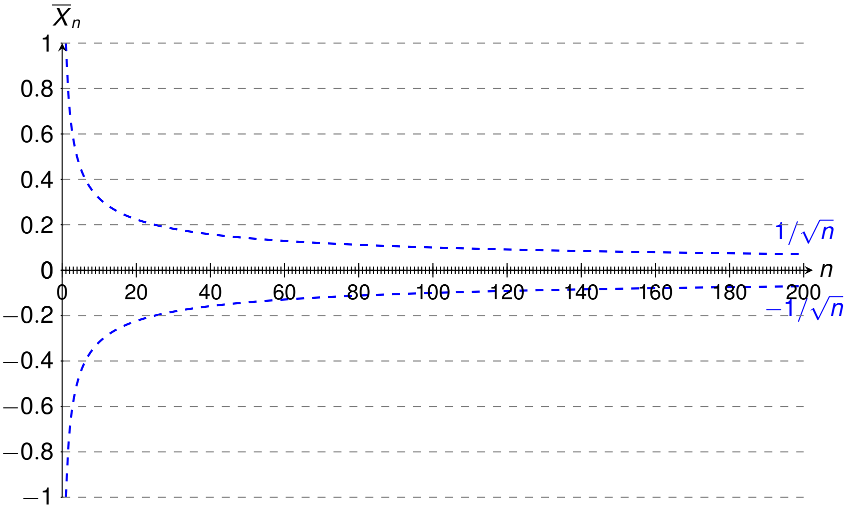

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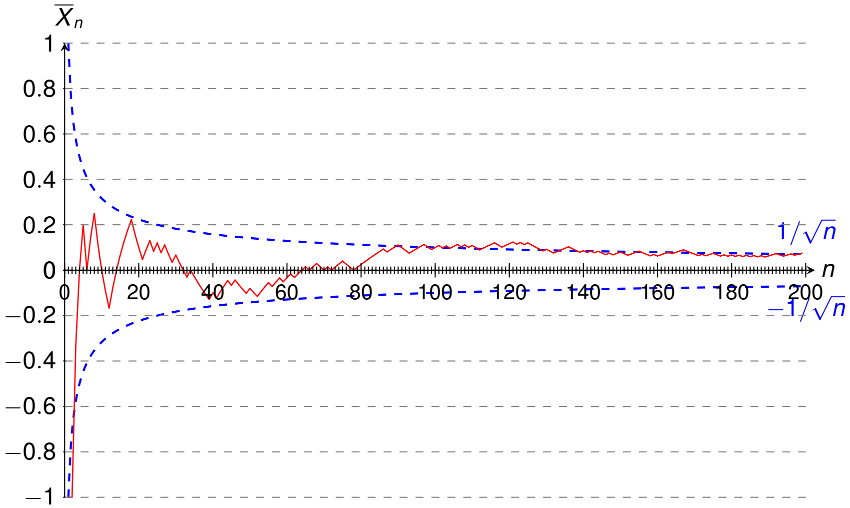

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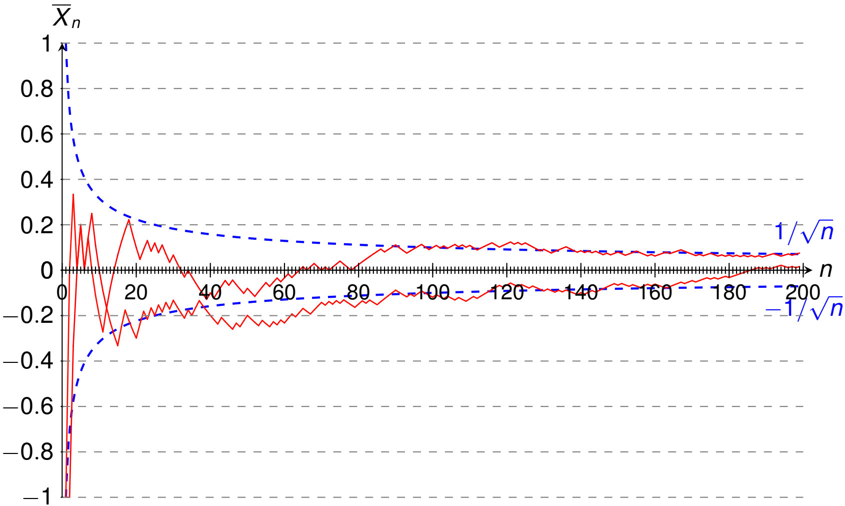

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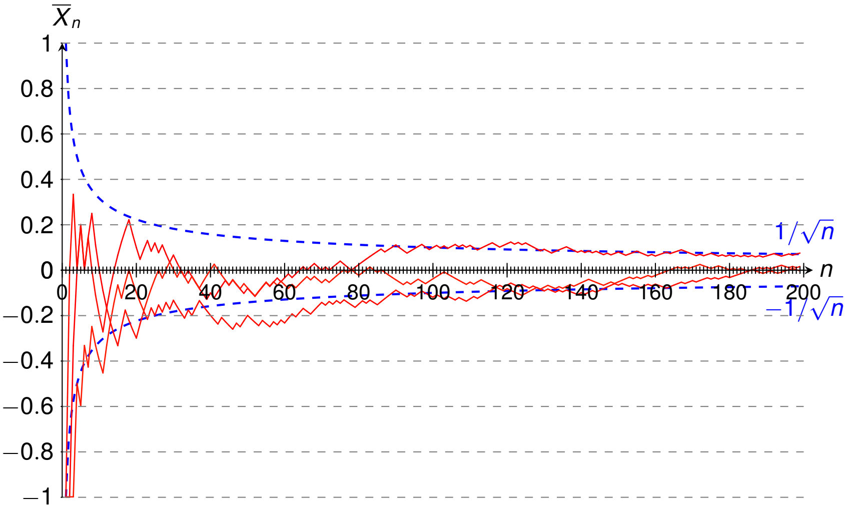

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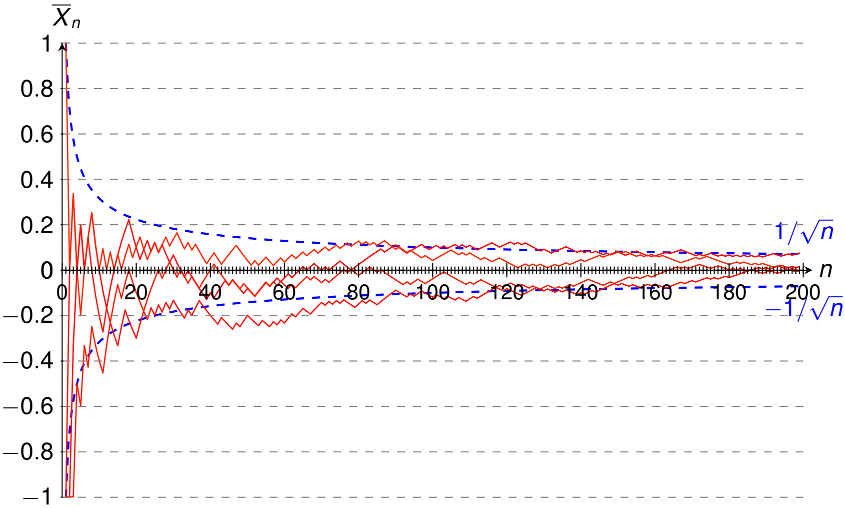

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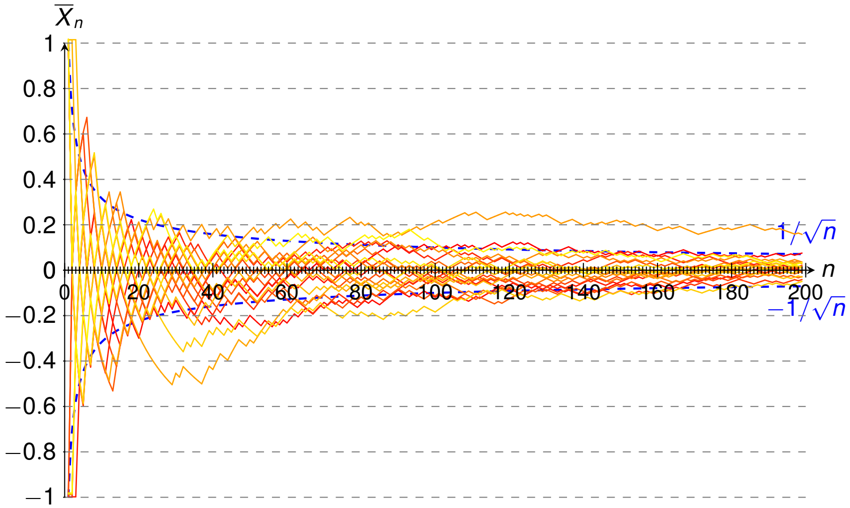

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- Applying Chebyshev's inequality yields:

$$\mathbf{P}\left[\left|\overline{X}_n - \mathbf{E}\left[\overline{X}_n\right]\right| > \epsilon\right] \leq \frac{1}{\epsilon^2} \cdot \mathbf{V}\left[\overline{X}_n\right]$$

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- For any (fixed) $\epsilon > 0$, the right hand side vanishes as $n \to \infty$.

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$$\lim_{n\to\infty} \mathbf{P}\left[\,|\overline{X}_n - \mu| > \epsilon\,\right] = 0$$

Proof

- Let $\overline{X}_n := 1/n \cdot \sum_{i=1}^{n} X_i$
- Then $\mathbf{E}\left[\overline{X}_n\right] = \mu$ and
  $\mathbf{V}\left[\overline{X}_n\right] = 1/n^2 \cdot \mathbf{V}\left[\sum_{i=1}^{n} X_i\right] = 1/n^2 \cdot \sum_{i=1}^{n} \mathbf{V}\left[X_i\right] = 1/n \cdot \sigma^2$.
- Applying Chebyshev's inequality yields:

$$\mathbf{P}\left[\left|\overline{X}_n - \mathbf{E}\left[\overline{X}_n\right]\right| > \epsilon\right] \le \frac{1}{\epsilon^2} \cdot \mathbf{V}\left[\overline{X}_n\right] = \frac{\sigma^2}{n\epsilon^2}.$$

- For any (fixed) $\epsilon > 0$, the right hand side vanishes as $n \to \infty$.

(Let $\epsilon > 0$, $\delta > 0$. Pick $N = \frac{\sigma^2}{\epsilon^2 \cdot \delta}$. Then for any $n \ge N$, the probability above is smaller than δ.)

Inferring Probabilities of an Event

Example 4

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$$p := \mathbf{P}[X \in (a, b]], \text{ where } a < b.$$

How can we use the Law of Large Numbers?

Answer

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Answer

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- Similarly, $\mathbf{V}\left[\, Y_i \,\right] = p(1 - p)$
- The random variables $Y_1, Y_2, \ldots, Y_n$ are i.i.d., so we can apply the Law of Large Numbers to $\overline{Y}_n$.