Introduction to Probability

Lecture 11: Estimators (Part II)

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Outline

Estimating Population Size (First Model)

- Suppose we have a sample of a few serial numbers (IDs) of some product
- We assume IDs are running from 1 to an unknown parameter N (so $N = \theta$)
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- Achieving unbiasedness alone is not a good strategy
- Improvement: find an estimator which always returns a value at least $\max(X_1, X_2, \ldots, X_n)$

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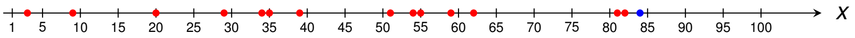

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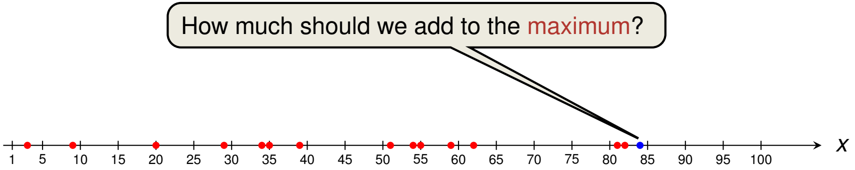

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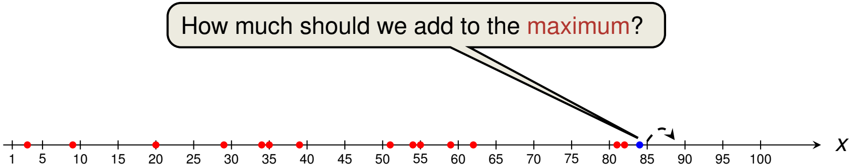

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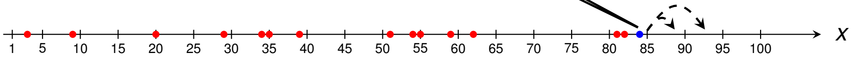

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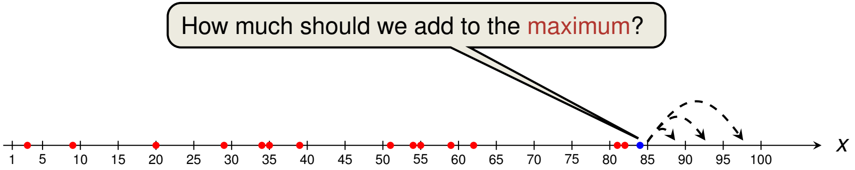

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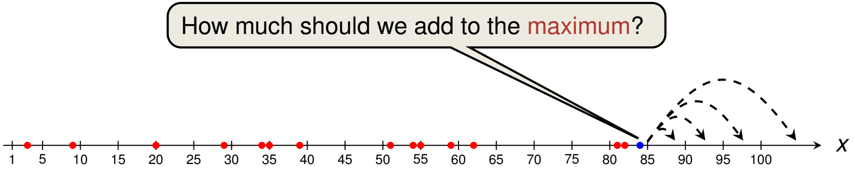

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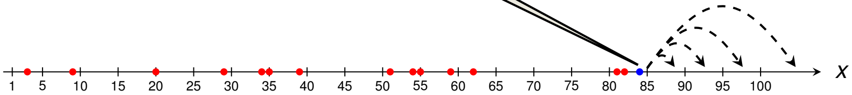

Rearrange the other 14 points equi-spaced between 0 and 84.

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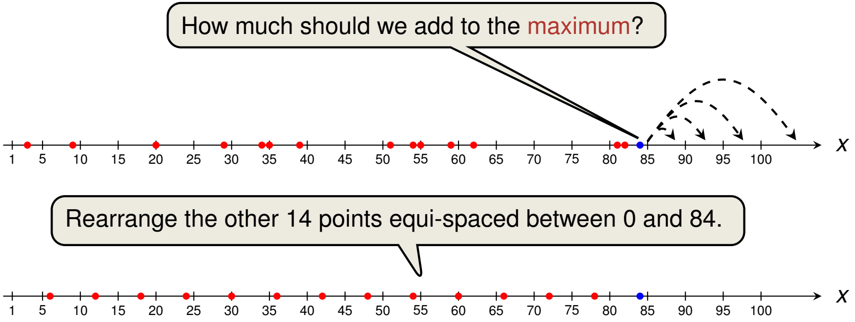

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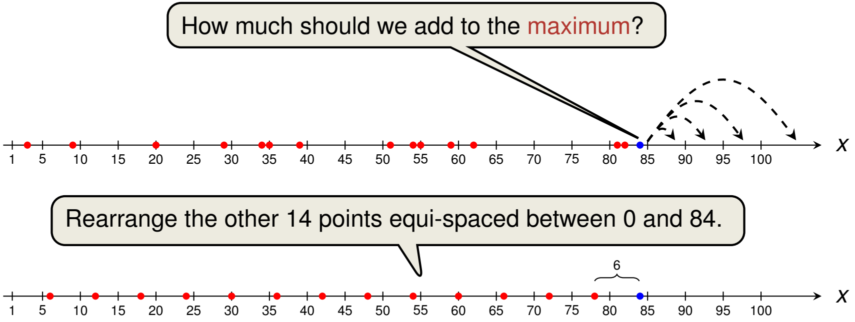

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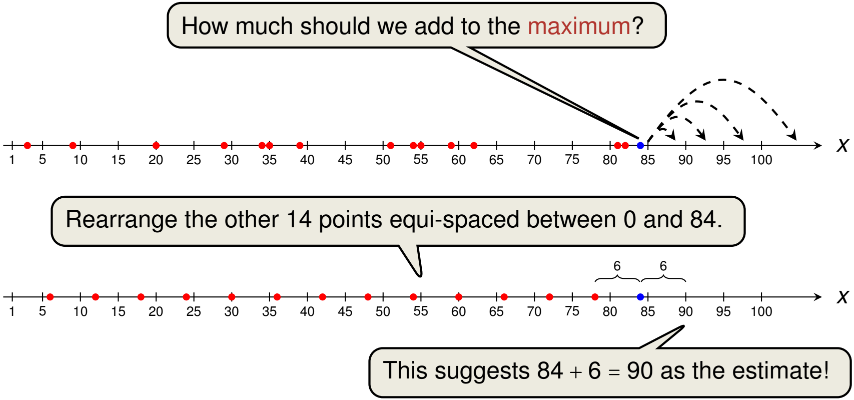

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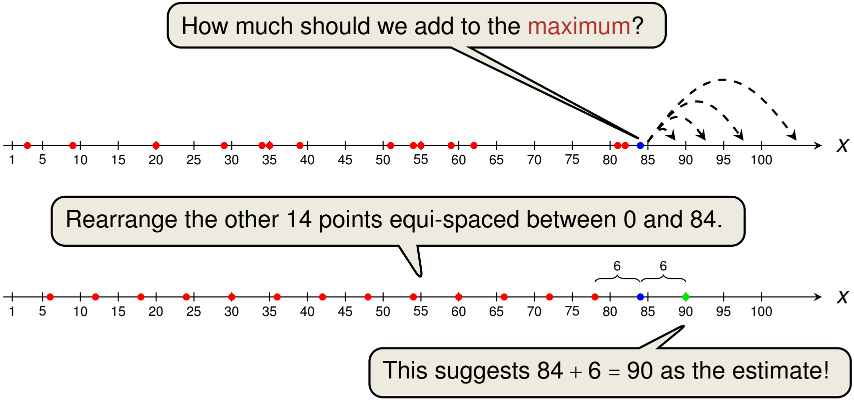

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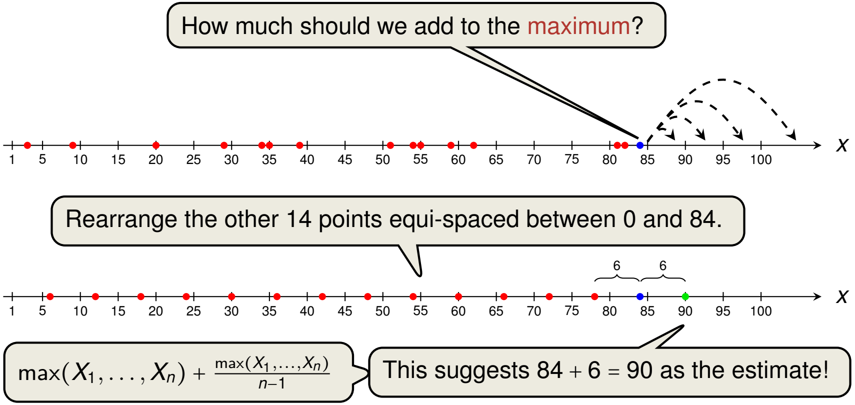

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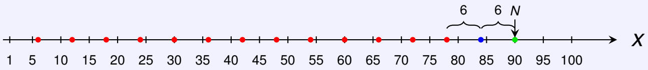

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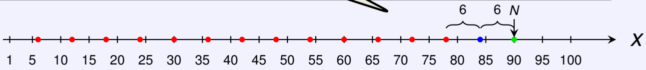

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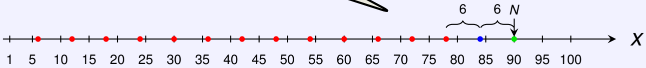

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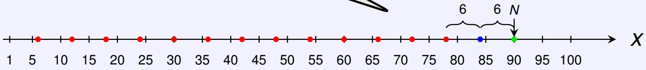

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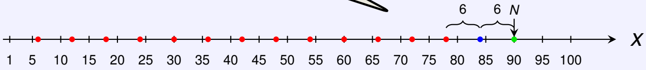

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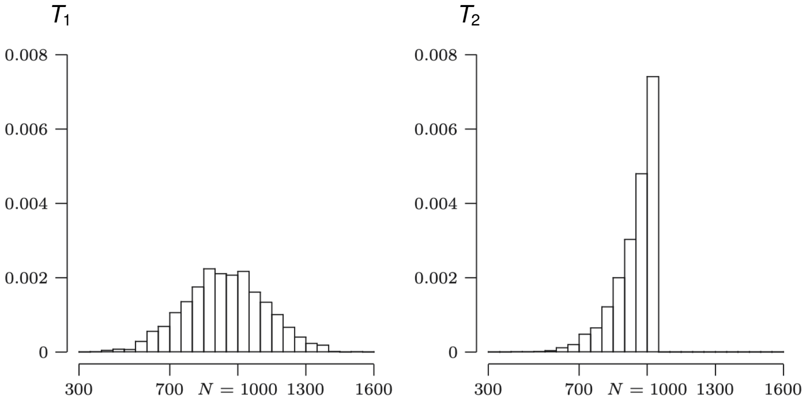
- For our samples before, we get $t_2 = \frac{16}{15} \cdot 84 - 1 = 88.6$.

Empirical Analysis of the two Estimators



Source: Modern Introduction to Statistics

Figure: Histogram of 2000 values for T_1 and T_2, when $N = 1000$ and $n = 10$.

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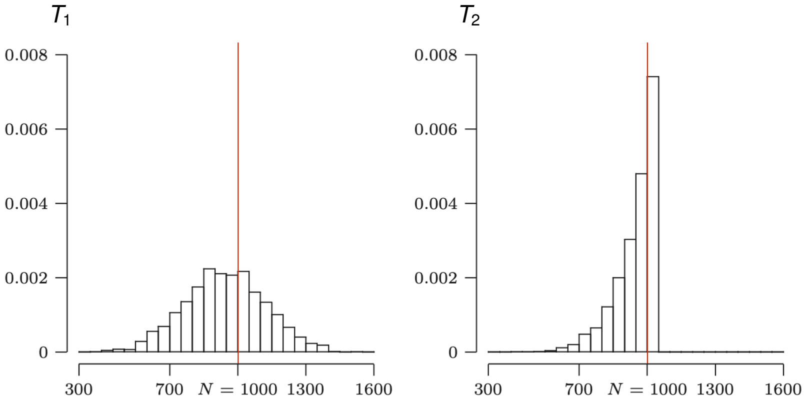

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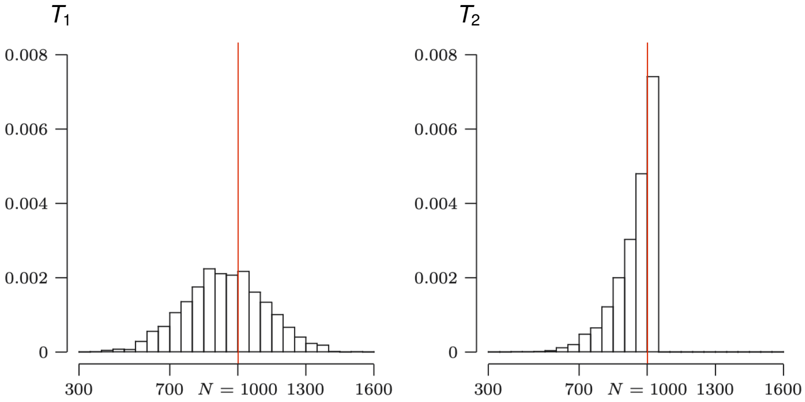

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Can we find a quantity that captures the superiority of T_2 over T_1?

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- If T_1 and T_2 are both unbiased, T_1 is better than T_2 iff $\mathbf{V}[\,T_1\,] < \mathbf{V}[\,T_2\,]$.

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$$\begin{aligned} \mathbf{MSE}[\,T\,] &= \mathbf{E}\left[\,(T-\theta)^2\,\right] \\ &= \mathbf{E}\left[\,T^2 - 2T\theta + \theta^2\,\right] \end{aligned}$$

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Bias-Variance Decomposition: Derivation

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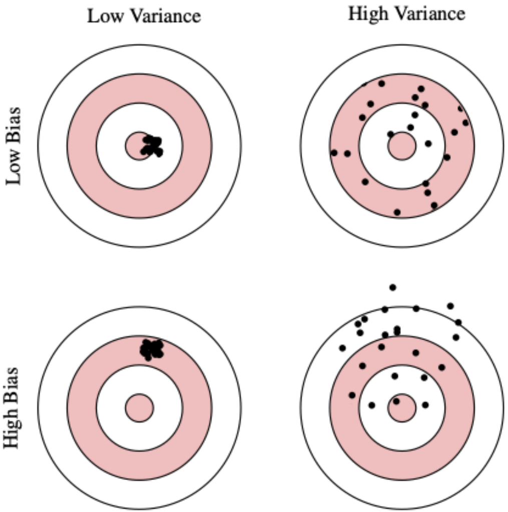
\end{aligned}$$

Bias-Variance Decomposition: Illustration



Source: Edwin Leuven (Point Estimation)

Example 4

It holds that **MSE** $[\ T_1\ ] = \Theta\left(\frac{N^2}{n}\right)$, where $T_1 = 2 \cdot \overline{X}_n - 1$.

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- Intuitively, X_1 and X_2 are negatively correlated, which would be sufficient to complete the proof. For a more rigorous and precise derivation (see Dekking et al.):

$$\mathbf{Cov}[X_1, X_2] = -\frac{1}{12}(N+1).$$

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- Rearranging and simplifying gives

$$\mathbf{V}[T_1] = \frac{(N+1)(N-n)}{3n}.$$

Analysis of the MSE for T_2 (non-examinable)

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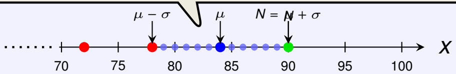
- One can prove:

For details see Dekking et al.

$$\mathbf{V}[\,\max(X_1, \ldots, X_n)\,] = \cdots = \frac{n(N+1)(N-n)}{(n+2)(n+1)^2} = \Theta\left(\frac{N^2}{n^2}\right)$$

Equi-spaced (idealised) configuration suggests a standard deviation of $\sigma \approx \frac{N}{n}$



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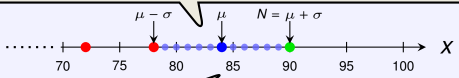

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- $\mathbf{MSE}\left[\, T_2 \,\right]$ is much lower than $\mathbf{MSE}\left[\, T_1 \,\right] = \Theta\left(\frac{N^2}{n}\right)$, i.e., $\frac{\mathbf{MSE}[\, T_1 \,]}{\mathbf{MSE}[\, T_2 \,]} = \frac{n+2}{3}$
- $\Rightarrow$ confirms simulations suggesting that T_2 is better than T_1!
- can be shown T_2 is the best unbiased estimator, i.e., it minimises MSE.

Outline

A New Estimation Problem

Previous Model

- Population/ID space $S = \{1, 2, \ldots, N\}$
- We take uniform samples from S without replacement
- **Goal:** Find estimator for N

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Similar idea applies to situations where elements are not labelled before we see them first time (Mark & Recapture Method)

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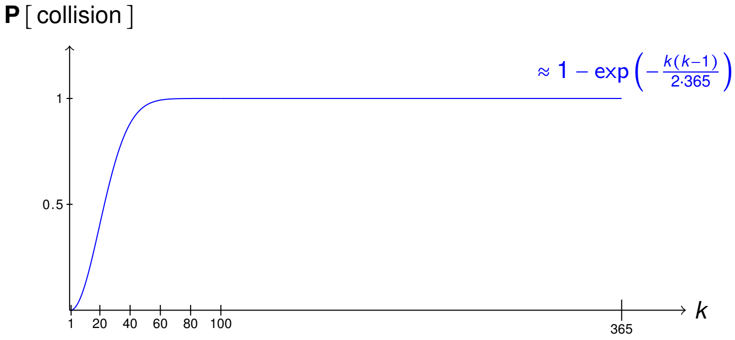

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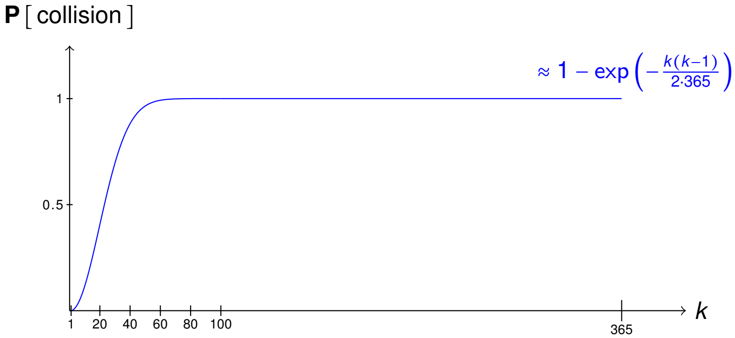

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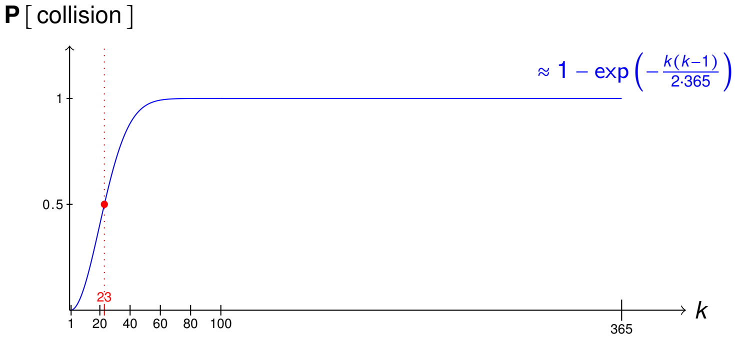

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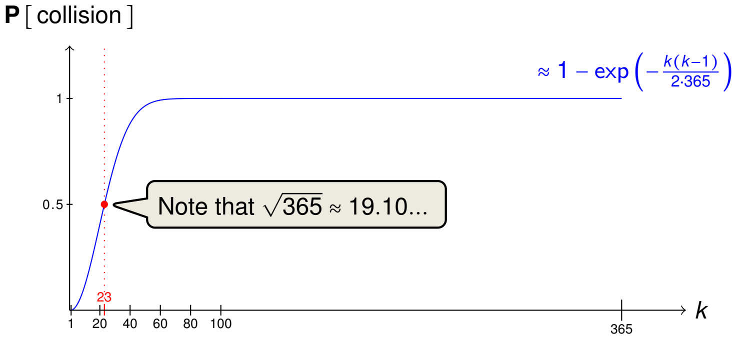

Estimation via Collision: The Algorithm

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C = ∅
For i = 1, 2, ...
    Take next i.i.d. sample X_i from S
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Exercise: Prove a bound of $\leq 2 \cdot \sqrt{N}$

Estimation via Collision: Getting the Estimator Unbiased

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One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[T] = |S|$ for any finite, non-empty set S.

Answer

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Estimation via Collision: Getting the Estimator Unbiased

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One can define $T(i)$, $i \in \mathbb{N}$, such that $\mathbf{E}[T] = |S|$ for any finite, non-empty set S.

Answer

- We outline a construction by induction.
- Case $|S| = 1$: Algo always stops after $i = 2$ rounds and returns $T(2)$. We want

$$1 = \mathbf{E}[T] = T(2) \qquad \Rightarrow \qquad T(2) = 1.$$

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