

Introduction to Probability

Lecture 10: Estimators (Part I)

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Outline

Introduction

Defining and Analysing Estimators

More Examples

Setting: We can take **random samples** in the form of **i.i.d. random variables** X_1, X_2, \dots, X_n from an **unknown distribution**.


- Taking enough samples allows us to estimate the **mean** (WLLN, CLT)
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Measurement = Quantity of Interest + Measurement Error

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Empirical Distribution Functions

Definition of Empirical Distribution Function (Empirical CDF)

Let X_1, X_2, \dots, X_n be i.i.d. samples, and F be the corresponding distribution function. For any $a \in \mathbb{R}$, define

$$F_n(a) := \frac{\text{number of } X_i \in (-\infty, a]}{n}.$$

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Thus by taking enough samples, we can estimate the entire distribution (including its expectation and variance).

Empirical Distribution Functions (Example 1/2)

Example 1

Consider throwing an unbiased dice 8 times, and let the **realisation** be:

$$(x_1, x_2, \dots, x_8) = (4, 1, 4, 3, 1, 6, 4, 1).$$

What is the Empirical Distribution Function $F_8(a)$?

Answer

Empirical Distribution Functions (Example 1/2)

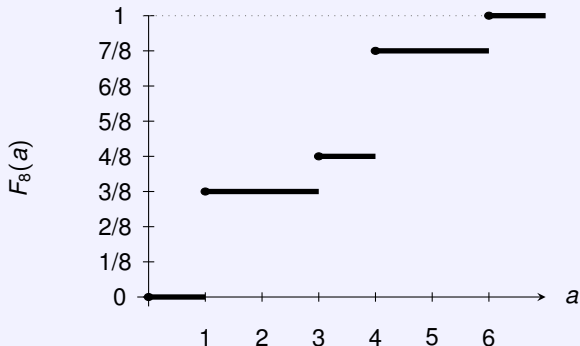
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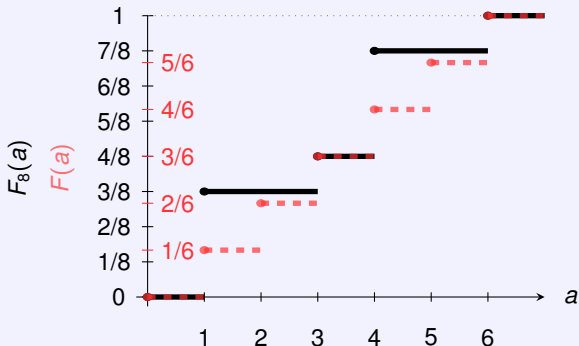
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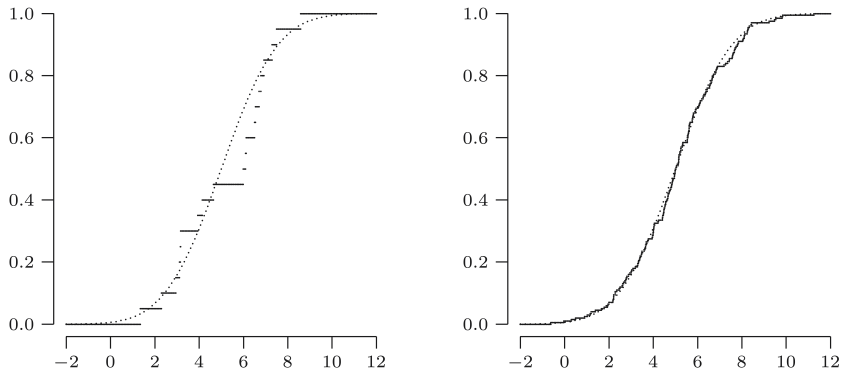
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Empirical Distribution Functions (Example 2/2)



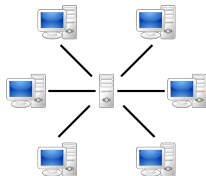
Source: Modern Introduction to Statistics

Figure: Empirical Distribution Functions of samples from a Normal Distribution $\mathcal{N}(5, 4)$ ($n = 20$ left, $n = 200$ right)

An Example of an Estimation Problem

Scenario

Consider the packages arriving at a network server.



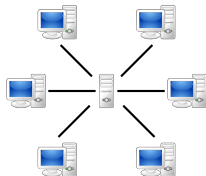
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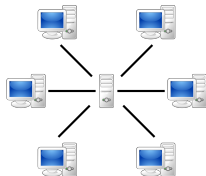
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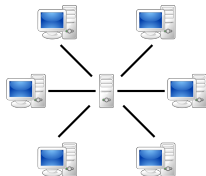
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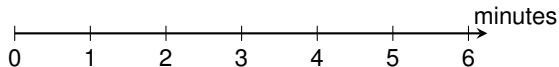
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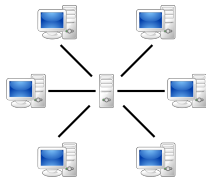


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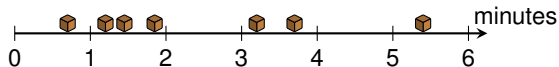
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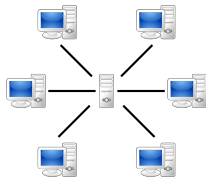


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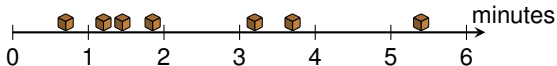
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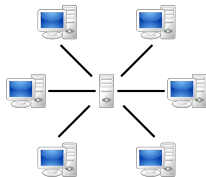


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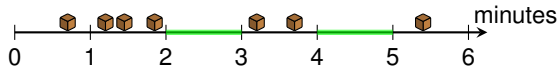
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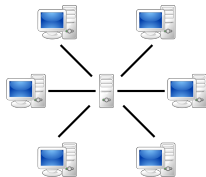


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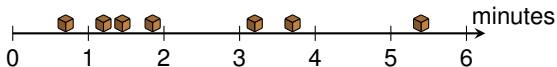
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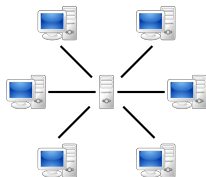


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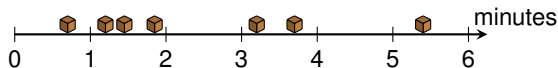
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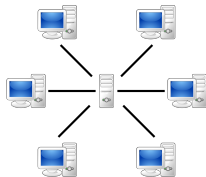


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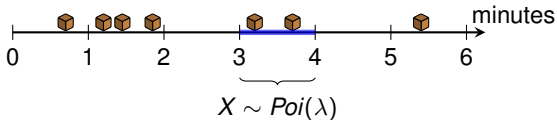
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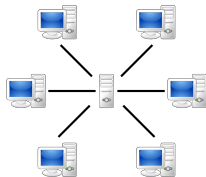


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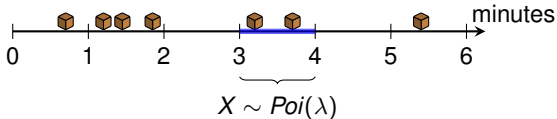
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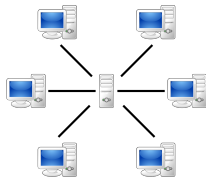
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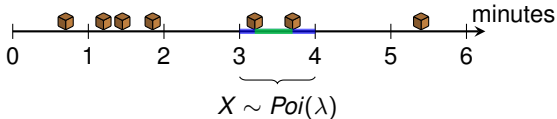
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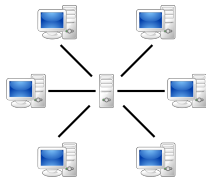
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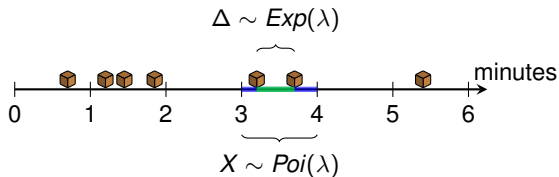
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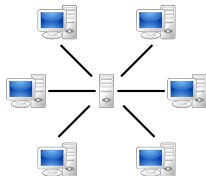
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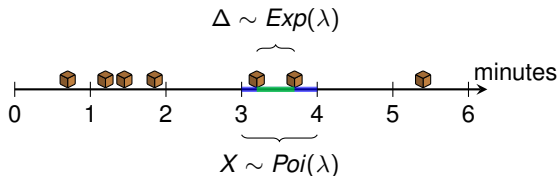
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Waiting Time (Lecture 5, Slide 22)



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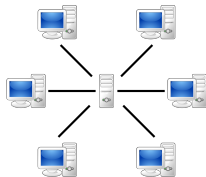
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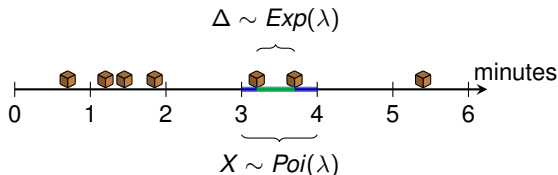
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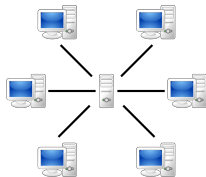
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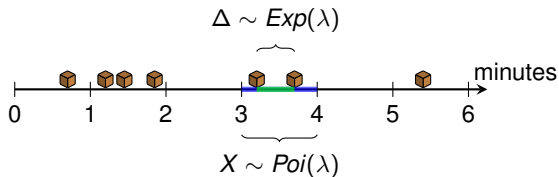
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Estimator for $e^{-\lambda}$

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Questions:

- What makes an **estimator** suitable? **unbiased** (later: **mean squared error**)
- Does an **unbiased estimator** always exist? How to compute it?
- If there are several **unbiased** estimators, which one to choose?

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More Examples

Example: Arrival of Packets (1/3)

- **Samples:** Given X_1, X_2, \dots, X_n i.i.d., $X_i \sim \text{Pois}(\lambda)$
- **Meaning:** X_i is the number of packets arriving in minute i



Example 2

Estimate λ by using the sample mean \bar{X}_n .

Answer

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Applying the **Weak Law of Large Numbers**:

$$\lim_{n \rightarrow \infty} \mathbf{P} \left[\left| \bar{X}_n - \lambda \right| > \epsilon \right] = 0 \quad \text{for any } \epsilon > 0.$$

Example: Arrival of Packets (2/3)

Example 3a

Define an estimator h_1 for the probability of zero arrivals, $e^{-\lambda}$.

_____ Answer _____

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and thus we can define an estimator by

$$h_1(X_1, X_2, \dots, X_n) := \frac{Y_1 + Y_2 + \dots + Y_n}{n} = \bar{Y}_n.$$

Example: Arrival of Packets (3/3)

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- Then $(y_1, y_2, y_3) = (0, 0, 1)$, and $h_1(x_1, x_2, x_3) = \frac{1}{3}$
- This seems **too large!** Also note that for the samples $(x_1, x_2, x_3) = (1, 1, 0)$, our estimator would give the same estimate

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This suggests to estimate $e^{-\lambda}$ by $e^{-\bar{X}_n}$. Hence our estimator is

$$h_2(X_1, X_2, \dots, X_n) := e^{-\bar{X}_n}.$$

- Suppose we have $n = 30$ and we want to estimate $e^{-\lambda}$
- Consider the **two estimators** $h_1(X_1, \dots, X_n)$ and $h_2(X_1, \dots, X_n)$.

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For most values of λ , both estimators will never return the **exact** value of $e^{-\lambda}$ on the basis of 30 observations.

Simulation of the two Estimators

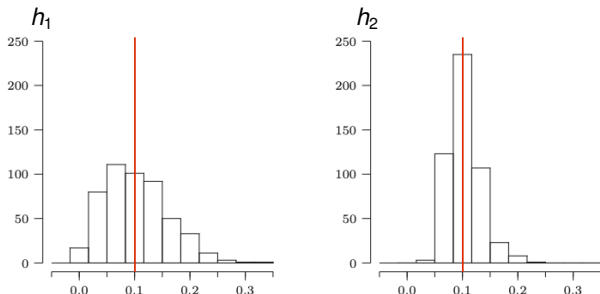
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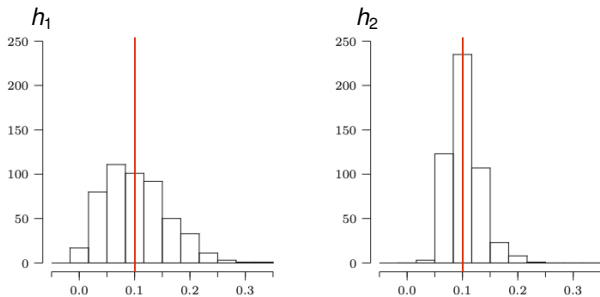
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Source: Modern Introduction to Statistics

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Both estimators concentrate around the true value 0.1, but the second estimator appears to be more concentrated.

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Unbiased Estimators and Bias

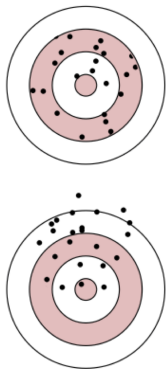
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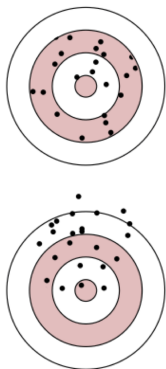
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Which of the two estimators h_1, h_2 are unbiased?



Example 4a

Is $h_1(X_1, X_2, \dots, X_n) = \frac{Y_1 + Y_2 + \dots + Y_n}{n}$ an unbiased estimator for $e^{-\lambda}$?

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Bias of the Second Estimator (and Jensen's Inequality)

Example 4b

Is $h_2(X_1, X_2, \dots, X_n) = e^{-\bar{X}_n}$ an unbiased estimator for $e^{-\lambda}$?

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Bias of the Second Estimator (and Jensen's Inequality)

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Jensen's Inequality

For any random variable X , and any **convex function** $g : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\mathbf{E}[g(X)] \geq g(\mathbf{E}[X]).$$

If g is **strictly convex** and X is not constant, then the inequality is strict.

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- This follows by **Jensen's inequality**, and the inequality is **strict** since $g : z \mapsto e^{-z}$ is **strictly convex** and \bar{X}_n is not constant.
- Thus $h_2(X_1, X_2, \dots, X_n)$ is not unbiased – it has **positive bias**.

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Asymptotic Bias of the Second Estimator (non-examinable)

Example 4c

$\mathbf{E}[h_2(X_1, \dots, X_n)] \xrightarrow{n \rightarrow \infty} e^{-\lambda}$ (hence it is **asymptotically unbiased**).

Answer

- Recall $h_2(X_1, \dots, X_n) = e^{-\bar{X}_n}$. For any $0 \leq k \leq n$,

$$\mathbf{P}\left[h_2(X_1, \dots, X_n) = e^{-k/n}\right] = \mathbf{P}\left[\sum_{i=1}^n X_i = k\right] = \mathbf{P}[Z = k],$$

where $Z \sim \text{Pois}(n \cdot \lambda)$ (since $\text{Pois}(\lambda_1) + \text{Pois}(\lambda_2) = \text{Pois}(\lambda_1 + \lambda_2)$)

$$\Rightarrow \mathbf{P}\left[h_2(X_1, \dots, X_n) = e^{-k/n}\right] = \frac{e^{-n\lambda} \cdot (n\lambda)^k}{k!}$$

$$\Rightarrow \mathbf{E}[h_2(X_1, \dots, X_n)] = \sum_{k=0}^{\infty} e^{-n\lambda} \cdot \frac{(n\lambda)^k}{k!} \cdot e^{-k/n}$$

By LOTUS

$$= e^{-n\lambda} \cdot e^{n\lambda e^{-1/n}} \sum_{k=0}^{\infty} e^{-n\lambda e^{-1/n}} \cdot \frac{(n\lambda e^{-1/n})^k}{k!}$$

$$= e^{-n\lambda \cdot (1 - e^{-1/n})} \cdot 1$$

since $e^x = 1 + x + O(x^2)$ for small x

$$\xrightarrow{n \rightarrow \infty} e^{-n\lambda \cdot (1 - 1/n + O(1/n^2))} = e^{-\lambda + O(\lambda/n)}.$$

Hence in the limit, the positive bias of h_2 diminishes.

Outline

Introduction

Defining and Analysing Estimators

More Examples

Unbiased Estimators for Expectation and Variance

Let X_1, X_2, \dots, X_n be **identically distributed** samples from a distribution with finite expectation μ and finite variance σ^2 .

- Then

$$\bar{X}_n := \frac{X_1 + X_2 + \dots + X_n}{n}$$

is an **unbiased** estimator for μ .

- Furthermore, for $n \geq 2$,

$$S_n = S_n(X_1, \dots, X_n) := \frac{1}{n-1} \cdot \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

is an **unbiased** estimator for σ^2 .

Example 5

We need to prove: $\mathbf{E}[S_n] = \sigma^2$.

Answer

Multiplying by $n - 1$ yields:

$$\begin{aligned}(n-1) \cdot S_n &= \sum_{i=1}^n (X_i - \bar{X}_n)^2 \\ &= \sum_{i=1}^n (X_i - \mu + \mu - \bar{X}_n)^2 \\ &= \sum_{i=1}^n (X_i - \mu)^2 + \sum_{i=1}^n (\bar{X}_n - \mu)^2 - 2 \sum_{i=1}^n (X_i - \mu) (\bar{X}_n - \mu) \\ &= \sum_{i=1}^n (X_i - \mu)^2 + n (\bar{X}_n - \mu)^2 - 2 (\bar{X}_n - \mu) \cdot n \cdot (\bar{X}_n - \mu) \\ &= \sum_{i=1}^n (X_i - \mu)^2 - n (\bar{X}_n - \mu)^2.\end{aligned}$$

Let us now take **expectations**:

By Lec. 8, Slide 21: $\mathbf{E}[(\bar{X}_n - \mu)^2] = \mathbf{V}[\bar{X}_n] = \sigma^2/n$

$$\begin{aligned}(n-1) \cdot \mathbf{E}[S_n] &= \sum_{i=1}^n \mathbf{E}[(X_i - \mu)^2] - n \cdot \mathbf{E}[(\bar{X}_n - \mu)^2] \\ &= n \cdot \sigma^2 - n \cdot \sigma^2/n \\ &= (n-1) \cdot \sigma^2.\end{aligned}$$

An Unbiased Estimator may not always exist

Example 6

Suppose that we have one sample $X \sim \text{Bin}(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is **no unbiased estimator** for $1/p$.

Answer

- First a simpler proof which exploits that p might be arbitrarily small
- **Intuition:** By making p smaller and smaller, we force $\max_{0 \leq k \leq n} T(k)$, $k \in \{0, 1, \dots, n\}$ to become bigger and bigger
- **Formal Argument:**
 - Fix any estimator $T(X)$
 - Define $M := \max_{0 \leq k \leq n} T(k)$. Then,

$$\begin{aligned} \mathbf{E}[T(X)] &= \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} \cdot T(k) \\ &\leq M \cdot \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} = M. \end{aligned}$$

- Hence this estimator does not work for $p < \frac{1}{M}$, since then $\mathbf{E}[T(X)] \leq M < \frac{1}{p}$ (negative bias!)
- The next proof will work even if $p \in [a, b]$ for $0 < a < b \leq 1$.

An Unbiased Estimator may not always exist (cntd. - non-examinable)

Example 6 (cntd.)

Suppose that we have one sample $X \sim \text{Bin}(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is **no unbiased estimator** for $1/p$.

Answer

- Suppose there exists an unbiased estimator with $\mathbf{E}[T(X)] = 1/p$.
- Then

$$\begin{aligned}1 &= p \cdot \mathbf{E}[T(X)] \\ &= p \cdot \sum_{k=0}^n \mathbf{P}[X = k] \cdot T(k) \\ &= p \cdot \sum_{k=0}^n \binom{n}{k} p^k \cdot (1-p)^{n-k} \cdot T(k)\end{aligned}$$

- Last term is a **polynomial of degree $n + 1$** with constant term zero
 $\Rightarrow p \cdot \mathbf{E}[T(X)] - 1$ is a **(non-zero) polynomial of degree $\leq n + 1$**
 \Rightarrow this polynomial has at most $n + 1$ roots
 $\Rightarrow \mathbf{E}[T(X)]$ can be equal to $1/p$ for at most $n + 1$ values of p , and thus cannot be an unbiased.