

# Introduction to Probability

Lecture 6: Marginals and Joint Distributions

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To this end, we will introduce:

- Joint/Marginal distribution of two (or more) variables
- Independence of two (or more) variables
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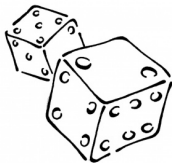
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For simplicity, we will mainly focus on **discrete** random variables.



## Warm-Up Exercise



### Example

Let  $X_1, X_2 \in \{1, 2, \dots, 6\}$  be two independent rolls of an unbiased die. Let  $S := X_1 + X_2$  and  $M := \max\{X_1, X_2\}$ . List the elements of the event  $\{S = 7, M \leq 5\}$  and deduce the probability.

\_\_\_\_\_ Answer \_\_\_\_\_

## Joint Probability

### Joint Probability Mass Function

The **joint probability mass function** of two **discrete** random variables  $X$  and  $Y$  is the function  $p : \mathbb{R}^2 \rightarrow [0, 1]$ , defined by:

$$p_{X,Y}(a, b) = \mathbf{P}[X = a, Y = b] \quad \text{for } -\infty < a, b < \infty.$$



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### Joint Distribution Function

The **joint distribution function** of two (**discrete or continuous**) random variables  $X$  and  $Y$  is the function  $F : \mathbb{R}^2 \rightarrow [0, 1]$ , defined by:

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### Marginal Distribution

Given a joint distribution  $F_{X,Y}$  of two random variables  $X, Y$ , one obtains the **marginal distribution** of  $X$  for any  $a$  as follows:

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Joint Distribution contains (much) more information than the two marginals!



## Discrete Example 1

### Example

Let  $X_1, X_2 \in \{1, 2, \dots, 6\}$  be independent rolls of an unbiased die. Let  $S := X_1 + X_2$  and  $M := \max\{X_1, X_2\}$ . Compute the **joint probability mass function**  $p$  of  $S$  and  $M$  and the **marginal distributions** of  $S$  and  $M$ .

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Answer



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Answer

$a$	$b$					
	1	2	3	4	5	6
2	1/36	0	0	0	0	0
3	0	2/36	0	0	0	0
4	0	1/36	2/36	0	0	0
5	0	0	2/36	2/36	0	0
6	0	0	1/36	2/36	2/36	0
7	0	0	0	2/36	2/36	2/36
8	0	0	0	1/36	2/36	2/36
9	0	0	0	0	2/36	2/36
10	0	0	0	0	1/36	2/36
11	0	0	0	0	0	2/36
12	0	0	0	0	0	1/36



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Answer

$a$	$b$						$p_S(a)$
	1	2	3	4	5	6	
2	1/36	0	0	0	0	0	1/36
3	0	2/36	0	0	0	0	2/36
4	0	1/36	2/36	0	0	0	3/36
5	0	0	2/36	2/36	0	0	4/36
6	0	0	1/36	2/36	2/36	0	5/36
7	0	0	0	2/36	2/36	2/36	6/36
8	0	0	0	1/36	2/36	2/36	5/36
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10	0	0	0	0	1/36	2/36	3/36
11	0	0	0	0	0	2/36	2/36
12	0	0	0	0	0	1/36	1/36
$p_M(b)$	1/36	3/36	5/36	7/36	9/36	11/36	1



## Discrete Example 2

### Example

Suppose an urn contains balls numbered  $1, 2, \dots, N$ . We draw  $1 \leq n \leq N$  balls uniformly and **without replacement** from the urn. Let  $X_i \in \{1, 2, \dots, N\}$  be the number of the ball drawn in the  $i$ -th step. What is the marginal distribution of  $X_i$ ?

Answer



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Answer

We first compute the **joint distribution**. For distinct  $a_1, a_2, \dots, a_n$ ,

Fix  $i$  and consider the **marginal distribution** of  $X_i$ :



### Definition

Random variables  $X$  and  $Y$  have a **joint continuous distribution** if for some function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  and for all numbers  $a_1 \leq b_1$  and  $a_2 \leq b_2$ ,

$$\mathbf{P}[a_1 \leq X \leq b_1, a_2 \leq Y \leq b_2] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y) dx dy.$$

The function  $f$  has to satisfy  $f(x, y) \geq 0$  for all  $x$  and  $y$ , and  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$ . We call  $f$  the **joint probability density**.



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As in one-dimensional case we switch from  $F$  to  $f$  by **differentiating** (or **integrating**):

$$F(a, b) = \int_{-\infty}^a \int_{-\infty}^b f(x, y) dx dy \quad \text{and} \quad f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$



## Example of a Joint Distribution of Continuous Random Variables

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- Consider the density:

$$f(x, y) = \frac{30}{\pi} \cdot e^{-50x^2 - 50y^2 + 80xy},$$

where  $-\infty < x, y < \infty$ .



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- This is an example of a so-called **bivariate normal probability density function**.



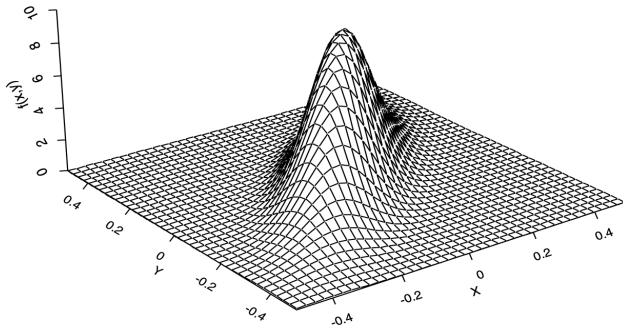
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Source: Modern Introduction to Statistics



## Dealing with Continuous Variables

### Example (1/2)

Suppose that the joint probability density of  $X$  and  $Y$  is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y} & \text{for } 0 < x < \infty, 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Compute (i)  $\mathbf{P}[X > 1, Y < 1]$  and (ii)  $\mathbf{P}[X < Y]$ .

Answer \_\_\_\_\_

(i) We first compute:

$$\mathbf{P}[X > 1, Y < 1] = \int_0^1 \int_1^{\infty} 2e^{-x}e^{-2y} dx dy$$



## Dealing with Continuous Variables (cont.)

### Example (2/2)

Suppose that the joint probability density of  $X$  and  $Y$  is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y} & \text{for } 0 < x < \infty, 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Compute (i)  $\mathbf{P}[X > 1, Y < 1]$  and (ii)  $\mathbf{P}[X < Y]$ .

Answer

(ii) We have:

$$\begin{aligned} \mathbf{P}[X < Y] &= \int_0^{\infty} \int_0^y 2e^{-x}e^{-2y} dx dy \\ &= \frac{1}{3}. \end{aligned}$$

