

Introduction to Probability

Lecture 6: Marginals and Joint Distributions

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Experiments often involve **several** random variables, and some of them may **influence** each other.

To this end, we will introduce:

- Joint/Marginal distribution of two (or more) variables
- Independence of two (or more) variables
- Covariance of two variables

For simplicity, we will mainly focus on **discrete** random variables.



Warm-Up Exercise



Example

Let $X_1, X_2 \in \{1, 2, \dots, 6\}$ be two independent rolls of an unbiased die. Let $S := X_1 + X_2$ and $M := \max\{X_1, X_2\}$. List the elements of the event $\{S = 7, M \leq 5\}$ and deduce the probability.

Answer

The elements are $\{(2, 5), (3, 4), (4, 3), (5, 2)\}$. Since each of these elements has a probability of $1/6 \cdot 1/6 = 1/36$, the sought probability is $4/36 = 1/9$.



Joint Probability

Joint Probability Mass Function

The **joint probability mass function** of two **discrete** random variables X and Y is the function $p : \mathbb{R}^2 \rightarrow [0, 1]$, defined by:

$$p_{X,Y}(a, b) = \mathbf{P}[X = a, Y = b] \quad \text{for } -\infty < a, b < \infty.$$

Joint Distribution Function

The **joint distribution function** of two (**discrete or continuous**) random variables X and Y is the function $F : \mathbb{R}^2 \rightarrow [0, 1]$, defined by:

$$F_{X,Y}(a, b) = \mathbf{P}[X \leq a, Y \leq b] \quad \text{for } -\infty < a, b < \infty.$$

Marginal Distribution

Given a joint distribution $F_{X,Y}$ of two random variables X, Y , one obtains the **marginal distribution** of X for any a as follows:

$$F_X(a) = \mathbf{P}[X \leq a] = \lim_{b \rightarrow \infty} F_{X,Y}(a, b).$$

Joint Distribution contains (much) more information than the two marginals!



Discrete Example 1

Example

Let $X_1, X_2 \in \{1, 2, \dots, 6\}$ be independent rolls of an unbiased die. Let $S := X_1 + X_2$ and $M := \max\{X_1, X_2\}$. Compute the **joint probability mass function** p of S and M and the **marginal distributions** of S and M .

Answer

a	b						$p_S(a)$
	1	2	3	4	5	6	
2	1/36	0	0	0	0	0	1/36
3	0	2/36	0	0	0	0	2/36
4	0	1/36	2/36	0	0	0	3/36
5	0	0	2/36	2/36	0	0	4/36
6	0	0	1/36	2/36	2/36	0	5/36
7	0	0	0	2/36	2/36	2/36	6/36
8	0	0	0	1/36	2/36	2/36	5/36
9	0	0	0	0	2/36	2/36	4/36
10	0	0	0	0	1/36	2/36	3/36
11	0	0	0	0	0	2/36	2/36
12	0	0	0	0	0	1/36	1/36
$p_M(b)$	1/36	3/36	5/36	7/36	9/36	11/36	1

$(X_1, X_2) = (4, 3)$ or
 $(X_1, X_2) = (3, 4)$

$$p_{S,M}(7, 4) = \frac{1}{36} + \frac{1}{36}$$

$\{S = 8\} =$
 $\{S = 8, M = 4\}$
 $\cup \{S = 8, M = 5\}$
 $\cup \{S = 8, M = 6\}$

$$p_S(8) = \frac{1}{36} + \frac{2}{36} + \frac{2}{36}$$



Discrete Example 2

Example

Suppose an urn contains balls numbered $1, 2, \dots, N$. We draw $1 \leq n \leq N$ balls uniformly and **without replacement** from the urn. Let $X_i \in \{1, 2, \dots, N\}$ be the number of the ball drawn in the i -th step. What is the marginal distribution of X_i ?

Answer

We first compute the **joint distribution**. For distinct a_1, a_2, \dots, a_n ,

$$\begin{aligned} p(a_1, a_2, \dots, a_n) &= \mathbf{P}[X_1 = a_1, X_2 = a_2, \dots, X_n = a_n] \\ &= \frac{1}{N(N-1) \cdots (N-n+1)}. \end{aligned}$$

Fix i and consider the **marginal distribution** of X_i :

$$\begin{aligned} p_{X_i}(k) &= \sum_{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n} p(a_1, \dots, a_{i-1}, k, a_{i+1}, \dots, a_n) \\ &= (N-1)(N-2) \cdots (N-n+1) \cdot \frac{1}{N(N-1) \cdots (N-n+1)} \end{aligned}$$

The X_i 's are **not** independent, yet their marginals are **identical**!

Same argument applies to the **hypergeometric distribution**, with balls of two different colours.



Joint Distributions of Continuous Variables

Definition

Random variables X and Y have a **joint continuous distribution** if for some function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and for all numbers $a_1 \leq b_1$ and $a_2 \leq b_2$,

$$\mathbf{P}[a_1 \leq X \leq b_1, a_2 \leq Y \leq b_2] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y) dx dy.$$

The function f has to satisfy $f(x, y) \geq 0$ for all x and y , and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$. We call f the **joint probability density**.

As in one-dimensional case we switch from F to f by **differentiating** (or **integrating**):

$$F(a, b) = \int_{-\infty}^a \int_{-\infty}^b f(x, y) dx dy \quad \text{and} \quad f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$



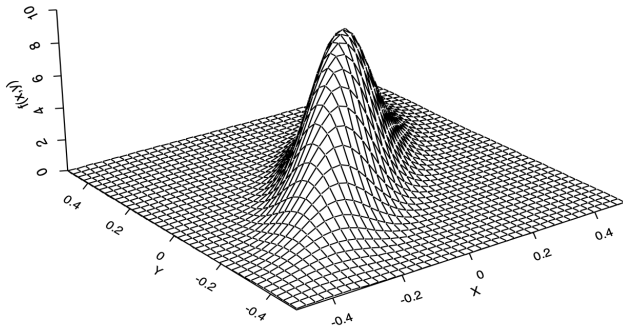
Example of a Joint Distribution of Continuous Random Variables

- Consider the density:

$$f(x, y) = \frac{30}{\pi} \cdot e^{-50x^2 - 50y^2 + 80xy},$$

where $-\infty < x, y < \infty$.

- This is an example of a so-called **bivariate normal probability density function**.



Source: Modern Introduction to Statistics



Dealing with Continuous Variables

Example (1/2)

Suppose that the joint probability density of X and Y is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y} & \text{for } 0 < x < \infty, 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Compute (i) $\mathbf{P}[X > 1, Y < 1]$ and (ii) $\mathbf{P}[X < Y]$.

Answer _____

(i) We first compute:

$$\begin{aligned} \mathbf{P}[X > 1, Y < 1] &= \int_0^1 \int_1^\infty 2e^{-x}e^{-2y} dx dy \\ &= \int_0^1 2e^{-2y}(-e^{-x}) \Big|_1^\infty dy = \int_0^1 2e^{-2y}(0 - (-e^{-1})) dy \\ &= e^{-1} \int_0^1 2e^{-2y} dy = e^{-1} \cdot (1 - e^{-2}). \end{aligned}$$



Dealing with Continuous Variables (cont.)

Example (2/2)

Suppose that the joint probability density of X and Y is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y} & \text{for } 0 < x < \infty, 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Compute (i) $\mathbf{P}[X > 1, Y < 1]$ and (ii) $\mathbf{P}[X < Y]$.

Answer

(ii) We have:

$$\begin{aligned} \mathbf{P}[X < Y] &= \int_0^{\infty} \int_0^y 2e^{-x}e^{-2y} dx dy \\ &= \int_0^{\infty} 2e^{-2y}(1 - e^{-y}) dy \\ &= \int_0^{\infty} 2e^{-2y} dy - \int_0^{\infty} 2e^{-3y} dy \\ &= 1 - \frac{2}{3} = \frac{1}{3}. \end{aligned}$$

