# Introduction to Probability

Lecture 4: More discrete distributions – Poisson, Geometric, Negative Binomial, Hypergeometric

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# Outline

Poisson discrete random variable

Geometric discrete random variable

Negative binomial discrete random variable

Hypergeometric discrete random variable

# Preliminaries:

## The natural exponent *e*

*e* is a mathematical constant AKA the Euler number. *e* is very important for exponential functions. Here are some important identities:

$$e \approx 2.71828$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$e^{-\lambda} = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n$$

$$e^r = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n$$

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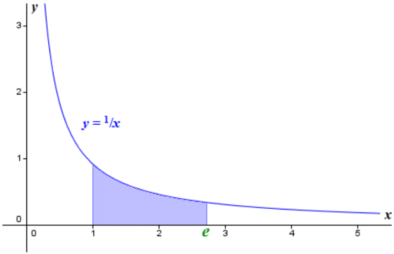

## Binomial RV example: large $n$, small $p$

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What if 2 people enter in the same minute?

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2. Break an hour into **milliseconds**.
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3. Break an hour into **infinitely small units**.
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as $n \to \infty$ $\quad \dfrac{n(n-1)\cdots(n-k+1)}{n^k} \approx \dfrac{n^k}{n^k} = 1$

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Therefore, in our store footfall example: the probability of $k$ people entering the store in the next 1 hour is:

$$\mathbf{P}[X=k] = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!}e^{-\lambda}$$

# Poisson

## Poisson discrete random variable

A Poisson RV $X$ approximates Binomial where $n$ is large, $p$ is small, and $\lambda = np$ is "moderate". Thus we no longer need to know $n$ and $p$, we only need to provide **rate** $\lambda$. $X$ is the number of successes over the duration of the experiment.

$$\mathbf{X} \sim \mathbf{Pois}(\lambda)$$

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\text{Range:}\quad & \{0, 1, 2, \ldots\} \\
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Key idea: Divide time into a **large number** of small increments. Assume that during each increment, there is some **small probability** of the event happening (independent of other increments).

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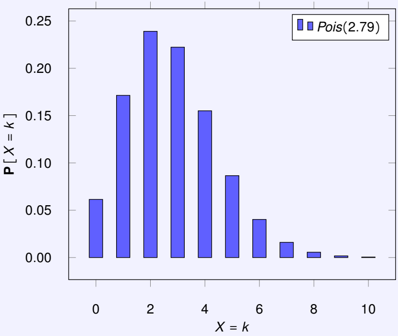

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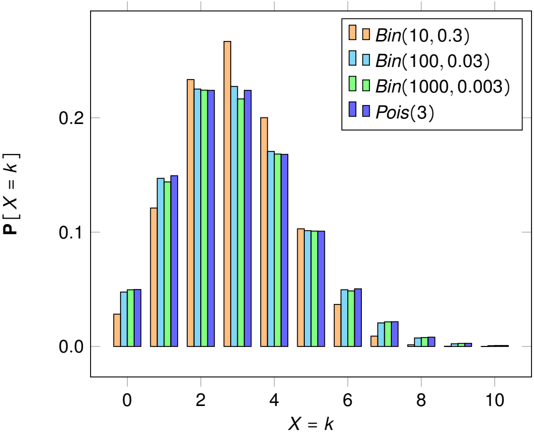

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$$\mathbf{E}\left[X^k\right] = \lambda \mathbf{E}\left[(X+1)^{k-1}\right] \quad \longleftarrow \text{ useful generalisation}$$

# Bernoulli, Poisson, and random processes

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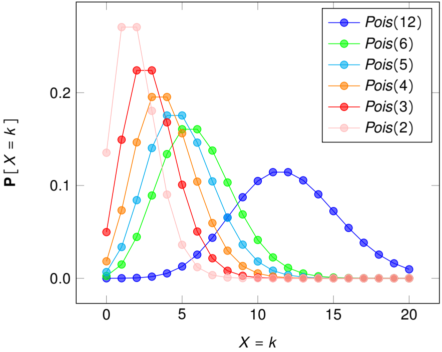

# Outline

Poisson discrete random variable

Geometric discrete random variable

Negative binomial discrete random variable

Hypergeometric discrete random variable

# Geometric

**Geometric discrete random variable**

$X$ is a geometric RV if $X$ is a number of independent Bernoulli trials until the first success, and $p$ is the probability of success on each Bernoulli trial.

$$\mathbf{X \sim Geo(p)}$$

$$\begin{aligned} \text{Range:} \quad & \{1, 2, \ldots\} \\ \text{PMF:} \quad & \mathbf{P}[\, X = n \,] = (1 - p)^{n-1} p \\ \text{Expectation:} \quad & \mathbf{E}[\, X \,] = \frac{1}{p} \\ \text{Variance:} \quad & \mathbf{V}[\, X \,] = \frac{1 - p}{p^2} \end{aligned}$$

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Examples: tossing a coin ($\mathbf{P}[\, head \,] = p$) until first heads appears, generating bits with $\mathbf{P}[\, bit = 1 \,] = p$ until first 1 is generated.

# Geometric RV properties

PMF ($E_i$ is the event that the $i$-th trial succeeds):

$$\mathbf{P}[X = n] = \mathbf{P}\left[E_1^c E_2^c \dots E_{n-1}^c E_n\right] =$$

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# Die example

Example

You roll a fair 6-sided die until it comes up with #6. What is the probability that it will take 3 rolls?

Answer

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Answer

Let $X$ be a RV for # of rolls. Probability for any # on die is $\frac{1}{6}$.

Define RVs: $X \sim Geo(\frac{1}{6})$, want $\mathbf{P}[\, X = 3\,]$.

Solve:

# Outline

# Negative binomial

**Negative binomial discrete random variable**

$X$ is a negative binomial RV if $X$ is the number of independent Bernoulli trials until $r$ successes and $p$ is the probability of success on each trial.

$$\mathbf{X} \sim \mathbf{NegBin}(\mathbf{r}, \mathbf{p})$$

Range: $\{r, r+1, \ldots\}$

PMF: $\mathbf{P}[X = n] = \binom{n-1}{r-1}(1-p)^{n-r}p^r$

Expectation: $\mathbf{E}[X] = \frac{r}{p}$

Variance: $\mathbf{V}[X] = \frac{r(1-p)}{p^2}$

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\end{aligned}$$

Examples: tossing a coin until $r$-th heads appears, generating bits until the first $r$ 1's are generated.

**Note:** $Geo(p) = NegBin(1, p)$.

# NegBin example

**Example (not real life!)**

A PhD student is expected to publish 2 papers to graduate. A conference accepts each paper randomly and independently with probability $p = 0.25$. On average, how many papers will the student need to submit to a conference in order to graduate?

Answer

# Adding NegBin example

Example

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of $Z$.

Answer

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- Need to show that $Z \sim NegBin(m + n, p)$.

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Answer

- Need to show that $Z \sim NegBin(m + n, p)$.
- Consider the sequence of independent events tossing a coin with $\mathbf{P}[\, heads \,] = p$.

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Answer

- Need to show that $Z \sim NegBin(m + n, p)$.
- Consider the sequence of independent events tossing a coin with $\mathbf{P}[\,heads\,] = p$.
- Let $X$ be a RV for # of coin tosses until $m$ heads are observed. Thus $X \sim NegBin(m, p)$.

## Adding NegBin example

**Example**

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of $Z$.

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- Now, continue to toss a coin after $m$ heads are observed, until $n$ more heads are observed. Thus, for this part of the sequence, $Y \sim NegBin(n, p)$.

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- Looking at it from the beginning we tossed independently the coin until we observed $m + n$ heads, thus $Z = X + Y$ and thus $Z \sim NegBin(m + n, p)$.

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- Now, continue to toss a coin after $m$ heads are observed, until $n$ more heads are observed. Thus, for this part of the sequence, $Y \sim NegBin(n, p)$.
- Looking at it from the beginning we tossed independently the coin until we observed $m + n$ heads, thus $Z = X + Y$ and thus $Z \sim NegBin(m + n, p)$.
- Note: if $X_1, X_2, \ldots X_m$ are $m$ independent $Geo(p)$ RVs, then the RV $X = X_1 + X_2 + \cdots + X_m$ has $NegBin(m, p)$ distribution.

# Outline

Poisson discrete random variable

Geometric discrete random variable

Negative binomial discrete random variable

Hypergeometric discrete random variable

# Hypergeometric

**Hypergeometric discrete random variable**

$X$ is a hypergeometric RV that samples $n$ objects, without replacement, with $i$ successes (random draw for which the object drawn has a specified feature), from a finite population of size $N$ that contains exactly $m$ objects with that feature.

$$\mathbf{X \sim Hyp(N, n, m)}$$

Range: $\{0, 1, \ldots, n\}$

PMF: $\mathbf{P}[\, X = i \,] = \dfrac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$

Expectation: $\mathbf{E}[\, X \,] = n\dfrac{m}{N}$

Variance: $\mathbf{V}[\, X \,] = n\dfrac{m}{N}\left(1 - \dfrac{m}{N}\right)\left(1 - \dfrac{n-1}{N-1}\right)$

Example: an urn has $N$ balls of which $m$ are white and $N - m$ are black; we take a random sample **without replacement** of size $n$ and measure $X$: # of white balls in the sample.

# Survey sampling

**Example**

A street has 40 houses of which 5 houses are inhabited by families with an income below the poverty line. In a survey, 7 houses are sampled at random from this street. What is the probability that: (a) none of the 5 families with income below poverty line are sampled? (b) 4 of them are sampled? (c) no more than 2 are sampled? (d) at least 3 are sampled?

Answer

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**Answer**

Let $X$: # of families sampled which are below the poverty line.

$X \sim Hyp(N = 40, n = 7, m = 5)$.

# Summary of discrete RV

| | $Ber(p)$ | $Bin(n,p)$ | $Pois(\lambda)$ | $Geo(p)$ | $NegBin(r,p)$ | $Hyp(N,n,m)$ |
|---|---|---|---|---|---|---|
| PMF | $\mathbf{P}[X=1]=p$ | $\mathbf{P}[X=k]=\binom{n}{k}p^k(1-p)^{n-k}$ | $\mathbf{P}[X=k]=\frac{\lambda^k}{k!}e^{-\lambda}$ | $\mathbf{P}[X=n]=(1-p)^{n-1}p$ | $\mathbf{P}[X=n]=\binom{n-1}{r-1}(1-p)^{n-r}p^r$ | $\mathbf{P}[X=i]=\frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$ |
| $\mathbf{E}[X]$ | $p$ | $np$ | $\lambda$ | $\frac{1}{p}$ | $\frac{r}{p}$ | $n\frac{m}{N}$ |
| $\mathbf{V}[X]$ | $p(1-p)$ | $np(1-p)$ | $\lambda$ | $\frac{1-p}{p^2}$ | $\frac{r(1-p)}{p^2}$ | $n\frac{m}{N}\left(1-\frac{m}{N}\right)\left(1-\frac{n-1}{N-1}\right)$ |
| Descr. | 1 experiment with prob $p$ of success | $n$ independent trials with prob $p$ of success | # successes over experiment duration, $\lambda = np$ rate of success | # independent trials until first success | # independent trials until $r$ successes | # successes of drawing item with a feature (without replacement) in a sample of size $n$ from a population of size $N$ with $m$ items with the feature |