

# Introduction to Probability

Lecture 2: Random variables, probability mass function, expectation

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# Outline

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Random variable

Probability mass function

Cumulative distribution function

Expectation



## What is a random variable?

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Random variable

A random variable  $X$  is a function from the sample space to the real numbers.

- We can interpret  $X$  as a quantity whose value depends on the outcome of an experiment (some probabilistic process).
  - Roll two dice,  $X$ : sum of dice
  - Toss 3 coins,  $X$ : number of heads
  - Give a student a test,  $X$ : score
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- Many different types of RV: indicator, binary, choice, Bernoulli, etc.
- Random variable can be **discrete** or continuous:
  - $X$  has finitely many possible values: discrete.
  - $X$  has every integer as a possible value: discrete.
  - $X$  amount of time it takes to finish a race: continuous (possible value:  $\{t : 0 \leq t < \infty\} = [0, \infty)$ ).



## Examples of random variables

### Example

We toss 3 fair coins. Let a **random variable**  $X$  be the total number of heads on the 3 coins. What are the probabilities of  $X$  taking on the following values:  $X = 0$ ,  $X = 1$ ,  $X = 2$ ,  $X = 3$ ,  $X \geq 4$ ?

Answer



## Random variables are NOT events

random variables  $\neq$  events

### Tossing 3 fair coins example

$X = x$	$\mathbf{P}[X = x]$	Set of outcomes	Possible event $E$
$X = 0$	$\frac{1}{8}$	$\{(T, T, T)\}$	Toss 0 heads
$X = 1$	$\frac{3}{8}$	$\{(H, T, T), (T, H, T), (T, T, H)\}$	Toss exactly 1 head
$X = 2$	$\frac{3}{8}$	$\{(H, H, T), (T, H, H), (H, T, H)\}$	Event where $X = 2$ Toss exactly 2 heads
$X = 3$	$\frac{1}{8}$	$\{(H, H, H)\}$	Toss 0 tails
$X \geq 4$	0	$\{\}$	Toss 4 or more heads

We can define events by condition of the value of a random variable (RV takes on values that satisfy a numerical test).



### Example

Tossing a coin has the probability  $p$  that it comes up heads. Toss a coin 5 times. Let  $X$ : the number of heads in 5 tosses. What is the **range** of  $X$  (i.e., what are the values that  $X$  can take on with non-zero probability)? What is  $\mathbf{P}[X = k]$  where  $k$  is in the range of  $X$ ?

Answer

- Notice that each coin toss is an independent trial.

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## Probability mass function definition (PMF)

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Discrete random variable

A random variable  $X$  is **discrete** if its range has countably many values

$$X = x \text{ where } x \in \{x_1, x_2, x_3, \dots\}$$



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The probability mass function (**PMF**) of a discrete random variable  $X$  is a function  $p(a)$  of  $X$  that maps possible outcomes of a random variable to the corresponding probabilities:

$$p(a) = \mathbf{P}[X = a] = p_X(a)$$



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Recall that probabilities must sum to 1:  $\sum_{i=1}^{\infty} p(a_i) = 1$ .



## Example for a single die

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- Let  $X$  be a RV representing a single die roll.
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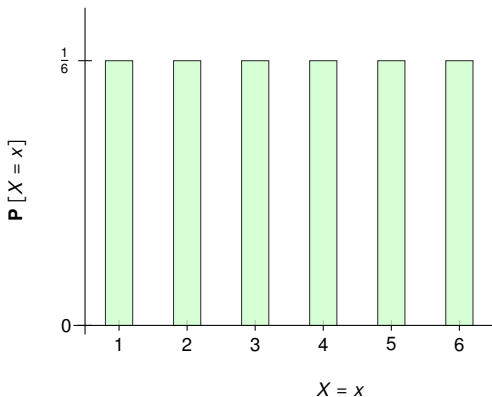
$$p(x) = \mathbf{P}[X = x] = \begin{cases} \frac{1}{6} & x \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases}$$



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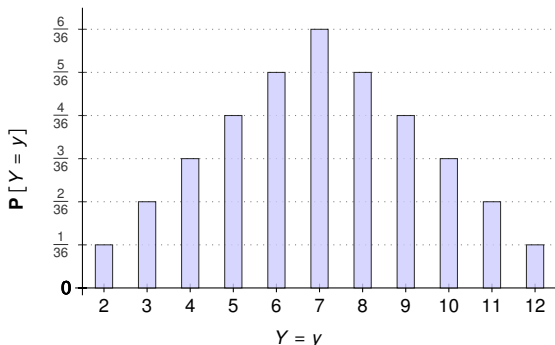


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$$\sum_{i=1}^{\infty} p(a_i) = \sum_{i=1}^{\infty} \mathbf{P}[X = a_i] = \mathbf{P}\left[\bigcup_{i=1}^{\infty} \{X = a_i\}\right] = \mathbf{P}[S] = 1$$



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4. Notice that everything to do with discrete RVs is expressed in terms of (finite or infinite) sum.
5. For continuous RVs, these sums are replaced by integrals.



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## Cumulative distribution function definition (CDF)

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Another useful way to analyse probabilities.

### Cumulative distribution function

The cumulative distribution function (CDF) of a random variable  $X$  is defined as

$$F(a) = F_X(a) = \mathbf{P}[X \leq a] \text{ where } -\infty < a < \infty$$

For a **discrete** random variable  $X$ , the CDF is

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Note that for a discrete RV the CDF is a step function, i.e., the value of  $F$  is constant in the intervals  $(x_{i-1}, x_i)$  and then takes a step of size  $p(x_i)$  at  $x_i$ .



## Example

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- Let the PMF for  $X$  be given by  $p(1) = \frac{1}{4}, p(2) = \frac{1}{2}, p(3) = \frac{1}{8}, p(4) = \frac{1}{8}$ .



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- Then CDF is:

$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \leq a < 2 \\ \frac{3}{4} & 2 \leq a < 3 \\ \frac{7}{8} & 3 \leq a < 4 \\ 1 & 4 \leq a \end{cases}$$

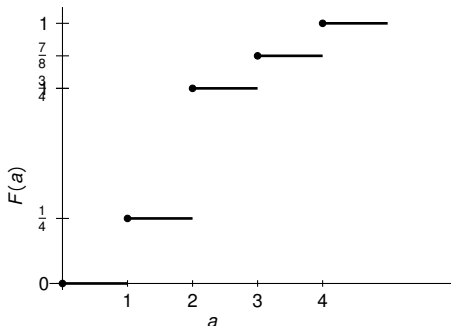


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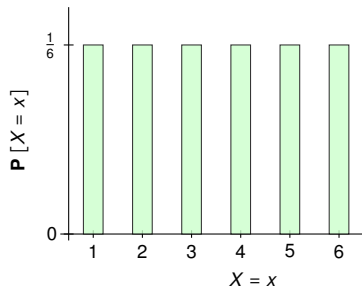
- Graphical depiction of function:



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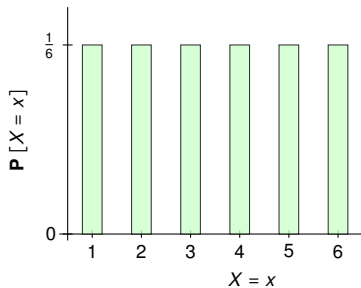
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PMF of  $X$

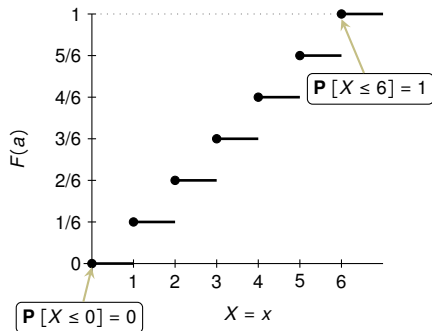


## Example for a single die

PMF of  $X$



CDF of  $X$



## Properties of CDF

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1.  $0 \leq F(x) \leq 1$  for all  $x$
2.  $\lim_{x \rightarrow -\infty} F(x) = 0$
3.  $\lim_{x \rightarrow \infty} F(x) = 1$
4.  $F(x)$  is a non-decreasing function of  $x$  (if  $x_1 < x_2$  then  $F(x_1) \leq F(x_2)$ )



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### Expectation

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$$\mathbf{E}[X] = \sum_{x:\mathbf{P}[x]>0} x\mathbf{P}[x]$$

- Expectation is the average value of the random variable over many repetitions of the experiment it represents.
- It is the sum over all values of  $X = x$  that have non-zero probability.
- AKA: mean, expected value, weighted average, centre of mass, first moment.



## Example of a die roll

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What is the expected value of a 6-sided die roll (i.e., what is the average value of a die roll)?

1. Define random variables:

$X =$  RV for value of roll

$$\mathbf{P}[X = x] = \begin{cases} \frac{1}{6} & x \in \{1, \dots, 6\} \\ 0 & \text{otherwise} \end{cases}$$

2. Solve:



## Example of school classes

### Example

A school has 3 classes with 5, 10 and 150 students. What is the average class size?

Answer

**Interpretation 1:** Randomly choose a class with equal probability. Thus,  $X$  = size of chosen class

**Interpretation 2:** Randomly choose a student with equal probability. Thus,  $Y$  = size of chosen class

This is a general phenomenon: it occurs because the more students are in a class, the more likely it is that a randomly chosen student would be in that class. As a result, bigger classes are given more weight than smaller classes.



## Example of Roulette Version 1

### Example

A roulette wheel has 36 places numbered from 1 to 36. In addition, 18 of them are coloured red and the other 18 are coloured black. A ball is thrown to take one of 36 places. A gambler can bet:

- on the colour of the place that the ball takes. A correct, either red or black, place wins them a 1 to 1 ratio payout;
- on the number of the place that the ball takes. A correct number wins them a 35 to 1 ratio payout.

What is the expected value if a gambler bets on

1. the colour of the place in the roulette;
2. the number of the place in the roulette that the ball will fall.

Are the two different betting games fair?

Answer



## Example of Roulette Version 1 Cont.

### Example

What is the expected value if a gambler bets on

1. the colour of the place in the roulette;
2. the number of the place in the roulette that the ball will fall.

Are the two different betting games fair?

Answer

1. Let  $E_X$  : bet on colour.

2. Let  $E_Y$  : bet on number.



## Example of Roulette Version 2

### Example

Change the game to add two green places, 0 and 00. Now there are a total of 38 places. Payouts are the same as before. What are the expected values now?

Answer

1. Let  $E_X$  : bet on red colour.
  
2. Let  $E_Y$  : bet on number 10.

