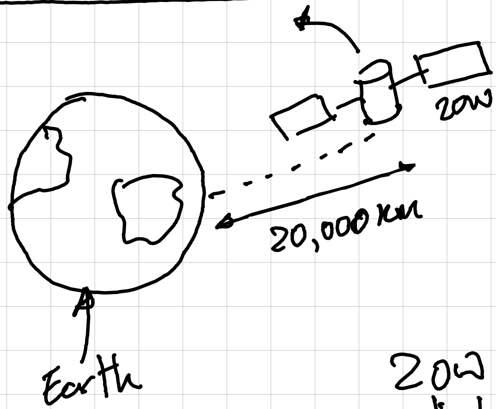
# Today's Conundrum

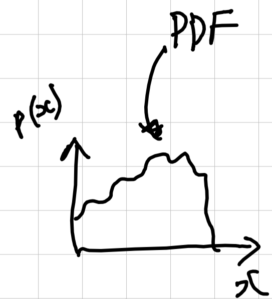



How can this possibly work

20W "light bulb" that can reliably communicate to almost all devices today.

# Moving to Continuous entropy

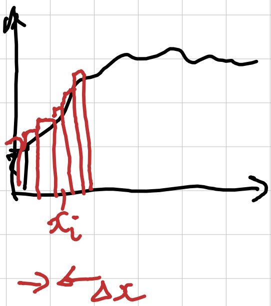
$$\boxed{H(X) = \sum_i p_i \log_2 \frac{1}{p_i}}$$ Discrete Entropy




$\Rightarrow$ How do we generalise

Idea: Could we use $$\int_x p(x) \log_2 \frac{1}{p(x)}\, dx$$

Does this work? Hint: No




Discretise our PDF

centre of bin

Define $P_i = P(x \text{ is in bin } i) = p(x_i)\Delta x$

Discretising

$$H(X^\Delta) = \sum P_i \log \frac{1}{P_i}$$

$$= \sum p(x_i)\Delta x \log \frac{1}{p(x_i)\Delta x}$$

$$= \sum p(x_i)\Delta x \log \frac{1}{p(x_i)} + \underbrace{\sum p(x_i)\Delta x}_{=1} \log \frac{1}{\Delta x}$$

$$= \sum p(x_i) \log \frac{1}{p(x_i)} \Delta x + \log_2 \frac{1}{\Delta x}$$

$\Delta x \to 0$ But $H(X) = \int p(x) \log \frac{1}{p(x)} dx + \underline{\underline{\infty}}$

Crap!

## Differential Entropy

$$\boxed{H_{dif}(X) = \int_x p(x) \log \frac{1}{p(x)}\, dx}$$

has no physical meaning.

## Continuous Mutual Information

$$I(X;Y) = H(Y) - H(Y|X)$$

$$= H_{dif}(Y) \cancel{+\infty} - H_{dif}(Y|X) - \cancel{\infty}$$

$$= H_{dif}(Y) - H_{dif}(Y|X)$$

## Capacity

$$C = \max_{p(x)} \left\{ I(X;Y) \right\}$$ makes sense for continuous signal.

$$= \max_{p(x)} \left\{ H_{dif}(Y) - H_{dif}(Y|X) \right\}$$

$\downarrow$ symmetric channel.

$$= \max_{p(x)} \left\{ H_{dif}(Y) \right\} - H_{dif}(\eta).$$

To reach capacity we want to maximise

$H(Y)$ or $H_{dif}(Y)$

Discrete

Cont.

$$X \longrightarrow \boxed{\text{Channel}} \longrightarrow Y$$

## Discrete Capacity

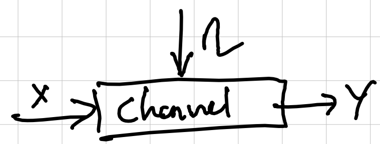

No noise.

$H(Y)$ max $\Rightarrow$ $P(y)$ is uniform (Lecture 2)

if $P(Y) = \mathcal{U}$ then $P(X)$ is uniform

what huffman codes did for us

## Continuous/Noisy capacity




$$Y = X + \eta$$

Q1. What $P(Y)$ makes $H_{dif}(Y)$ maximal?

Q2. What $P(X)$ gives us the $P(Y)$ we want?

## Max $H_{dif}$

$$L = \int p(x) \log \frac{1}{p(x)} \, dx \; - \lambda_1 \left( 1 - \int p(x) \, dx \right) - \lambda_2 K$$

This is not enough for a real cont. signal

Most common addition is to constrain your power

# Power and Variance

Power $\propto$ signal$^2$

Voltage signal $P = \frac{V^2}{R} = I^2 R$

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Variance of a signal

$$\sigma^2 = \sum \frac{1}{N}(x_i - \mu)^2$$

$$\Rightarrow \quad \sigma^2 \propto \text{signal}^2 \propto \text{Power}$$

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$$L = \int P(x) \log_2 \frac{1}{P(x)} dx - \lambda_1 \left(1 - \int P(x) dx\right) - \lambda_2 \left(\sigma^2 - \int (x - \bar{x})^2 dx\right)$$

Solve lagrangian $\Rightarrow$ $\boxed{\begin{array}{l} p(X) \text{ for } \max H_{dif}(X) \\ \text{is Gaussian} \end{array}}$

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## The "Gaussian channel"

$$X \rightarrow \boxed{\text{channel}} \xleftarrow{\eta} \rightarrow Y$$

want $P(Y)$ to be gaussian.

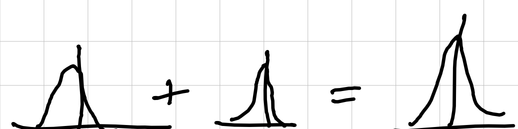
CLT

$P(\eta)$ will be gaussian for the Gaussian channel

$$P(X) \quad X + \underset{\sim N}{\eta} = \underset{\sim N}{Y}$$

Need $P(X) \sim N(\sigma^2, \mu)$

**Fun fact**



Gaussian + Gaussian = Gaussian

$\sigma_a^2$ $\quad$ $\sigma_b^2$ $\quad$ $\sigma_c^2 = \sigma_a^2 + \sigma_b^2$

# Capacity of Gaussian Channel

$$P(X) \sim N(\sigma_x^2, \mu_x)$$
$$P(N) \sim N(\sigma_n^2, \mu_N)$$
$$P(Y) \sim N(\sigma_y^2, \mu_Y)$$

Standard result

$$H(z \sim N(\sigma^2, \mu)) = \int P(z) \log \frac{1}{P(z)} dx$$
$$= \frac{1}{2} \log_2 (2\pi e \sigma^2)$$

$$C = H(Y) - H(N) \quad \text{(symm)}$$
$$= \frac{1}{2} \log_2 (2\pi e \sigma_Y^2) - \frac{1}{2} \log_2 (2\pi e \sigma_N^2)$$
$$= \frac{1}{2} \log_2 \left( \frac{\sigma_y^2}{\sigma_N^2} \right)$$

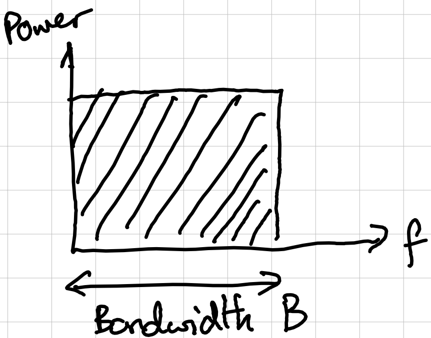
But $\sigma_y^2 = \sigma_x^2 + \sigma_N^2$

$$= \frac{1}{2} \log_2 \left( 1 + \frac{\sigma_x^2}{\sigma_N^2} \right)$$

Ratio of transmit power to noise power

SNR - Signal to noise ratio

# Bandwidth Limited Signals




Nyquist - must sample at 2x the freq to represent it.

$$C = \frac{1}{2}\log\left(1+\frac{S}{N}\right)$$ per symbol

$$C = \underbrace{\overset{\text{Nyquist}}{2B}}_{\text{samples per sec}} \times \frac{1}{2}\log\left(1+\frac{S}{N}\right)$$

$$\boxed{C = B\log_2\left(1+\frac{S}{N}\right) \text{ per second}}$$

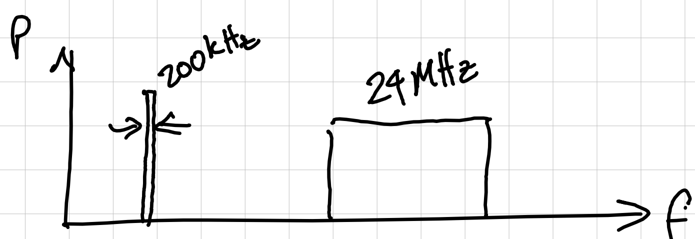
Shannon-Hartley Theorem.

# Back to GPS

$$C = B \log_2 \left(1 + \frac{S}{N}\right)$$

Just being louder is <u>not</u> a great way to increase capacity (log)

Increasing bandwidth is the way to go.



UWB Ultra Wideband

Bandwidth $\sim 10\,GHz$

$$C = B \log_2\left(1 + \frac{S}{N}\right)$$

Big

Tiny $\sim 1$