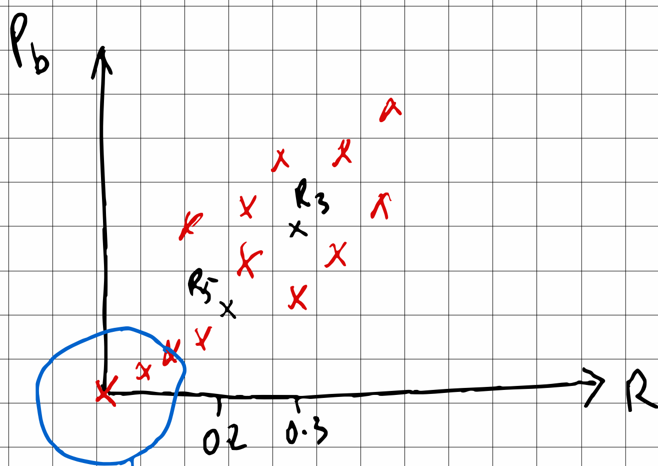
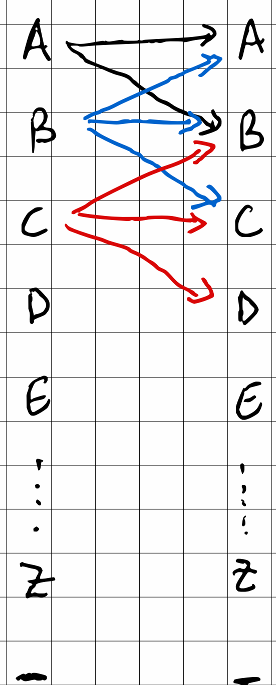
Mutual Information $\longrightarrow$ Capacity

$$\boxed{C = \max_{P(x)} I(X;Y)}$$

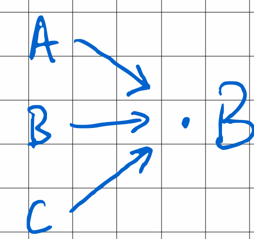


Perfect comms over noisy channels meant your rate $\Rightarrow 0$

Noisy Typewriter Channel

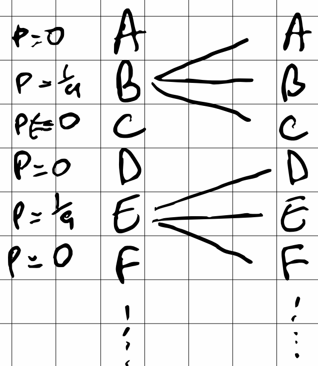


$$P(y=A \mid x=B) = 1/3$$
$$P(y=B \mid x=B) = 1/3$$
$$P(y=C \mid x=B) = 1/3$$
$$P(y=\text{other} \mid x=B) = 0$$



$$\begin{aligned}
I(X;Y) &= H(X) - H(X \mid Y) \\
&= H(X) - \sum_y P(y) \sum_x P(x \mid y) \log_2 \frac{1}{P(x \mid y)} \\
&= H(X) - \sum_y P(y) \left(3 \cdot \tfrac{1}{3} \cdot \log_2 3\right) \\
&= H(X) - \log_2 3 \\
&= \sum \tfrac{1}{27} \log_2 27 - \log_2 3 \\
&= \log_2 3 + \log_2 9 - \log_2 3 \\
&= \log_2 9 = C
\end{aligned}$$

Special block code for N.T. Channel



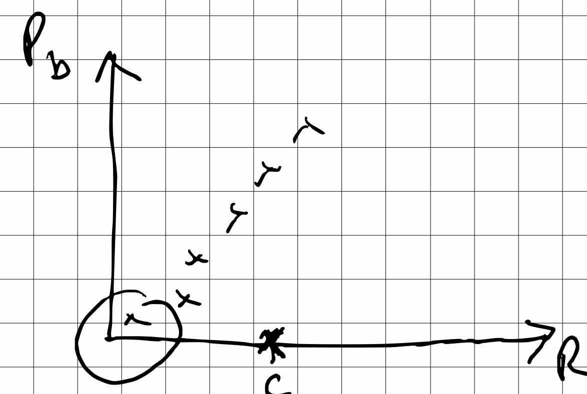
Block code $N=1$

We have perfect error-free comm.

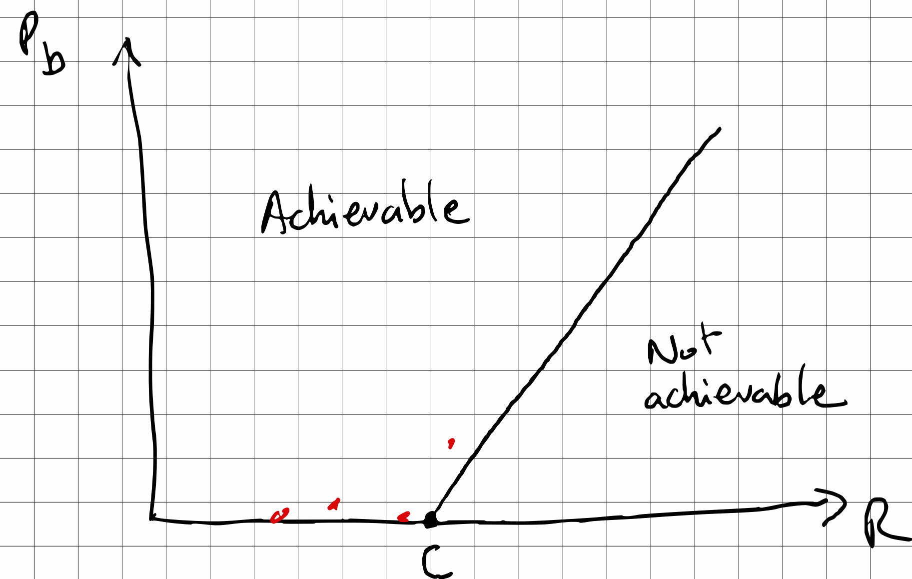
$$I(X;Y) = H(X) - \cancel{H(X|Y)} \to 0$$

$$= \sum \frac{1}{q} \log_2 q = \log_2 q = C$$

- A noisy channel
- Applied a code
- Achieved error-free comm.
- Rate = Capacity $\neq 0$



Noisy Channel Coding Theorem



Health warning!

The Return of Typical Set

$\underline{x}$ typical $\quad \left| \frac{1}{N} \log \frac{1}{P(x)} - H(X) \right| < \beta$

$\underline{y}$ typical $\quad \left| \frac{1}{N} \log \frac{1}{P(y)} - H(Y) \right| < \beta$

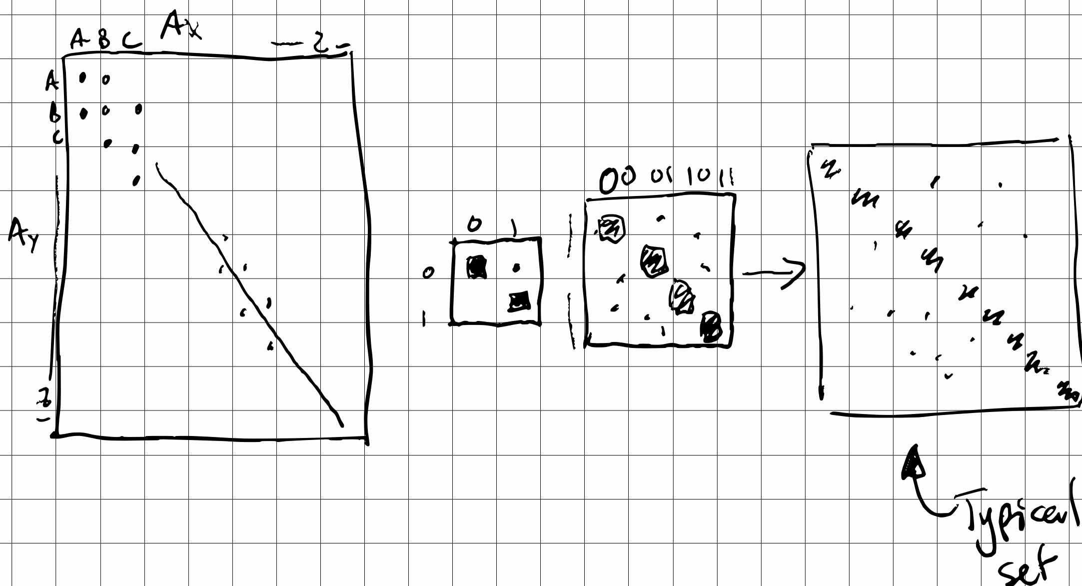
$\underline{x}, \underline{y}$ typical $\quad \left| \frac{1}{N} \log_2 \frac{1}{P(x,y)} - H(X,Y) \right| < \beta$

Jointly Typical Set

1. The prob of x, y being typical $\to 1$ as $N \to \infty$
2. $|J_{N\beta}| \leq 2^{N(H(X,Y)+\beta)}$
3. $x' \sim X^N \quad y' \sim Y^N \qquad P((x', y') \in J_{N\beta}) \leq 2^{N(I(X;Y)-3\beta)}$

Mackay Pg 163

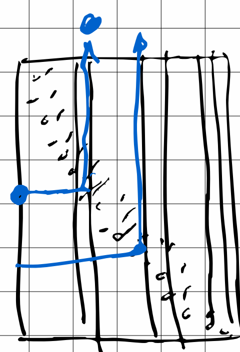
Intuition



Theoretically represent a code + decoder

1. Random code

2. Typical set decoder



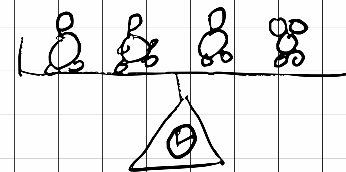
Not optimal but we are able to reason about it

⇒ Good enough for the proof!

Mackay
pg 165, 166

Block Error

$\cancel{\langle P_b \rangle}$ $\quad \overline{P_b}$



10x babies < 1kg.

$$P\left((x^{(s')}, y) \in J_{N\beta}\right) \leq 2^{-N(I(X;Y) - 3\beta)}$$

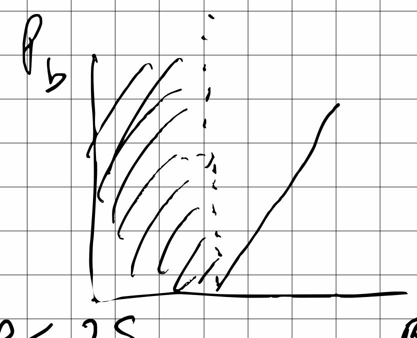
$\bar{w} < 10$kg $\Rightarrow$ at least one of these babies is underweight

confusable codes (s') $\longrightarrow$ $2^{NR'} - 1$

$$\overline{P_b} \leq \delta + \sum_{\text{confusable codes}} 2^{-N(I(X;Y) - 3\beta)}$$

$$\leq \delta + 2^{-N(I(X;Y) - R' - 3\beta)}$$

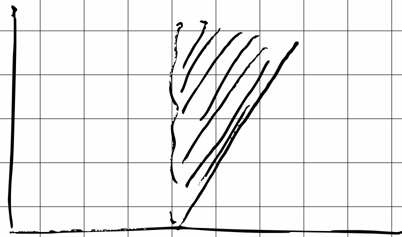
$$\overline{P_b} \leq 2\delta \Rightarrow R' < I(X;Y) - 3\beta < C - 3\beta$$



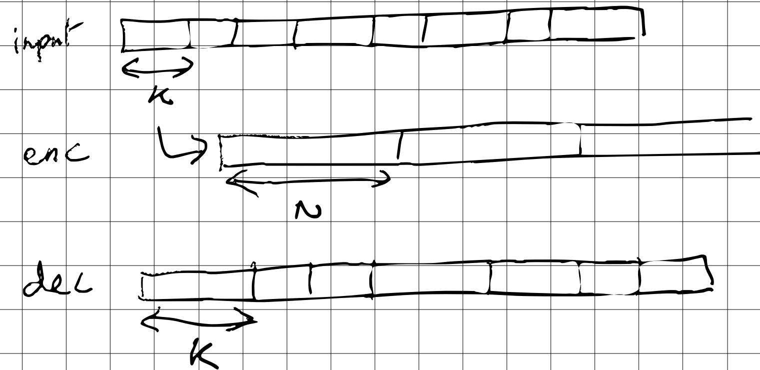
$\therefore$ There is at least 1 code with $P_b < 2\delta$

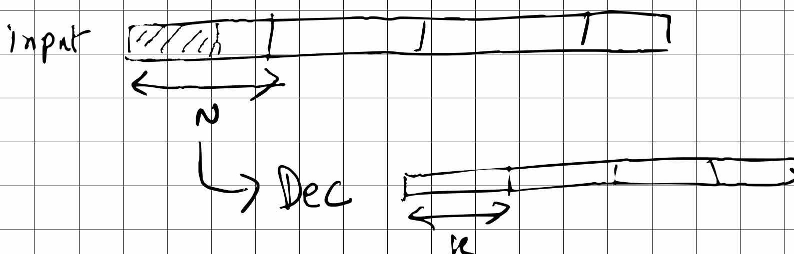
Mackay

pg 166



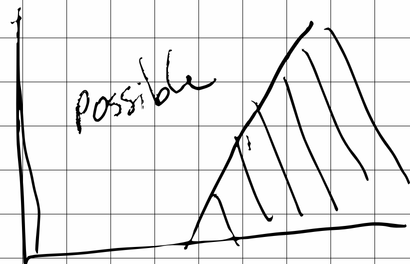
(N, k) $K \to N$





$$\boxed{R = \frac{C}{1 - H_2(p_b)}}$$

Pg 168.



MacKay

pg. 168

Theorem

1. for capacity $C = \max_{P(X)} I(X;Y)$ then there exists at least one code of length N and rate $R < C$ such that $P_b < \epsilon$ for any $\epsilon > 0$, as $N \to \infty$

2. Rates up to $R(P_b) = \frac{C}{1 - H_2(P_b)}$ are achievable for block error P_b.

3. For any P_b $R > R(P_b)$ are not achievable