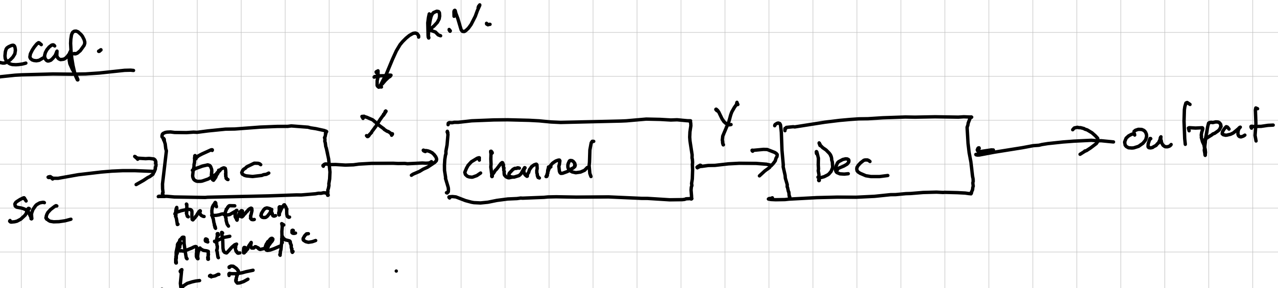
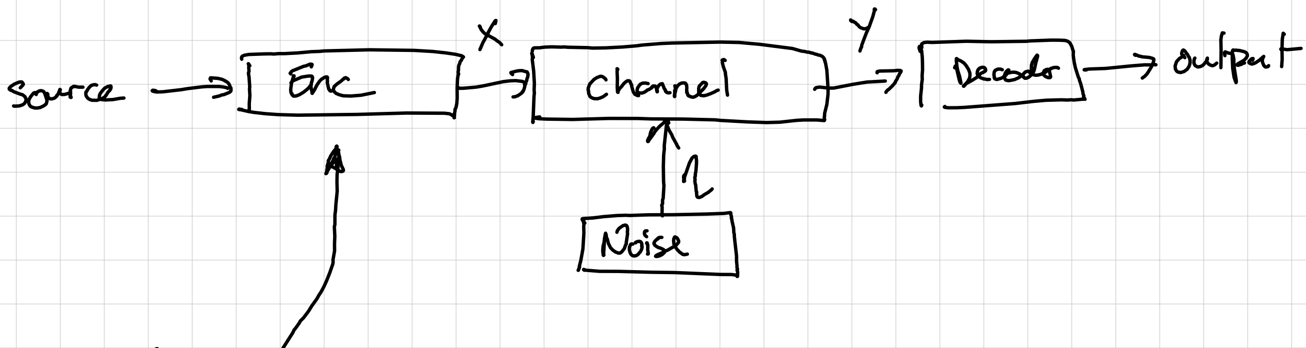
Recap.



Assumed no noise. $Y = X$

- Stripped out redundancy
- Can't go below source entropy without risking loss of info.

From Today we introduce evil noise



Encoder now seeks to <u>add</u> redundancy.

Block Codes

$\boxed{\text{An } (N,K) \text{ block code uses } N \text{ bits to transmit } K \text{ source bits}}$

$N > K$

E.g. $\underline{s} = s_0 s_1 s_2 \ldots s_n$

bit

$= 10101\ 11001\ 0$

$(7,5)$ code

$\underline{t} = 1010111\ 1100101$

Repetition code

$\boxed{\text{An } R_k \text{ repetition code repeats each source bit } k \text{ times}}$ $(k,1)$ block code.

$\underline{s} = s_0 s_1 s_2 \ldots s_n$

$R_2 \Rightarrow \underline{t} = s_0 s_0 s_1 s_1 \ldots s_n s_n$

$R_3 \Rightarrow \underline{t} = s_0 s_0 s_0\, s_1 s_1 s_1 \ldots s_n s_n s_n$

Decoding

n^{th} trans bit

k-vector recv

$$P(t_n \mid \underline{r}_n) = \frac{P(\underline{r}_n \mid t_n)\, P(t_n)}{P(\underline{r}_n)}$$

Need to model channels

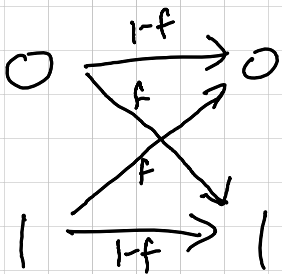
Decision ratio $\quad D_n = \dfrac{P(t_n=1 \mid \underline{r}_n)}{P(t_n=0 \mid \underline{r}_n)} = \dfrac{P(t_n=1)}{P(t_n=0)} \times \dfrac{P(\underline{r}_n \mid t_n=1)}{P(\underline{r}_n \mid t_n=0)}$

$D_n > 1 \Rightarrow s_n = 1$

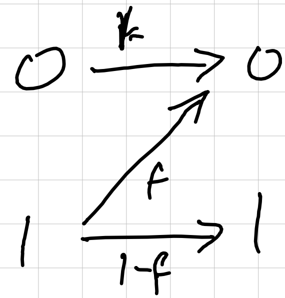
$D_n < 1 \Rightarrow s_n = 0$

Common channel models

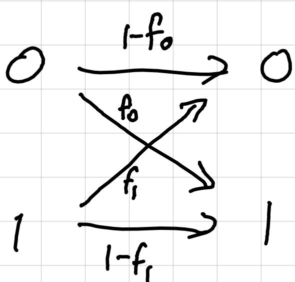
$$P(\underline{r}_n \mid t_n = 0,1) = \prod_i P(r_n^i \mid t_n = 0,1)$$



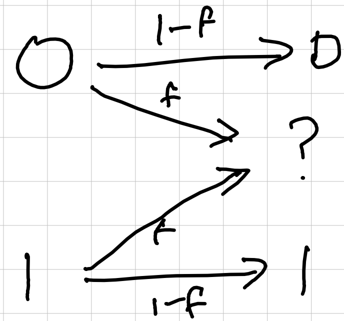
Binary Symmetric channel (BSC)



Z-channel



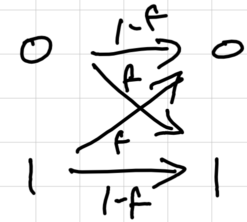
Binary Asymmetric channel.



Binary Erasure channel

Decode Example (BSC)

$$D = \underbrace{\frac{P(t_n=1)}{P(t_n=0)}}_{\alpha} \times \prod_i \frac{P(r_n^i \mid t_n=1)}{P(r_n^i \mid t_n=0)}$$



r_n	D	t_n $(\alpha=1)$
000	$\alpha\left(\frac{f}{1-f}\right)^3 = \alpha\gamma^{-3}$	0
001	$\alpha \, \frac{f^2(1-f)}{(1-f)^2 f} = \alpha\gamma^{-1}$	0
010	$\alpha\gamma^{-1}$	0
011	$\alpha\gamma$	1
100	$\alpha\gamma^{-1}$	0
101	$\alpha\gamma$	1
110	$\alpha\gamma$	1
111	$\alpha\gamma^3$	1

$$\gamma = \frac{1-f}{f}$$

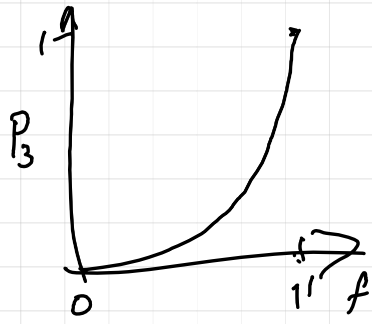
$$\gamma > 1$$

We see decode errors for 2 or 3 bit flips

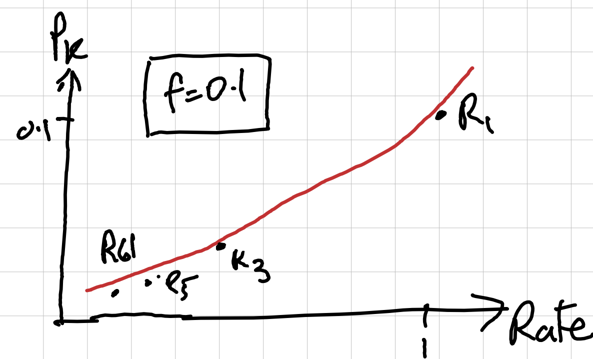
Data Rates

Let $P_K = P(\text{error for an } R_K \text{ code on a BSC})$

$$P_3 = P(2 \text{ flips}) + P(3 \text{ flips})$$
$$3 \times f^2(1-f) + f^3 = 3f^2 - 2f^3$$



$$P_K = \sum_{n=\frac{K+1}{2}}^{K} {}^{K}C_n f^n (1-f)^{K-n} \quad \text{for } K \text{ odd}$$

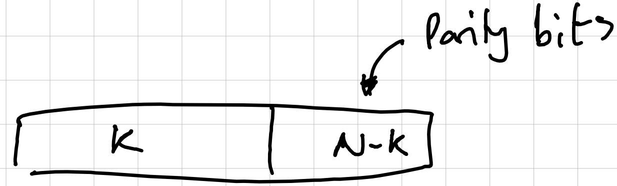


Intuition. ⇒ Perfect comm. requires ∞ repetition
⇒ Rate will go to zero

E.g. 7.1

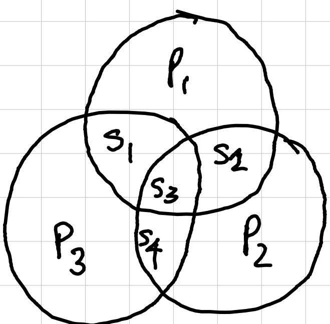
Hamming Code

(N, k) block code



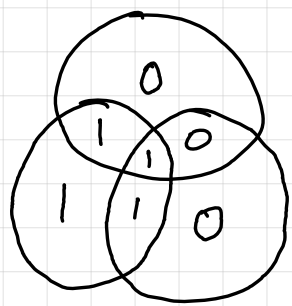
$(7,4)$ hamming code

Idea: Parity bits from diff. subsets of block.



$S_1 S_2 S_3 S_4 P_1 P_2 P_3$

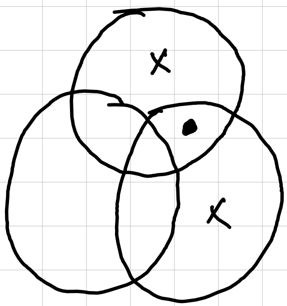
$\underline{s} = 1011$



$\underline{t} = 1011\,\underline{001}$

(7,4) Hamming Codewords

0000	000	1000	101
0001	011	1001	110
0010	111	1010	010
0011	100	1011	001
0100	110	1100	011
0101	101	1101	000
0110	001	1110	100
0111	010	1111	111



Any 1-bit flip affects at least 2 circles

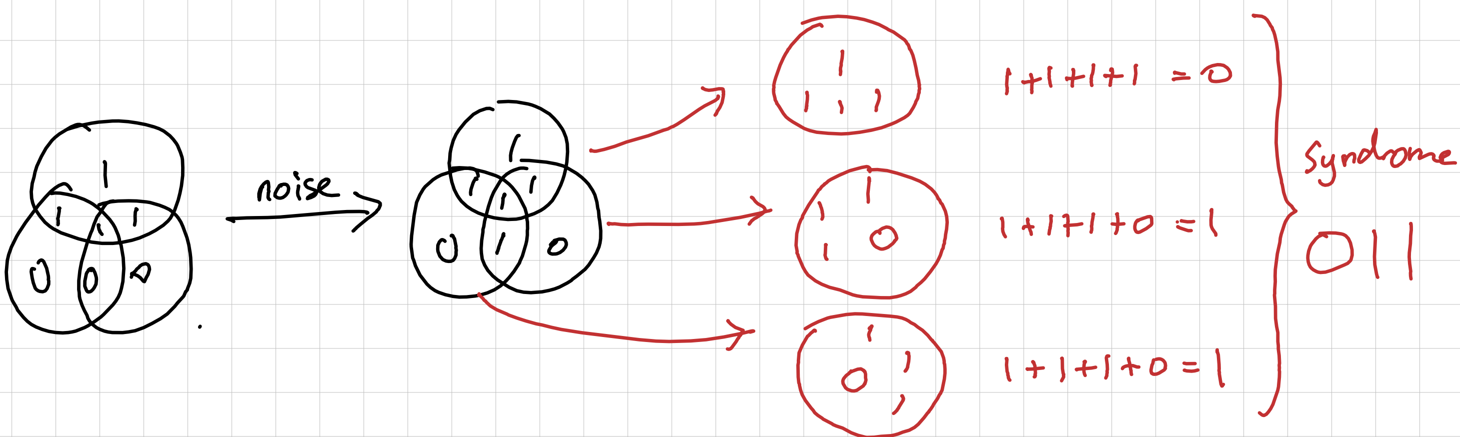
Hamming dist = no. of differing bits per codewords

= 3 for (7,4) ⇒ at least 3 differ

Decoding

Option 1 compute the hamming dist between r_n and every codeword $\Rightarrow$ take the min.

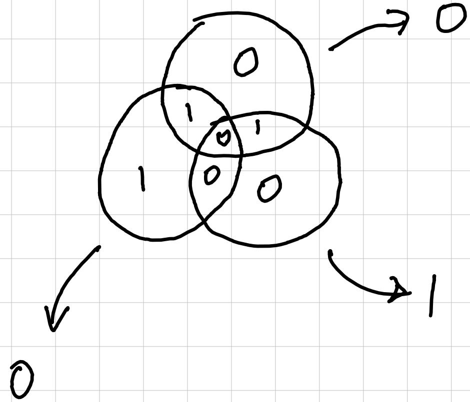
Option 2 "Syndrome" decoding. Sum the bits in each circle



Syndrome	000	001	010	011	100	101	110	111
decode	–	r_7	r_6	r_4	r_5	r_1	r_2	r_3

Examples

$\underline{r} = 1100001$



$\underline{t} = 1100\ [011]$

Syndrome $010 \Rightarrow$ flip r_6

$\therefore$ Transmit was 1100001

$\rightarrow 1100$ message

E.g. 7.2

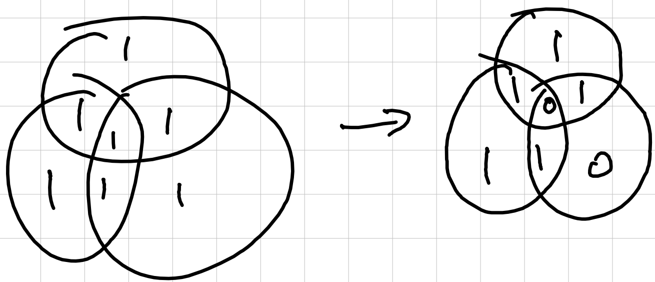
$\underline{r} = 1000010$

$r = 0111111$

$\underline{r} = 0001111$

Syndrome	000	001	010	011	100	101	110	111
decode	–	r_7	r_6	r_4	r_5	r_1	r_2	r_3

2 bits flipped



syndrome
$z = 101$
flip r_1
$\Rightarrow$ 3 bits of error!

A (7,4) H.C. can detect and correct 1 bit error
" " " " detect (only) a 2 bit error

$\Rightarrow$ f is typically low $\ll 1$ $\therefore$ this is ok.

$\Rightarrow$ Whatever we do, a hamming code isn't perfect

$\Rightarrow$ we need ∞-length codes, but rate $\to 0$

Generalising

$$2^p \geqslant d + p + 1$$

source bits per block

parity bits per block

$p = 3 \Rightarrow \quad 2^3 = 8 \geqslant d + 3 + 1 \quad \Rightarrow \quad d \leqslant 4 \qquad (7,4)$

$p = 4 \Rightarrow \quad 2^4 = 16 \geqslant d + 4 + 1 \quad \Rightarrow \quad d \leqslant 11 \qquad (15,11)$