Unordering

Pairing axiom

For every a and b, there is a set with a and b as its only elements.

$\mathcal{F}(a,b) = \{a, b\}$ $\qquad$ $\mathcal{F}(a,b) = \mathcal{F}(b,a)$

defined by

$$\forall x.\, x \in \{a, b\} \iff (x = a \lor x = b)$$

NB The set $\{a, a\}$ is abbreviated as $\{a\}$, and referred to as a *singleton*.

Example: $(0,1) \neq (1,0)$

Ordered pairing

Notation:

$$(a, b) \quad \text{or} \quad \langle a, b \rangle$$

Fundamental property:

$$(a, b) = (x, y) \implies a = x \wedge b = y$$

A construction:

For every pair a and b,

$$\langle a, b \rangle = \big\{ \{ a \}, \{ a, b \} \big\}$$

defines an *ordered pairing* of a and b.

Products

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{\, x \mid \exists\, a \in A, b \in B.\, x = (a, b) \,\}$$

where

$$\begin{aligned} &\forall a_1, a_2 \in A, b_1, b_2 \in B. \\ &\quad (a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \wedge b_1 = b_2) \quad . \end{aligned}$$

Thus,

$$\forall\, x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$

Pattern-matching notation

Example: The subset of ordered pairs from a set A with equal components is formally

$$\{ x \in A \times A \mid \exists\, a_1 \in A.\, \exists\, a_2 \in A.\, x = (a_1, a_2) \ \wedge\ a_1 = a_2 \}$$

but often abbreviated using *pattern-matching notation* as

$$\{ (a_1, a_2) \in A \times A \mid a_1 = a_2 \} \ .$$

Notation: For a property $P(a, b)$ with a ranging over a set A and b ranging over a set B,

$$\{ (a, b) \in A \times B \mid P(a, b) \}$$

abbreviates

$$\{ x \in A \times B \mid \exists\, a \in A.\, \exists\, b \in B.\, x = (a, b) \ \wedge\ P(a, b) \} \ .$$

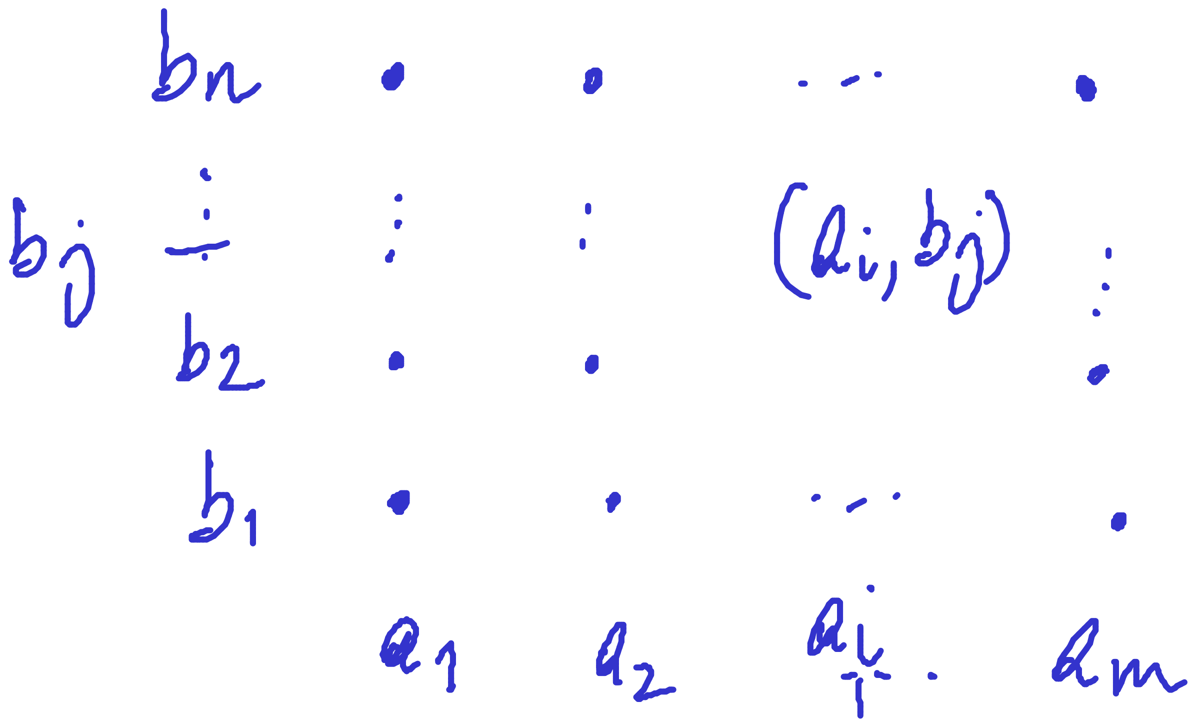
Proposition 110 *For all finite sets* A *and* B,

$$\#(\mathrm{A} \times \mathrm{B}) = \#\mathrm{A} \cdot \#\mathrm{B} \quad .$$

PROOF IDEA:

$\#(A \times B) = m \cdot n$

$\#B = n$



$\#A = m$

Sets and logic

$\mathcal{P}(\mathbb{U})$	$\{\,\textbf{false}\,,\ \textbf{true}\,\}$
$\emptyset$	**false**
$\mathbb{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{\mathrm{c}}$	$\neg(\cdot)$
$\bigcup$	$\exists$
$\bigcap$	$\forall$

Big unions

Example: $\bigcup \{A_1, A_2, \ldots, A_n\} = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$

- Consider the family of sets

$$\mathcal{T} = \left\{ T \subseteq [5] \;\middle|\; \begin{array}{l}\text{the sum of the elements of} \\ T \text{ is less than or equal } 2\end{array} \right\}$$

$$= \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}\}$$

- The *big union* of the family $\mathcal{T}$ is the set $\bigcup \mathcal{T}$ given by the union of the sets in $\mathcal{T}$:

$$n \in \bigcup \mathcal{T} \iff \exists T \in \mathcal{T}. n \in T \quad .$$

Hence, $\bigcup \mathcal{T} = \{0, 1, 2\}$.

$\underline{NB}$: $\mathcal{F} \in \mathcal{P}(\mathcal{P}\,\mathcal{U}) \iff \mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$

$A \in \mathcal{P}(\mathcal{U}) \Rightarrow A \subseteq \mathcal{U}$

Definition 111 *Let* $\mathcal{U}$ *be a set. For a collection of sets* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$, *we let the* big union *(relative to* $\mathcal{U}$*) be defined as*

$$\bigcup \mathcal{F} = \{ x \in \mathcal{U} \mid \exists A \in \mathcal{F}.\, x \in A \} \in \mathcal{P}(\mathcal{U}) \quad .$$

$$x \in \bigcup \mathcal{F} \overset{\text{def}}{\iff} x \in \mathcal{U} \wedge \exists A \in \mathcal{F}.\, x \in A.$$

$$\iff \exists A.\, A \in \mathcal{F} \wedge x \in A$$

$\mathcal{F} \subseteq \mathcal{P}(\mathcal{P}\mathcal{U})$

$\shortparallel$

$\{ \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_i, \ldots \}$ $\qquad$ $\mathcal{A}_i \subseteq \mathcal{P}\mathcal{U}$

Proposition 112 *For all* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{U})))$,

$$\bigcup\Big(\bigcup \mathcal{F}\Big) = \bigcup\Big\{ \bigcup \mathcal{A} \in \mathcal{P}(\mathbb{U}) \;\Big|\; \mathcal{A} \in \mathcal{F} \Big\} \in \mathcal{P}(\mathbb{U}) .$$

PROOF:

$\bigcup \{ \bigcup \mathcal{A}_1, \bigcup \mathcal{A}_2, \ldots, \bigcup \mathcal{A}_i, \ldots \}$

$\shortparallel$

$\bigcup \{ \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_i \cup \ldots \}$

$$x \in \bigcup(\bigcup \mathcal{F})$$

$$\iff \exists S.\ S \in \bigcup \mathcal{F} \wedge x \in S$$

$$\iff \exists S.\ (\exists A.\ A \in \mathcal{F} \wedge S \in A) \wedge x \in S$$

$$\iff \exists A.\ A \in \mathcal{F} \wedge \exists S.\ S \in A \wedge x \in S$$

exercise

$$\iff \exists A.\ A \in \mathcal{F} \wedge\ x \in \bigcup A$$

$\bigcap\{A_1, A_2, \ldots, A_n\} = A_1 \cap A_2 \cap \ldots \cap A_n$

Big intersections

Example:

- Consider the family of sets

$$\mathcal{S} = \left\{ S \subseteq [5] \;\middle|\; \text{the sum of the elements of } S \text{ is } 6 \right\}$$

$$= \{\{2,4\},\{0,2,4\},\{1,2,3\},\{0,1,2,3\}\}$$

- The *big intersection* of the family $\mathcal{S}$ is the set $\bigcap\mathcal{S}$ given by the intersection of the sets in $\mathcal{S}$:

$$n \in \bigcap\mathcal{S} \iff \forall S \in \mathcal{S}.\, n \in S \quad .$$

Hence, $\bigcap\mathcal{S} = \{2\}$.

Definition 113 *Let* $\mathcal{U}$ *be a set. For a collection of sets* $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$, *we let the* big intersection *(relative to* $\mathcal{U}$*) be defined as*

$$\bigcap \mathcal{F} = \{ x \in \mathcal{U} \mid \forall A \in \mathcal{F}.\, x \in A \} \quad .$$

$$x \in \bigcap \mathcal{F} \iff [x \in \mathcal{U} \wedge \forall A.\, A \in \mathcal{F} \Rightarrow x \in A]$$

$$\iff [\forall A.\, A \in \mathcal{F} \Rightarrow x \in A]$$

Theorem 114 *Let*

$$\mathcal{F} = \Big\{ S \subseteq \mathbb{R} \;\Big|\; \overbrace{(0 \in S) \wedge \underbrace{(\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S)}_{\text{S is closed under successor}}}^{\text{closure property}} \Big\} \,.$$

Then, $(\mathrm{i})\ \mathbb{N} \in \mathcal{F}$ *and* $(\mathrm{ii})\ \mathbb{N} \subseteq \underbrace{\bigcap}_{\text{The biggest set contained in all sets in } \mathcal{F}} \mathcal{F}$. *Hence,* $\bigcap \mathcal{F} = \mathbb{N}$.

PROOF:

$\mathbb{N} \in \mathcal{F}$ ✓

$\mathbb{Q} \in \mathcal{F}$ ✓

$\mathbb{R} \in \mathcal{F}$ ✓

$\emptyset \notin \mathcal{F}$

$$\mathbb{N} \subseteq \bigcap \mathcal{F}$$

$$\iff \forall n \in \mathbb{N}.\ n \in \bigcap \mathcal{F}.$$

$$\iff \forall n \in \mathbb{N}.\ \forall S \in \mathcal{F}.\ n \in S$$

} __exercise__: proof by induction.

Proposition 115 *Let U be a set and let $\mathcal{F} \subseteq \mathcal{P}(U)$ be a family of subsets of U.*

Proof Principle or Technique for proving a set is a big union or a big intersection.

1. *For all* $S \in \mathcal{P}(U)$,

$$S = \bigcup \mathcal{F}$$

iff

$$[\forall A \in \mathcal{F}.\, A \subseteq S]$$
$$\wedge \ [\forall X \in \mathcal{P}(U).\, (\forall A \in \mathcal{F}.\, A \subseteq X) \Rightarrow S \subseteq X]$$

2. *For all* $T \in \mathcal{P}(U)$,

$$T = \bigcap \mathcal{F}$$

iff

$$① \ [\forall A \in \mathcal{F}.\, T \subseteq A]$$
$$\wedge ② \ [\forall Y \in \mathcal{P}(U).\, (\forall A \in \mathcal{F}.\, Y \subseteq A) \Rightarrow Y \subseteq T]$$

Union axiom

$\boxed{\text{Every collection of sets has a union.}}$

$$\bigcup \mathcal{F}$$

$$x \in \bigcup \mathcal{F} \iff \exists X \in \mathcal{F}.\, x \in X$$

For *non-empty* $\mathcal{F}$ we also have

$$\bigcap \mathcal{F}$$

defined by

$$\forall x.\ x \in \bigcap \mathcal{F} \iff \left(\forall X \in \mathcal{F}. x \in X\right) \quad .$$

$\{1\} \times A = \{ (1,a) \mid a \in A \}$ $\{2\} \times B = \{ (2,b) \mid b \in B \}$

$(\{1\} \times A) \cap (\{2\} \times B) = \emptyset$

Disjoint unions

Definition 116 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B \;=\; \underbrace{(\{1\} \times A)}_{A \text{ tagged by } 1} \cup \underbrace{(\{2\} \times B)}_{B \text{ tagged by } 2} \quad .$$

Thus,

$$\forall x.\, x \in (A \uplus B) \iff \big(\exists a \in A.\, x = (1,a)\big) \vee \big(\exists b \in B.\, x = (2,b)\big) .$$

X and Y disjoint $\overset{}{\underset{\text{def}}{\iff}}$ $X \cap Y = \emptyset$

datatype

(α, β) sum = one of α | two of β

$A \uplus B \qquad (\{one\} \times A) \cup (\{two\} \times B)$

Proposition 118 *For all finite sets* A *and* B*,*

$$A \cap B = \emptyset \implies \#(A \cup B) = \#A + \#B \quad .$$

PROOF IDEA:

$A = \{ a_1 \dots a_m \}$

$B = \{ b_1 \dots b_n \}$

$A \cup B = \{ a_1 \dots a_m \, b_1 \dots b_n \}$

Corollary 119 *For all finite sets* A *and* B*,*

$$\#(A \uplus B) = \#A + \#B \quad .$$