Denotational Semantics

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Lectures for Part II CST 2025/2026

Extensionality

Characterizing formal approximation

Contextual preorder $\Gamma \vdash t \leq_{\text{ctx}} t' : \tau$ is the one-sided version of contextual equivalence: for all $\mathcal{C}$ such that $\cdot \vdash_{\Gamma,\tau} \mathcal{C} : \gamma$ where $\gamma \in \{\texttt{nat}, \texttt{bool}\}$ and for all values v,

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Contextual equivalence is recovered by the contextual preorder via:

$$\Gamma \vdash t \cong_{\text{ctx}} t' : \tau \quad \Longleftrightarrow \quad (\Gamma \vdash t \leq_{\text{ctx}} t' : \tau \ \wedge \ \Gamma \vdash t' \leq_{\text{ctx}} t : \tau)$$

Application contexts

To characterise contextual preorder between closed terms, applicative contexts are enough.

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Lemma

Let $\cdot \vdash t_1, t_2 : \tau$ be closed terms. Then $t_1 \leq_{\mathrm{ctx}} t_2 : \tau$ if and only if

$$\forall f \in \mathrm{PCF}_{\tau \to \mathtt{bool}} \,.\, (f\ t_1 \Downarrow_{\mathtt{bool}} \mathtt{true} \implies f\ t_2 \Downarrow_{\mathtt{bool}} \mathtt{true})$$

Contextual preorder and formal approximation

Formal approximation **captures** contextual preorder.

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Proposition

For all Pcf types τ and all closed terms $t_1, t_2 \in \text{Pcf}_\tau$

$$t_1 \leq_{\text{ctx}} t_2 : \tau \quad \iff \quad [\![t_1]\!] \lhd_\tau t_2.$$

Extensionality properties of contextual preorder

For $\gamma = \texttt{bool}$ or $\texttt{nat}$, $t_1 \leq_{\mathrm{ctx}} t_2 : \gamma$ holds if and only if

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At a function type $\tau \to \tau'$, $t_1 \leq_{\text{ctx}} t_2 : \tau \to \tau'$ holds if and only if

$$\forall t \in \text{PCF}_\tau\,.\,(t_1\ t \leq_{\text{ctx}} t_2\ t : \tau').$$

Full abstraction

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Failure of full abstraction

Recap

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- Compositionality: $[\![t]\!] = [\![t']\!] \implies [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$ for all terms and contexts
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Full abstraction

A denotational model is called **fully abstract** when

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It is a form of **completeness** of semantic equivalence wrt. contextual equivalence.

The domain model of PCF is **not** fully abstract.

Parallel or

The *parallel or* function $\mathbf{por} : \mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$ is defined as given by the following table:

por	true	false	$\perp$
true	true	true	true
false	true	false	$\perp$
$\perp$	true	$\perp$	$\perp$

Left sequential or

The (left) sequential or function $\mathbf{or} : \mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$ is defined as

$$\mathrm{or} \stackrel{\mathrm{def}}{=} [\![\texttt{fun}\, x\colon \texttt{bool}.\ \texttt{fun}\, y\colon \texttt{bool}.\ \texttt{if}\ x\ \texttt{then}\ \texttt{true}\ \texttt{else}\ y]\!]$$

It is given by the following table:

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true	true	true	true
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Parallel vs sequential or

por	true	false	⊥
true	true	true	true
false	true	false	⊥
⊥	true	⊥	⊥

or	true	false	⊥
true	true	true	true
false	true	false	⊥
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or is sequential, but por is not.

Undefinability or parallel or

Theorem

There is no closed Pcf term

$$t : \texttt{bool} \to \texttt{bool} \to \texttt{bool}$$

satisfying

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The proof – originally by Plotkin – is beyond the scope of this course.

Failure of full abstraction

The denotational model of Pcf in domains and continuous functions is not fully abstract.

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For well-chosen $T_{\mathtt{true}}$ and $T_{\mathtt{false}}$,

$$T_{\mathtt{true}} \cong_{\mathrm{ctx}} T_{\mathtt{false}} : (\mathtt{bool} \to \mathtt{bool} \to \mathtt{bool}) \to \mathtt{bool}$$

$$[\![T_{\mathtt{true}}]\!] \neq [\![T_{\mathtt{false}}]\!] \in (\mathbb{B} \to \mathbb{B} \to \mathbb{B}) \to \mathbb{B}$$

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Idea:

- for all $f \in PCF_{\texttt{bool}\to\texttt{bool}\to\texttt{bool}}$, ensure $T_b\ f \Uparrow_{\texttt{bool}}$...
- but $[\![T_b]\!](\text{por}) = [\![b]\!]$.

Example of full abstraction failure

$$
\begin{aligned}
T_b \stackrel{\text{def}}{=}\ & \texttt{fun}\ f\colon \texttt{bool} \to (\texttt{bool} \to \texttt{bool}).\\
& \quad \texttt{if}(f\ \texttt{true}\ \Omega_{\texttt{bool}})\ \texttt{then}\\
& \quad\quad \texttt{if}\ (f\ \Omega_{\texttt{bool}}\ \texttt{true})\ \texttt{then}\\
& \quad\quad\quad \texttt{if}\ (f\ \texttt{false}\ \texttt{false})\ \texttt{then}\ \Omega_{\texttt{bool}}\ \texttt{else}\ b\\
& \quad\quad \texttt{else}\ \Omega_{\texttt{bool}}\\
& \quad \texttt{else}\ \Omega_{\texttt{bool}}
\end{aligned}
$$