Denotational Semantics

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Adequacy

Proposition

For any closed PCF term t and value v of ground type $\gamma \in \{\mathtt{nat}, \mathtt{bool}\}$

$$[\![t]\!] = [\![v]\!] \quad \Rightarrow \quad t \Downarrow_\gamma v$$

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- We introduced the **formal approximation** relation for every type τ and context Γ:

$$\lhd_\tau \subseteq [\![\tau]\!] \times \textsc{Pcf}_\tau \qquad\qquad \lhd_\Gamma \subseteq [\![\Gamma]\!] \times \textsc{Pcf}_\Gamma$$

  where $\textsc{Pcf}_\Gamma$ is the set of substitutions $\cdot \vdash \sigma : \Gamma$
- We reduced the proof of adequacy to the fundamental property

Fundamental Property

$[\![t]\!] \lhd_\tau t$ for every $t \in \textsc{Pcf}_\tau$

For $\cdot \vdash t : \tau$ and $d \in [\![\tau]\!]$ we define:

$$d \lhd_{\mathtt{nat}} t \quad \overset{\text{def}}{\Leftrightarrow} \quad (d \in \mathbb{N} \Rightarrow t \Downarrow_{\mathtt{nat}} \underline{d})$$

$$d \lhd_{\mathtt{bool}} t \quad \overset{\text{def}}{\Leftrightarrow} \quad (d = \mathrm{true} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{true}) \wedge (d = \mathrm{false} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{false})$$

$$d \lhd_{\tau \to \tau'} t \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall e \in [\![\tau]\!].\, \forall u \in \mathrm{PCF}_{\tau}\,.\,(e \lhd_{\tau} u \Rightarrow d(e) \lhd_{\tau'} t\, u)$$

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while for $\cdot \vdash \sigma : \Gamma$ and $\rho \in [\![\Gamma]\!]$, we define:

$$\rho \lhd_{\Gamma} \sigma \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall x \in \mathrm{dom}(\Gamma).\ \rho(x) \lhd_{\Gamma(x)} \sigma(x)$$

Properties of formal approximation

1. For every closed term $\cdot \vdash t : \tau$, the set $S = \{d \in [\![\tau]\!] \mid d \lhd_\tau t\}$

   1.1 contains the bottom element $\bot_{[\![\tau]\!]}$;
   1.2 it is down-closed ($d \in S \wedge d' \sqsubseteq d \implies d' \in S$);
   1.3 it is chain-closed.

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2. For every type τ and element $d \in [\![\tau]\!]$,
   2.1 If $\forall v. (t \Downarrow_\tau v \implies t' \Downarrow_\tau v)$ then $d \lhd_\tau t \implies d \lhd_\tau t'$
   2.2 If $t' \rightsquigarrow_\tau t$ then $d \lhd_\tau t \implies d \lhd_\tau t'$

The fundamental property

For any

- context Γ and type τ
- term t such that $\Gamma \vdash t : \tau$
- environment $\rho \in [\![\Gamma]\!]$
- substitution $\cdot \vdash \sigma : \Gamma$

we have that

$$\rho \lhd_\Gamma \sigma \implies [\![t]\!](\rho) \lhd_\tau t[\sigma].$$

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Proof

We proceed by induction on $\Gamma \vdash t : \tau$ to show

$$\Psi(\Gamma, t, \tau) \overset{\text{def}}{\Leftrightarrow} \forall \rho.\, \forall \sigma.\, (\rho \lhd_{\Gamma} \sigma \implies [\![t]\!](\rho) \lhd_{\tau} t[\sigma])$$

The typing derivation

$$\textsc{Zero}\ \frac{}{\Gamma \vdash \mathtt{0} : \mathtt{nat}}$$

$$\textsc{Zero}\ \frac{}{\Gamma \vdash \mathtt{0} : \mathtt{nat}} \qquad \textsc{True}\ \frac{}{\Gamma \vdash \mathtt{true} : \mathtt{bool}}$$

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$$\textsc{Succ}\ \frac{\Gamma \vdash t : \mathtt{nat}}{\Gamma \vdash \mathtt{succ}\,(t) : \mathtt{nat}}$$

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$$\textsc{If}\ \frac{\Gamma \vdash b : \mathtt{bool} \qquad \Gamma \vdash t : \tau \qquad \Gamma \vdash t' : \tau}{\Gamma \vdash \mathtt{if}\ b\ \mathtt{then}\ t\ \mathtt{else}\ t' : \tau}$$

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