Denotational Semantics

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Lectures for Part II CST 2025/2026

- a mapping of PCF types τ to domains $[\![\tau]\!]$;
- a mapping of PCF contexts Γ to domains $[\![\Gamma]\!]$;
- a mapping of closed, well-typed PCF terms $\cdot \vdash t : \tau$ to elements $[\![t]\!] \in [\![\tau]\!]$;
- denotation of open terms $\Gamma \vdash t : \tau$ will be continuous functions $[\![\Gamma]\!] \to [\![\tau]\!]$

Compositionality: $[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$.

Soundness: for any type τ, $t \Downarrow_\tau v \Rightarrow [\![t]\!] = [\![v]\!]$.

Adequacy: for $\gamma = \texttt{bool}$ or $\texttt{nat}$, if $t \in \text{PCF}_\gamma$ and $[\![t]\!] = [\![v]\!]$ then $t \Downarrow_\gamma v$.

Recap: Types and contexts

$$[\![\texttt{nat}]\!] \stackrel{\text{def}}{=} \mathbb{N}_\bot \qquad \text{(flat domain)}$$

$$[\![\texttt{bool}]\!] \stackrel{\text{def}}{=} \mathbb{B}_\bot \qquad \text{(flat domain)}$$

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!] \qquad \text{(function domain)}$$

$$[\![\cdot]\!] = \mathbb{1} \qquad \text{(one element set)}$$

$$[\![\Gamma, x{:}\,\tau]\!] = [\![\Gamma]\!] \times [\![\tau]\!] \qquad \text{(product domain)}$$

$$
\begin{aligned}
[\![\texttt{0}]\!] &= \lambda\,\rho \in [\![\Gamma]\!].\,0 \\
[\![\texttt{true}]\!] &= \lambda\,\rho \in [\![\Gamma]\!].\,\mathrm{true} \\
[\![\texttt{false}]\!] &= \lambda\,\rho \in [\![\Gamma]\!].\,\mathrm{false} \\
[\![\texttt{succ}\,(t)]\!] &= \mathrm{succ}_\perp \circ [\![t]\!] \\
[\![\texttt{pred}\,(t)]\!] &= \mathrm{pred}_\perp \circ [\![t]\!] \\
[\![\texttt{zero?}\,(t)]\!] &= \mathrm{zero?}_\perp \circ [\![t]\!]
\end{aligned}
$$

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\begin{aligned}
[\![\texttt{if}\ b\ \texttt{then}\ t\ \texttt{else}\ t']\!] &= \mathrm{if} \circ \langle [\![b]\!], \langle [\![t]\!], [\![t']\!] \rangle \rangle \\
[\![x]\!] &= \pi_x \\
[\![t_1\ t_2]\!] &= \mathrm{eval} \circ \langle [\![t_1]\!], [\![t_2]\!] \rangle \\
[\![\texttt{fun}\ x\colon \tau.\ t]\!] &= \mathrm{cur}([\![t]\!]) \\
[\![\texttt{fix}\ f]\!] &= \mathrm{fix} \circ [\![f]\!]
\end{aligned}
$$

We define also denotation for evaluation contexts $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$ to be functions

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

such that

$$[\![\mathcal{C}[t]]\!] = [\![\mathcal{C}]\!]([\![t]\!])$$

This gives us compositionality for free:

$$[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$$

for every evaluation context $\mathcal{C}$.

Adequacy

Proposition (Soundness)

For all Pcf types τ and all closed $t, v \in \text{Pcf}_\tau$ with v a value, if $t \Downarrow_\tau v$ is derivable, then

$$[\![t]\!] = [\![v]\!] \in [\![\tau]\!]$$

Proposition (Adequacy)

For any closed Pcf term t and value v of ground type $\gamma \in \{\texttt{nat}, \texttt{bool}\}$

$$[\![t]\!] = [\![v]\!] \Rightarrow t \Downarrow_\gamma v$$

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A more serious example for $\Gamma = (f : \texttt{nat} \to \texttt{nat})$

$$[\![\texttt{fun}\, x\colon \texttt{nat}.\, (\texttt{if zero?}\,(f\ x)\ \texttt{then true else true})]\!]$$

$$\stackrel{?}{=} \quad [\![\texttt{fun}\, x\colon \texttt{nat}.\, \texttt{true}]\!]$$

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This denotational equality holds exactly when f is a total function. But there is no hope that we can decide what the first expression should evaluate to: this would mean solving the halting problem for f!

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The same proof shows adequacy for `bool`.

Formal approximation at base types

We define the **formal approximation** relation recursively on the type τ:

$$\lhd_\tau \subseteq [\![\tau]\!] \times \textsc{Pcf}_\tau$$

On base types, we let:

$$\begin{aligned} d \lhd_{\mathtt{nat}} t &\overset{\text{def}}{\Leftrightarrow} (d \in \mathbb{N} \Rightarrow t \Downarrow_{\mathtt{nat}} \underline{d}) \\ d \lhd_{\mathtt{bool}} t &\overset{\text{def}}{\Leftrightarrow} (d = \text{true} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{true}) \\ &\qquad \wedge (d = \text{false} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{false}) \end{aligned}$$

- Exactly what we asked for in the previous slide!
- Note though that $\bot \lhd_{\mathtt{nat}} t$ **for all** $t \in \textsc{Pcf}_{\mathtt{nat}}$.

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By definition! We let:

$$d \lhd_{\tau\to\tau'} t \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \forall e \in [\![\tau]\!].\ \forall u \in \textsc{Pcf}_\tau\,.\ (e \lhd_\tau u \Rightarrow d(e) \lhd_{\tau'} t\ u)$$

$$\textsc{Abs}\ \frac{\Gamma, x{:}\tau \vdash t : \tau'}{\Gamma \vdash \mathtt{fun}\, x{:}\tau.\, t : \tau \to \tau'}$$

To prove the fundamental property, we also need to talk about **open** terms.

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Parallel substitution: $\cdot \vdash \sigma : \Gamma$ assigns to each $x \in \mathrm{dom}(\Gamma)$ a term $\sigma(x) \in \textsc{Pcf}_{\Gamma(x)}$
We define also for $\rho \in [\![\Gamma]\!]$:

$$\rho \lhd_\Gamma \sigma \overset{\text{def}}{\Leftrightarrow} \forall x \in \mathrm{dom}(\Gamma), \rho(x) \lhd_{\Gamma(x)} \sigma(x)$$

For any

- context Γ and type τ
- term t such that $\Gamma \vdash t : \tau$
- environment $\rho \in [\![\Gamma]\!]$
- substitution $\cdot \vdash \sigma : \Gamma$

we have that

$$\rho \lhd_\Gamma \sigma \qquad \Rightarrow \qquad [\![t]\!](\rho) \lhd_\tau t[\sigma].$$

Corollary

For every term $t \in \textsc{Pcf}_\tau$, we have $[\![t]\!] \lhd_\tau t$.