Denotational Semantics

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Lectures for Part II CST 2025/2026

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We saw that continuous functions $f : D \to D$ on a domain have a **least (pre)fixed point**.

We saw methods for constructing domains:

$X_\perp$	$\prod_{i \in I} D_i$	$D \to E$
flat domains	product domains	function domains

The following operations on continuous functions are well-defined and continuous:

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- Composition

$$\begin{aligned} \circ &: (E \to F) \times (D \to E) \to (D \to F) \\ f \circ g &= \lambda d \in D.\ g(f(d)) \end{aligned}$$

Continuity of the fixed point operator

Proposition

The least fixed point operator $\mathrm{fix} : (D \to D) \to D$ is continuous.

Back to the introduction

The semantics of a while loop

$$[\![\texttt{while}\ X > 0\ \texttt{do}\ (Y := X * Y; X := X - 1)]\!]$$

is a fixed point of the following $F : D \to D$, where D is $(\mathbf{State} \rightharpoonup \mathbf{State})$:

$$F(w)([X \mapsto x, Y \mapsto y]) = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ w([X \mapsto x - 1, Y \mapsto x \cdot y]) & \text{if } x > 0 \end{cases}$$

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$$F(\perp) = \perp$$

This is continuous and $\mathrm{State}_\perp \to \mathrm{State}_\perp$ is a domain!

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We **can** compute explicitly

$$w_\infty[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ [X \mapsto 0, Y \mapsto (x!) \cdot y] & \text{if } x > 0 \end{cases}$$

We can **check** this agrees with the operational semantics.

Scott Induction

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Let D be a domain, $f\colon D \to D$ be continuous, and $S \subseteq D$. If the set S

(i) contains $\bot$,

(ii) is chain-closed, *i.e.* the lub of any chain of elements of S is also in S,

(iii) is stable for f, *i.e.* $f(S) \subseteq S$,

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$$\textsc{ScottInd}\ \frac{\Phi(\bot) \qquad \Phi(x) \Rightarrow \Phi(f(x)) \qquad (\forall i \in \mathbb{N}.\ \Phi(x_i)) \Rightarrow \Phi(\bigsqcup_{i\in\mathbb{N}} x_i)}{\Phi(\mathrm{fix}(f))}$$

Building chain-closed sets

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In other words, any formula built using $\vee, \wedge, \forall, \sqsubseteq, =$ and continuous f defines chain-closed subsets.

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Any formula written using:

- signature: continuous functions + constants
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Given any set I, domains D, E, functions $(f_i)_{i \in I}, g \colon D \to E$, $e \in E$,

$$\Phi(x) := \forall y \in E, (\forall i \in I, f_i(x) \sqsubseteq y) \vee g(x) = e$$

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Example: downset

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Assume $f(d) \sqsubseteq d$, *i.e.* d is a pre-fixed point of a continuous $f : D \to D$. Then $\mathrm{fix}(f) \sqsubseteq d$.

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Proof by Scott induction on $d \downarrow$

Example: partial correctness

Let $w_\infty : \mathrm{State}_\perp \to \mathrm{State}_\perp$ be the denotation of

$$\texttt{while}\ X > 0\ \texttt{do}\ (Y := X * Y; X := X - 1)$$

Recall that $w_\infty = \mathrm{fix}(F)$ where

$$F(w)(x, y) = \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

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Claim:

$$\forall x.\ \forall y \geq 0.\ w_\infty(x,y) \Downarrow \implies \pi_Y\left(w_\infty(x,y)\right) \geq 0$$

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Proof: by Scott induction!