Denotational Semantics

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Lectures for Part II CST 2025/2026

Domains and Fixed Points

Domains and Fixed Points

Posets and monotone functions

Partially ordered set

A partial order on a set D is a binary relation $\sqsubseteq$ that is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

antisymmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

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A pair $(D, \sqsubseteq)$ is called a partially ordered set, or simply poset.

Example: Partial functions $X \rightharpoonup Y$

Set: Partial functions $f : X \rightharpoonup Y$, *i.e.*
Total functions $f : A \to Y$ where $A \subseteq X$

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- $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and
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The agrees with the order that we defined on $\mathrm{State} \rightharpoonup \mathrm{State}$ to give semantics for while loops.

A function $f : D \to E$ between posets is monotone if it preserves the order:

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Monotonicity

A function $f\colon D \to E$ between posets is **monotone** if it preserves the order:

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Exercise: Check that the function $F_{b,c}$ is monotone.

Domains and Fixed Points

Least elements and least upper bounds

Least element

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Note that $\bot_S$ is always an element of S!

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- Lubs are also known as joins, supremums or limits
- The lub of S does *not* need to be in S!

Least upper bounds of chains

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The **least upper bound** of $\{d_n : n \in \mathbb{N}\}$ will be denoted by $\bigsqcup_{n \in \mathbb{N}} d_n$ or simply $\bigsqcup_n d_n$. It satisfies:

$$\text{LUB-BOUND}\ \frac{}{\forall i \in \mathbb{N}\,.\, d_i \sqsubseteq \bigsqcup_{n \in \mathbb{N}} d_n} \qquad \text{LUB-LEAST}\ \frac{\forall i \in \mathbb{N}\,.\, d_i \sqsubseteq d}{\bigsqcup_{n \in \mathbb{N}} d_n \sqsubseteq d}$$

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The lub of a sequence does *not* need to be an element of the sequence!

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- We can discard elements: $\bigsqcup_n d_n = \bigsqcup_n d_{n+k}$ for any $k \in \mathbb{N}$
- If $d_k = d_{k+1} = d_{k+2} = \ldots$ for some $k \in \mathbb{N}$, then $\bigsqcup_{n \in \mathbb{N}} d_n = d_k$

Let D be a poset and $(d_{m,n})_{m,n\in\mathbb{N}}$ be an increasing doubly-indexed sequence in D:

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Then, assuming they exist, the lubs form two chains

$$\bigsqcup_{n\in\mathbb{N}} d_{0,n} \sqsubseteq \bigsqcup_{n\in\mathbb{N}} d_{1,n} \sqsubseteq \bigsqcup_{n\in\mathbb{N}} d_{2,n} \sqsubseteq \dots$$

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Moreover, again assuming the lubs of these chains exist,

$$\bigsqcup_{m\in\mathbb{N}}\left(\bigsqcup_{n\in\mathbb{N}} d_{m,n}\right) = \bigsqcup_{n\in\mathbb{N}}\left(\bigsqcup_{m\in\mathbb{N}} d_{m,n}\right) = \bigsqcup_{k\in\mathbb{N}} d_{k,k}$$

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Show the equality by proving that they are all lubs for the set $\{d_{m,n} \,:\, m, n \in \mathbb{N}\}$

Domains and Fixed Points

Complete partial orders and domains

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We will see that these are the ingredients we need to construct least fixed points.

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$$f(x) = \begin{cases} f_n(x) & \text{if } x \in \mathbf{dom}(f_n) \text{ for some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Equivalently, the partial function f has graph the union of the graphs of the f_n.

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Beware: the definition of $\bigsqcup_n f_n$ is a partial function only if the f_n form a chain!

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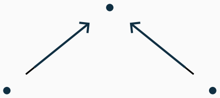
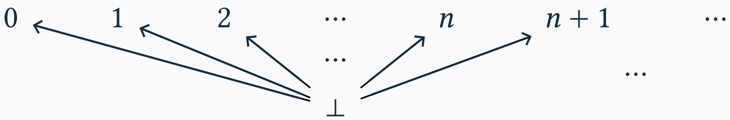


Example: The flat natural numbers $\mathbb{N}_\perp$



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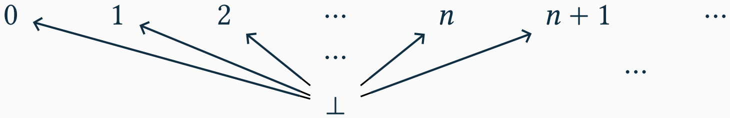


Every chain in $\mathbb{N}_\perp$ is eventually constant.

Example: Vertical natural numbers

Is $(\mathbb{N}, \leq)$ a domain?

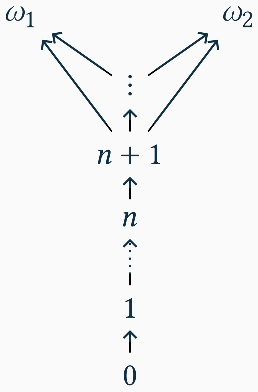
No! (Why?)

What if we add a greatest element?

$$\begin{array}{c} \omega \\ \uparrow \\ \vdots \\ n+1 \\ \uparrow \\ n \\ \uparrow \\ \vdots \\ 1 \\ \uparrow \\ 0 \end{array}$$

Yes!

What if we add two greatest elements?



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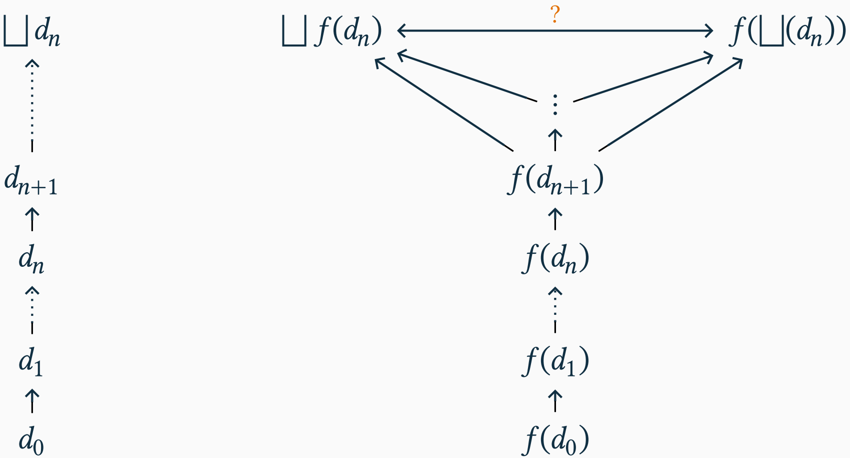
Domains and Fixed Points

Continuous functions

Monotone functions and lubs

Let $f : D \to E$ monotone function between cpos and $(d_n)_{n \in \mathbb{N}}$ a chain in D.



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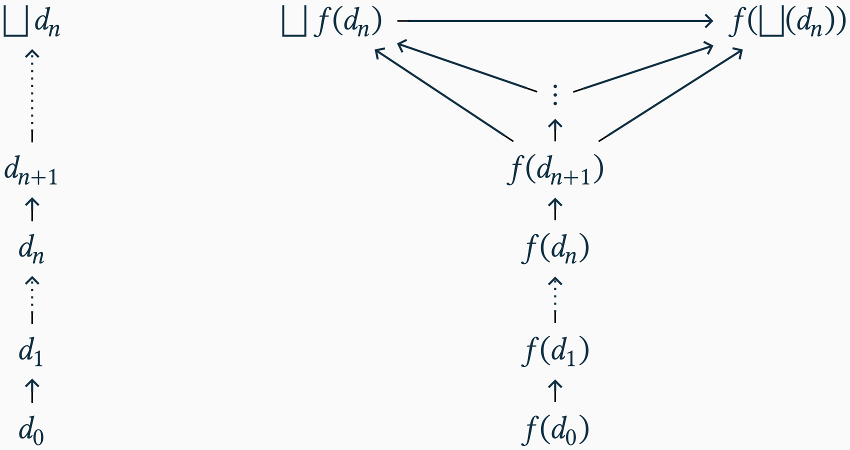

Continuity and strictness

Given two cpos D and E, a function $f: D \to E$ is called continuous if

- it is monotone, and
- it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, we have

$$\bigsqcup f(d_n) = f(\bigsqcup d_n) \qquad \text{(} \sqsubseteq \text{ is automatic)}$$

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A continuous function $f : D \to E$ between domains is strict when $f(\bot_D) = \bot_E$.

All computable functions are continuous.

Thesis

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Typical non-continuous function: "is a sequence the constant 0"? $(\mathbb{N} \rightharpoonup \mathbb{B}) \rightharpoonup \mathbb{B}$

$$
\begin{array}{llllll}
0 & 0 & \bot & \dots & & \mapsto \bot \\
0 & 0 & 0 & 0 & 1 & \dots \quad \mapsto 1 \\
\\
0 & 0 & 0 & 0 & 0 & \overline{0} \quad \mapsto 0
\end{array}
$$

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0 & 0 & 0 & 0 & 0 & \ldots & \mapsto ? \\
 & & & & & & \\
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Later in the course: we show the thesis… by giving a denotational semantics.

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Fixed points

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It is (uniquely) specified by the two properties:

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- $\mathrm{fix}(f)$ is a pre-fixed point
- To prove $\mathrm{fix}(f) \sqsubseteq d$, it is enough to show $f(d) \sqsubseteq d$.

$\mathrm{fix}(f)$ IS A FIXED POINT

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Application: If $f : D \to D$ is monotone, then $\mathrm{fix}(f)$ is a fixed point (if it exists)

$\mathrm{fix}(f)$ IS A FIXED POINT

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Application: If $f\colon D \to D$ is monotone, then $\mathrm{fix}(f)$ is a fixed point (if it exists)

$$\textsc{ASym}\ \frac{f(\mathrm{fix}(f)) \sqsubseteq \mathrm{fix}(f) \qquad \dfrac{}{\mathrm{fix}(f) \sqsubseteq f(\mathrm{fix}(f))}}{f(\mathrm{fix}(f)) = \mathrm{fix}(f)}$$

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Domains and Fixed Points

Kleene's fixed point theorem

Theorem

Let $f\colon D \to D$ be a continuous function on a domain D. Then f possesses a least pre-fixed point, given by

$$\mathrm{fix}(f) = \bigsqcup_{n\in\mathbb{N}} f^n(\bot).$$

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We need to check that:

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Question: What is $\mathrm{fix}(f)$ when f is strict?

Constructions on Domains

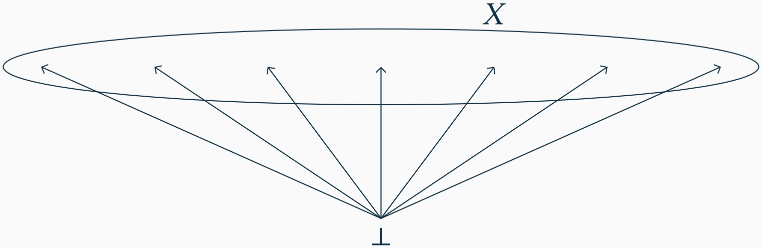
Constructions on Domains

Flat domains

Flat domain on X

The flat domain on a set X is defined by:

- its underlying set $X \uplus \{\bot\}$ *i.e.* X extended with a new element $\bot$;
- $x \sqsubseteq x'$ if either $x = \bot$ or $x = x'$.



Let $f : X \rightharpoonup Y$ be a partial function between two sets. Then

$$
\begin{aligned}
f_\perp : \quad X_\perp \quad &\to \quad Y_\perp \\
d \quad &\mapsto \begin{cases} f(d) & \text{if } d \in X \text{ and } f \text{ is defined at } d \\ \perp & \text{if } d \in X \text{ and } f \text{ is not defined at } d \\ \perp & \text{if } d = \perp \end{cases}
\end{aligned}
$$

defines a strict continuous function.

Constructions on Domains

Binary products

Binary product of posets

The **product** of two posets $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \wedge d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined componentwise:

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \wedge d_2 \sqsubseteq_2 d_2'$$

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$$\textsc{po}\times \; \frac{d_1 \sqsubseteq_1 d_1' \qquad d_2 \sqsubseteq_2 d_2'}{(d_1, d_2) \sqsubseteq (d_1', d_2')}$$

Lubs of chains are computed componentwise:

$$\bigsqcup_n (d_{1,n}, d_{2,n}) = \left(\bigsqcup_i d_{1,i}, \bigsqcup_j d_{2,j}\right).$$

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$$\bot_{D_1 \times D_2} = (\bot_{D_1}, \bot_{D_2})$$

Binary product of cpos and domains

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Therefore, products of cpos are cpos, and products of domains are domains.

Functions of two arguments

A function $f : (D \times E) \to F$ is monotone exaclty when it is monotone in each argument:

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \Rightarrow f(d,e) \sqsubseteq f(d',e)$$
$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \Rightarrow f(d,e) \sqsubseteq f(d,e').$$

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It is continuous if and only if it preserves lubs in each argument separately:

$$f(\bigsqcup_m d_m\,,\, e) = \bigsqcup_m f(d_m, e) \qquad\qquad f(d\,,\, \bigsqcup_n e_n) = \bigsqcup_n f(d, e_n).$$

Derived rules for functions of two arguments

$$\text{MON}\times \ \frac{f \text{ monotone} \qquad x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x, y) \sqsubseteq f(x', y')}$$

$$f\left(\bigsqcup_m x_m, \bigsqcup_n y_n\right) = \bigsqcup_m \bigsqcup_n f(x_m, y_n) = \bigsqcup_k f(x_k, y_k)$$

Projections and pairing

Let D_1 and D_2 be cpos (domains). The projections

$$\begin{aligned} \pi_1 : \quad D_1 \times D_2 &\to D_1 \\ (d_1, d_2) &\mapsto d_1 \end{aligned} \qquad\qquad \begin{aligned} \pi_2 : \quad D_1 \times D_2 &\to D_2 \\ (d_1, d_2) &\mapsto d_2 \end{aligned}$$

are (strict) continuous functions.

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are (strict) continuous functions.

If $f_1 : D \to D_1$ and $f_2 : D \to D_2$ are (strict) continuous functions from a cpo (domain) D, then their pairing:

$$\begin{array}{rrcl} \langle f_1, f_2 \rangle : & D & \to & D_1 \times D_2 \\ & d & \mapsto & (f_1(d), f_2(d)) \end{array}$$

is (strict) continuous.

For any domain D, the conditional function

$$
\begin{aligned}
\mathbf{if} : \quad \mathbb{B}_\perp \times (D \times D) \quad &\to \quad D \\
(x, d) \qquad &\mapsto \quad \begin{cases} \pi_1(d) & \text{if } x = \mathrm{true} \\ \pi_2(d) & \text{if } x = \mathrm{false} \\ \perp_D & \text{if } x = \perp_{\mathbb{B}} \end{cases}
\end{aligned}
$$

is (strict) continuous.

Constructions on Domains

General products

General product

The (cartesian) **product** of a family of sets $(X_i)_{i \in I}$ indexed by a set I is the set

$$\prod_{i \in I} X_i = \left\{ p \colon I \to \bigcup_{i \in I} X_i \;\middle|\; \forall i \in I \,.\, p(i) \in X_i \right\}$$

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We can think of its elements equivalently either as

- I-indexed tuples: $(\ldots, x_i, \ldots)_{i \in I}$ such that $x_i \in X_i$;
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It is equipped with projection functions (for any $i \in I$) and pairing:

$$\pi_i : \left(\prod_{i \in I} X_i \right) \to X_i \qquad \langle - \rangle_{i \in I} : \prod_{i \in I} (X \to X_i) \to \left(X \to \prod_{i \in I} X_i \right)$$

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$$p \sqsubseteq p' \overset{\text{def}}{\Leftrightarrow} \forall i \in I.\, p(i) \sqsubseteq_i p'(i).$$

Lubs of chains and bottom elements are computed componentwise, so the product of cpos (domains) is again a cpo (domain)

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The projections are (strict) continuous and the pairing of (strict) continuous functions is (strict) continuous.

Constructions on Domains

Function domains

CPO/DOMAIN OF CONTINUOUS FUNCTIONS

Given two cpos (domains) $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$\{f : D \to E \mid f \text{ is a } \textit{continuous} \text{ function}\}$$

equipped with the pointwise order:

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$$\frac{f \sqsubseteq_{D \to E} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq_E g(y)}$$

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Argumentwise least elements and lubs:

$$\bot_{D \to E}(d) = \bot_E \qquad \left(\bigsqcup_{n \in \mathbb{N}} f_n\right)(d) = \bigsqcup_{n \in \mathbb{N}} f_n(d)$$