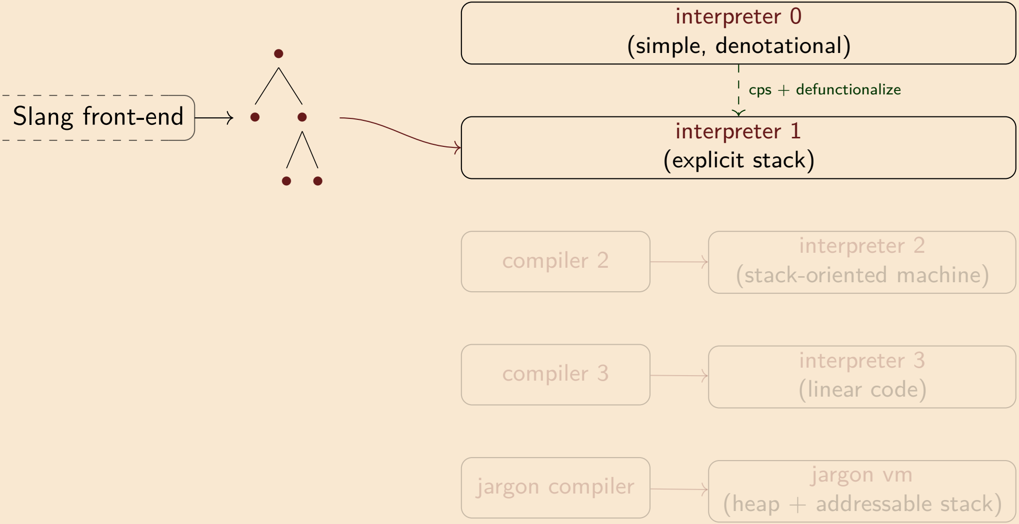
Compiler Construction

Lecture 8: CPS & defunctionalization

Jeremy Yallop

jeremy.yallop@cl.cam.ac.uk

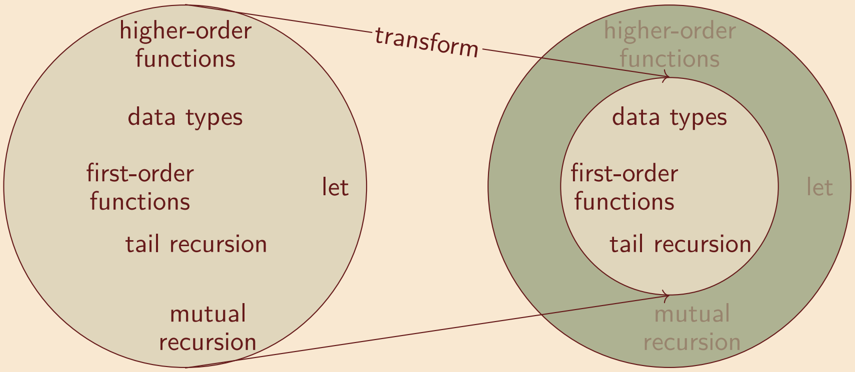
Lent 2026



interpreter 0
(simple, denotational)
cps + defunctionalize
Slang front-end
interpreter 1
(explicit stack)
compiler 2
interpreter 2
(stack-oriented machine)
compiler 3
interpreter 3
(linear code)
jargon compiler
jargon vm
(heap + addressable stack)

Source-to-source transformations

Source-to-source transformations map programs to a subset of the input language



Source-to-source transformations can show that some constructs are inessential

Interpreter 0 uses OCaml's stack and higher-order functions to implement Slang:

```
let rec interpret (e, env, store) =
match e with
...
| Lambda(x, e) → FUN (fun (v, s) →
                        interpret(e, update(env, (x, v)), s)), store
| App(e1, e2) →
   let (v2, store1) = interpret(e2, env, store) in
   let (v1, store2) = interpret(e1, env, store1) in
...
```

Our aim: **transform the interpreter** so it doesn't use these features

Illustrating on `fib`

Illustrate ideas on `fib` function

Aim: apply ideas to interpreter 0

Fibonacci function

cps

d17n

indexed recursion

Fibonacci machine

Continuation-Passing Style

Continuation-passing style: motivation

Programs in **continuation-passing style** have some useful properties:

Evaluation order is explicit

Every call is a tail call

$$\mathsf{f}\ (\mathsf{g}\ \mathsf{x}) \rightsquigarrow \mathsf{g}\ \mathsf{x}\ (\mathsf{fun}\ \mathsf{y} \rightarrow \mathsf{f}\ \mathsf{y}\ \mathsf{k})$$

Every intermediate result is named

Every *continuation* is reified

CPS conversion of `fib`

Source to source

CPS

D17n

CPS + D17n

Mutual recursion

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

let-bind function calls

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else let a = fib (m-1) in
      let b = fib (m-2) in
      a+b
```

CPS convert

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a →
      fib_cps (m-2) (fun b →
      k (a+b)))
```

CPS conversion of `fib`: details

1. Add a continuation parameter `k` to each function
2. Apply `k` to values returned by the function
3. Replace each application `let` binding with a continuation argument

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else let a = fib (m-1) in
      let b = fib (m-2) in
      a+b
```

CPS convert →

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Use the identity continuation

`fib_cps` has the type `int → (int → int) → int`.

To recover a function of type `int →int`, pass the identity continuation `fun x →x`:

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m−1) (fun a →
      fib_cps (m−2) (fun b →
      k (a+b)))

let fib_1 x = fib_cps x (fun x → x)
```

Now `fib_1` can be used like `fib`:

```
List.map fib_1 [0; 1; 2; 3; 4; 5; 6; 7; 8; 9; 10]
⇝ [1; 1; 2; 3; 5; 8; 13; 21; 34; 55; 89]
```

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Claim

For all `m` ≥ 0,
 for all `k : int -> int`,
  `fib_cps m k = k (fib m)`.

Proof

By strong induction on `m`.

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

fib_cps (m+1) k

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Claim
For all `m` ≥ 0,
  for all `k : int -> int`,
    `fib_cps m k = k (fib m)`.

Proof
By strong induction on `m`.

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

fib_cps (m+1) k

$\equiv$ (expand `fib_cps`) ...

if m+1 = 1 then k 1 else fib_cps ((m+1)-1) (fun a → fib_cps ((m+1)-2) (fun b → k (a+b)))

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`

Claim
For all `m` ≥ 0,
for all `k : int -> int`,
`fib_cps m k = k (fib m)`.

Proof
By strong induction on `m`.

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

```
if m+1 = 1 then k 1 else fib_cps ((m+1)-1) (fun a -> fib_cps ((m+1)-2) (fun b -> k (a+b)))
```

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`


```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Claim

For all `m` ≥ 0,
for all `k : int ->int`,
`fib_cps m k = k (fib m)`.

Proof

By strong induction on `m`.

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

```
if m+1 = 1 then k 1 else fib_cps ((m+1)-1) (fun a -> fib_cps ((m+1)-2) (fun b -> k (a+b)))
≡ (arithmetic) ...
if m+1 = 1 then k 1 else fib_cps m (fun a -> fib_cps (m-1) (fun b -> k (a+b)))
```

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`

Source to source

CPS

D17n

CPS + D17n

Mutual recursion

Claim

For all `m` ≥ 0,
for all `k : int → int`,
`fib_cps m k = k (fib m)`.

Proof

By strong induction on `m`.

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a →
      fib_cps (m-2) (fun b →
      k (a+b)))
```

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

```
if m+1 = 1 then k 1 else fib_cps m (fun a → fib_cps (m-1) (fun b → k (a+b)))
```

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`

Claim
For all `m` ≥ 0,
for all `k : int → int`,
`fib_cps m k = k (fib m)`.

Proof
By strong induction on `m`.

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m−1) + fib (m−2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m−1) (fun a →
      fib_cps (m−2) (fun b →
      k (a+b)))
```

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

if m+1 = 1 then k 1 else fib_cps m (fun a → fib_cps (m−1) (fun b → k (a+b)))

≡ (inductive assumption for $m-1$ and `k = (fun b → k (a+b))`) ...

if m+1 = 1 then k 1 else fib_cps m (fun a → (fun b → k (a+b)) (fib (m−1)))

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for fib

Claim
For all `m` ≥ 0,
  for all `k : int → int`,
    `fib_cps m k = k (fib m)`.

Proof
By strong induction on `m`.

```
let rec fib m =
  if m = 0 then 1
  else if m = 1 then 1
  else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
  if m = 0 then k 1
  else if m = 1 then k 1
  else fib_cps (m-1) (fun a ->
       fib_cps (m-2) (fun b ->
       k (a+b)))
```

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

```
if m+1 = 1 then k 1 else fib_cps m (fun a -> (fun b -> k (a+b)) (fib (m-1)))
```

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`

Claim
For all `m` ≥ 0,
for all `k : int → int`,
`fib_cps m k = k (fib m)`.

Proof
By strong induction on `m`.

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

if m+1 = 1 then k 1 else fib_cps m (fun a → (fun b → k (a+b)) (fib (m−1)))

≡ (inductive assumption for *m* and `k = (fun a → (fun b → k (a+b)) (fib (m-1)))`) ...

if m+1 = 1 then k 1 else (fun a → (fun b → k (a+b)) (fib (m−1))) (fib m)

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`


```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Claim

For all `m` ≥ 0,
 for all `k : int -> int`,
  `fib_cps m k = k (fib m)`.

Proof

By strong induction on `m`.

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

```
if m+1 = 1 then k 1 else (fun a -> (fun b -> k (a+b)) (fib (m-1))) (fib m)
```

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`

Claim

For all `m` ≥ 0,
for all `k : int → int`,
`fib_cps m k = k (fib m)`.

Proof

By strong induction on `m`.

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m−1) + fib (m−2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m−1) (fun a →
      fib_cps (m−2) (fun b →
      k (a+b)))
```

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:

Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.

We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

```
if m+1 = 1 then k 1 else (fun a → (fun b → k (a+b)) (fib (m−1))) (fib m)
```

$\equiv$ (beta reduction $\times 2$) ...

```
if m+1 = 1 then k 1 else k (fib m + fib (m−1))
```

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m−1) + fib (m−2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m−1) (fun a →
      fib_cps (m−2) (fun b →
      k (a+b)))
```

Claim

For all `m` ≥ 0,
for all `k : int → int`,
`fib_cps m k = k (fib m)`.

Proof

By strong induction on `m`.

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

`if m+1 = 1 then k 1 else k (fib m + fib (m−1))`

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`

Claim
For all `m` ≥ 0,
  for all `k : int -> int`,
    `fib_cps m k = k (fib m)`.

Proof
By strong induction on `m`.

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

```
if m+1 = 1 then k 1 else k (fib m + fib (m-1))
≡ (if e1 then k e2 else k e3 ≡ k (if e1 then e2 else e3)) ...
k (if m+1 = 1 then 1 else fib m + fib (m-1))
```

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Claim
For all `m` ≥ 0,
for all `k : int -> int`,
`fib_cps m k = k (fib m)`.

Proof
By strong induction on `m`.

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

`k (if m+1 = 1 then 1 else fib m + fib (m-1))`

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for fib

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Claim
For all `m` ≥ 0,
for all `k : int -> int`,
`fib_cps m k = k (fib m)`.

Proof
By strong induction on `m`.

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

k (if m+1 = 1 then 1 else fib m + fib (m−1))

≡ (definition of `fib`) ...

k (fib (m+1))

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Correctness of CPS conversion for `fib`

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Claim
For all `m` ≥ 0,
  for all `k : int -> int`,
    `fib_cps m k = k (fib m)`.

Proof
By strong induction on `m`.

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

`k (fib (m+1))`

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

```
let rec fib m =
 if m = 0 then 1
 else if m = 1 then 1
 else fib (m-1) + fib (m-2)
```

```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m-1) (fun a ->
      fib_cps (m-2) (fun b ->
      k (a+b)))
```

Claim
For all `m` ≥ 0,
for all `k : int -> int`,
`fib_cps m k = k (fib m)`.

Proof
By strong induction on `m`.

Base case (`m = 0`): `fib_cps 0 k = k 1 = k (fib 0)`.

Inductive step:
Assume for all `n` $\leq$ `m`, `k (fib n) = fib_cps n k`.
We want to show: `fib_cps (m+1) k = k (fib (m+1))`.

```
k (fib (m+1))
```

QED

NB: We approximate OCaml functions by mathematical functions, ignoring side effects etc.

Defunctionalization

Defunctionalization properties

Defunctionalized programs have some useful properties:

No higher-order functions

All values are data

$\xcancel{\text{fun } x \rightarrow e}$

All control-flow is first order

Every function is named

Defunctionalization: example

1. Add a constructor to `fn` for each `fun`
2. Replace each `fun` with its constructor
3. Add a case to `apply` for each `fun`
4. Replace each application `p x` with `apply p x`

```
let rec filter p l =
  match l with
  | [] → []
  | x :: xs →
      if p x
      then x :: filter p xs
      else filter p xs

let f l y =
    filter (fun x → x < 3) l
  @ filter (fun x → x > y) l
```

```
type fn = Lt_three
        | Gt of int

let apply fn x =
  match fn, x with
  | Lt_three, x → x < 3
  | Gt y, x → x > y

let rec filter p l =
  match l with
  | [] → []
  | x :: xs →
      if apply p x
      then x :: filter p xs
      else filter p xs

let f l y =
    filter Lt_three l
  @ filter (Gt y) l
```

Defunctionalization: example

1. Add a constructor to `fn` for each `fun`
2. Replace each `fun` with its constructor
3. Add a case to `apply` for each `fun`
4. Replace each application `p x` with `apply p x`

```
let rec filter p l =
  match l with
  | [] → []
  | x :: xs →
      if p x
      then x :: filter p xs
      else filter p xs

let f l y =
    filter (fun x → x < 3) l
  @ filter (fun x → x > y) l
```

```
type fn = Lt_three
        | Gt of int

let apply fn x =
  match fn, x with
  | Lt_three, x → x < 3
  | Gt y, x → x > y

let rec filter p l =
  match l with
  | [] → []
  | x :: xs →
      if apply p x
      then x :: filter p xs
      else filter p xs

let f l y =
    filter Lt_three l
  @ filter (Gt y) l
```

Combining CPS & defunctionalization

Defunctionalizing fib_cps


```
let rec fib_cps m k =
 if m = 0 then k 1
 else if m = 1 then k 1
 else fib_cps (m−1) (fun a → (* K1 *)
      fib_cps (m−2) (fun b → (* K2 *)
      k (a+b)))

let fib_1 x = fib_cps x (fun x → x) (* ID *)
```

To defunctionalize `fib_cps`, define a constructor for each `fun`:

```
type cont = ID | K1 of int * cont | K2 of int * cont
```

Constructor arguments are free variables, and we treat `k2` as free in `k1`:

```
let k2 = fun a b → k (a+b)
let k1 = fun a → fib_cps (m−2) (k2 a)
```

Defunctionalized fib_cps

Now define an apply function of type `cont → int → int`

```
type cont = ID | K1 of int * cont | K2 of int * cont

let rec apply_cont k v = match k, v with
| ID, a → a
| K1 (m, k), a → fib_cps_defun (m−2) (K2 (a, k))
| K2 (a, k), b → apply_cont k (a+b)
```

and call `apply_cont` at every application of a continuation:

```
and fib_cps_defun m k =
  if m = 0 then apply_cont k 1
  else if m = 1 then apply_cont k 1
  else fib_cps_defun (m −1) (K1 (m, k))

let fib_2 m = fib_cps_defun m ID
```

Correctness of `fib_cps` defunctionalization

Claim
Let <c> represent a continuation c : int → int constructed by fib_cps.
Then

apply_cont <c> m = c m

and

fib_cps n c = fib_cps_defun n <c>

(**Proof** left as an exercise)

Observation: continuations have list (stack) structure

```
type int_list =
    NIL
  | CONS of int * int_list
```

```
type cont =
    ID                   (* 'Nil' *)
  | K1 of int * cont     (* 'Cons' *)
  | K2 of int * cont     (* 'Cons' *)
```

Idea: replace cont with standard lists:

```
type tag = SUB2 of int   (* K1: k (a+b) *)
         | PLUS of int   (* K2: fib_cps (m-2) (k2 a) *)

type tag_list_cont = tag list
```

fib_cps_defun revisited, using lists for continuations


```
type tag = SUB2 of int | PLUS of int
type tag_list_cont = tag list

let rec apply_tag_list_cont k v = match k, v with
  | [], a → a
  | SUB2 m :: k, a → fib_cps_defun_tags (m−2) (PLUS a :: k)
  | PLUS a :: k, b → apply_tag_list_cont k (a+b)

and fib_cps_defun_tags m k =
  if m = 0 then apply_tag_list_cont k 1
  else if m = 1 then apply_tag_list_cont k 1
  else fib_cps_defun_tags (m−1) (SUB2 m :: k)

let fib_3 m = fib_cps_defun_tags m []
```

Mutual recursion

Mutual recursion ⇝ single recursion

Mutual recursion can be eliminated using **indexing**.

Given a set of mutually-recursive functions:

```
let rec is_even n = n = 0 || is_odd (n - 1)
    and is_odd n = n <> 0 && is_even (n - 1)
```

define an index datatype with one constructor for each function:

```
type eo = Even | Odd
```

and define a function that maps an index argument to a corresponding body:

```
let rec is f n =
  match f with
  | Even -> n = 0 || is Odd (n - 1)
  | Odd -> n <> 0 && is Even (n - 1)
```

Mutual recursion ⇝ single recursion for `fib`


```
type state_type =
| FIB (* for right-hand-sides starting with fib_ *)
| APP (* for right-hand-sides starting with apply_ *)

type state = (state_type * int * tag_list_cont) → int

(* eval acts as either apply_tag_list_cont or fib_cps_defun_tags *)
let rec eval = function
| FIB, 0, k → eval (APP, 1, k)
| FIB, 1, k → eval (APP, 1, k)
| FIB, m, k → eval (FIB, m-1, SUB2 m :: k)
| APP, a, SUB2 m :: k → eval (FIB, m-2, PLUS a :: k)
| APP, b, PLUS a :: k → eval (APP, a+b, k)
| APP, a, [] → a

let fib_4 m = eval (FIB, m, [])
```

Eliminate tail recursion to obtain *The Fibonacci Machine*


```
(* step : state → state *)
let step = function
| FIB, 0, k → (APP, 1, k)
| FIB, 1, k → (APP, 1, k)
| FIB, m, k → (FIB, m−1, SUB2 m :: k)
| APP, a, SUB2 m :: k → (FIB, m−2, PLUS a :: k)
| APP, b, PLUS a :: k → (APP, a+b, k)
| _ → failwith "step : runtime error!"

let rec driver = function (* clearly tail recursive! *)
| APP, a, [] → a
| state → driver (step state)

(* fib_5 : int → int *)
let fib_5 m = driver (FIB, m, [])
```

(This version makes the tail-recursive structure very explicit.)

Tracing of fib_5 4

```
FIB  4  []
FIB  3  [SUB2 4]
FIB  2  [SUB2 3; SUB2 4]
FIB  1  [SUB2 2; SUB2 3; SUB2 4]
APP  1  [SUB2 2; SUB2 3; SUB2 4]
FIB  0  [PLUS 1; SUB2 3; SUB2 4]
APP  1  [PLUS 1; SUB2 3; SUB2 4]
APP  2  [SUB2 3; SUB2 4]
FIB  1  [PLUS 2; SUB2 4]
APP  1  [PLUS 2; SUB2 4]
APP  3  [SUB2 4]
FIB  2  [PLUS 3]
FIB  1  [SUB2 2; PLUS 3]
APP  1  [SUB2 2; PLUS 3]
FIB  0  [PLUS 1; PLUS 3]
APP  1  [PLUS 1; PLUS 3]
APP  2  [PLUS 3]
APP  5  []
```

```
let step = function
| FIB, 0, k → (APP, 1, k)
| FIB, 1, k → (APP, 1, k)
| FIB, m, k → (FIB, m−1, SUB2 m :: k)
| APP, a, SUB2 m :: k → (FIB, m−2, PLUS a :: k)
| APP, b, PLUS a :: k → (APP, a+b, k)
```

Current state

We turned the recursive `fib` into a function that uses no OCaml stack space

The transformed `fib` function carries its own stack as an extra argument

We transformed `fib` incrementally, with each step easily proved correct

Next time: application to **interpreter 0**