Compiler Construction

Lecture 5: Foundations of LR parsing

Jeremy Yallop

jeremy.yallop@cl.cam.ac.uk

Lent 2026

Derivations

Recap: example grammars

G_1

$E \to E + E \mid E * E \mid (E) \mid \text{id}$

G_2 (from G_1: eliminate ambiguity)

$E \to E + T \mid T$
$T \to T * F \mid F$
$F \to (E) \mid \text{id}$

G_3 (from G_2: eliminate left recursion)

$E \to T\,E'$
$E' \to +\,T\,E' \mid \epsilon$
$T \to F\,T'$
$T' \to *\,F\,T' \mid \epsilon$
$F \to (\,E\,) \mid \text{id}$

G_1' (from G_1: add S)

$S \to E$
$E \to E + E \mid E * E \mid (E) \mid \text{id}$

G_2' (from G_2: add S, \$; from G_1': eliminate ambiguity)

$S \to E\,\$$
$E \to E + T \mid T$
$T \to T * F \mid F$
$F \to (E) \mid \text{id}$

Today's running example (G_2')

G_3' (from G_3: add S, \$; from G_2': eliminate left recursion)

$S \to E\,\$$
$E \to T\,E'$
$E' \to +\,T\,E' \mid \epsilon$
$T \to F\,T'$
$T' \to *\,F\,T' \mid \epsilon$
$F \to (\,E\,) \mid \text{id}$

Leftmost vs rightmost derivations

Leftmost derivation step:

$$wA\alpha \;\Rightarrow_{lm}\; w\beta\alpha$$

(basis of top-down (L**L**) parsing)

Rightmost derivation step:

$$\alpha Aw \;\Rightarrow_{rm}\; \alpha\beta w$$

(basis of bottom-up (L**R**) parsing)

where

$$w \in T*$$
$$\alpha, \beta \in (N \cup T)^*$$
$$A \rightarrow \beta \in P$$

Bottom-up parsers perform the derivation in reverse

Rightmost derivation

$$
\begin{aligned}
S &\Rightarrow_{rm} \mathbf{E} \\
&\Rightarrow_{rm} \mathbf{T} \\
&\Rightarrow_{rm} \mathbf{F} \\
&\Rightarrow_{rm} (\mathbf{E}) \\
&\Rightarrow_{rm} (E+\mathbf{T}) \\
&\Rightarrow_{rm} (E+\mathbf{F}) \\
&\Rightarrow_{rm} (\mathbf{E}+y) \\
&\Rightarrow_{rm} (\mathbf{T}+y) \\
&\Rightarrow_{rm} (\mathbf{F}+y) \\
&\Rightarrow_{rm} (x+y)
\end{aligned}
$$

flip!

Reversed rightmost derivation

$$
\begin{aligned}
(x+y) &\Leftarrow \\
(\mathbf{F}+y) &\Leftarrow \\
(\mathbf{T}+y) &\Leftarrow \\
(\mathbf{E}+y) &\Leftarrow \\
(E+\mathbf{F}) &\Leftarrow \\
(E+\mathbf{T}) &\Leftarrow \\
(\mathbf{E}) &\Leftarrow \\
\mathbf{F} &\Leftarrow \\
\mathbf{T} &\Leftarrow \\
\mathbf{E} &\Leftarrow S
\end{aligned}
$$

parsing direction

Backwards derivation $\rightsquigarrow$ stack machine execution

G_2'

$$S \to E\,\$ \qquad T \to T * F \mid F$$
$$E \to E + T \mid T \qquad F \to (E) \mid \mathrm{id}$$

$$\begin{array}{rcl}
(x+y) & \Leftarrow & \\
(F+y) & \Leftarrow & \\
(T+y) & \Leftarrow & \\
(E+y) & \Leftarrow & \\
(E+F) & \Leftarrow & \\
(E+T) & \Leftarrow & \\
(E) & \Leftarrow & \\
F & \Leftarrow & \\
T & \Leftarrow & \\
E & \Leftarrow & S
\end{array}$$

— View reversed derivation as a stack machine ⟶

stack	input
$\$$	$(x+y)\$$
$\$(F$	$+y)\$$
$\$(T$	$+y)\$$
$\$(E$	$+y)\$$
$\$(E+F$	$)\$$
$\$(E+T$	$)\$$
$\$(E)$	$\$$
$\$F$	$\$$
$\$T$	$\$$
$\$E$	$\$$
$\$S$	$\$$

Formalisation

LR parser configurations

An **LR parser configuration** has the form

$$\$\alpha, w\$$$

stack α

remaining input w

The configuration is **valid** when there exists a rightmost derivation of the form

$$S \Rightarrow^*_{rm} \alpha\, w$$

(NB: stacks now grow *rightwards*.)

There may be **lots of possible steps** from each configuration.

Suppose:

$$\alpha A w \;\Rightarrow_{rm}\; \alpha \beta B z w$$

One possible step between configurations replaces $\beta B z$ with A on top of the stack:

$$\$\alpha\beta B z, w\$ \;\xrightarrow[A \to \beta B z]{\text{reduce}}\; \$\alpha A, w\$$$

This action is called a **reduction** using production $A \to \beta B z$.

Reductions are not sufficient

Suppose we have the derivation:

$$\begin{aligned} & \alpha A w \\ \Rightarrow_{rm} \quad & \alpha\beta B z w \quad (\text{using } A \to \beta B z) \\ \Rightarrow_{rm} \quad & \alpha\beta\gamma z w \quad (\text{using } B \to \gamma) \end{aligned}$$

The reverse simulation gets stuck:

$$\begin{aligned} & \$\alpha\beta\gamma, zw\$ \\ \xrightarrow[B\to\gamma]{\text{reduce}} \quad & \$\alpha\beta B, zw\$ \\ \xrightarrow[???]{} \quad & ??? \end{aligned}$$

We have βB on top of the stack, but we want $\beta B z$ on top of the stack.

We also need *shift*

A **shift** action shifts a terminal onto the stack.

$$
\begin{aligned}
& \alpha A w \\
\Rightarrow_{rm} \quad & \alpha\beta B z w \quad (\text{using } A \to \beta B z) \\
\Rightarrow_{rm} \quad & \alpha\beta\gamma z w \quad (\text{using } B \to \gamma)
\end{aligned}
$$

$$
\begin{aligned}
& \$\alpha\beta\gamma, zw\$ \\
\xrightarrow[B\to\gamma]{\text{reduce}} \; & \$\alpha\beta B, zw\$ \\
\xrightarrow[z]{\textbf{shift}} \; & \$\alpha\beta B z, w\$ \\
\xrightarrow[A\to\beta B z]{\text{reduce}} \; & \$\alpha A, w\$
\end{aligned}
$$

Q: *How do we know when to stop shifting?*
(e.g. here we don't want to shift w)

Derivation

$$\begin{array}{lll} & \alpha B w A z & \\ \Rightarrow_{rm} & \alpha B w y z & (\text{using } A \to y) \\ \Rightarrow_{rm} & \alpha \gamma w y z & (\text{using } B \to \gamma) \end{array}$$

Our parser's possible actions:

$$\begin{array}{ll} & \$\alpha\gamma,\ wyz\$ \\ \xrightarrow[B\to\gamma]{\text{reduce}} & \$\alpha B,\ wyz\$ \\ \xrightarrow[w]{\text{shift}} & \$\alpha Bw,\ yz\$ \\ \xrightarrow[y]{\text{shift}} & \$\alpha Bwy,\ z\$ \\ \xrightarrow[A\to y]{\text{reduce}} & \$\alpha BwA,\ z\$ \end{array}$$

Again: *how do we know when to reduce and when to stop shifting?*

Shift & reduce

Shift and reduce are sufficient

It appears that if we have a derivation

$$S \Rightarrow^*_{rm} w$$

we can always "replay" it in reverse using shift/reduce actions:

$$\$, w\$ \rightarrow^* \$S, \$$$

i.e. **shift and reduce are sufficient**.

However, we have used the desired derivation to guide the "replay".
When parsing there is no derivation available in advance.
So **our parser is non-deterministic**: it must *guess* what the future holds.

Replay parsing of $(x + y)$ using shift/reduce actions

$S \to E\,\$$
$E \to E + T \mid T$
$T \to T * F \mid F$
$F \to (E) \mid \text{id}$

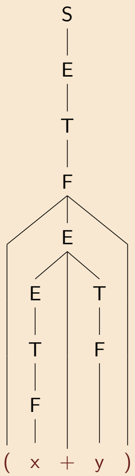
top of stack X — next input token a

stack	input	action$[X, a]$
$\$$	$(x + y)\$$	shift (
$\$($	$x + y)\$$	shift x
$\$(x$	$+y)\$$	reduce $F \to id$
$\$(F$	$+y)\$$	reduce $T \to F$
$\$(T$	$+y)\$$	reduce $E \to T$
$\$(E$	$+y)\$$	shift $+$
$\$(E+$	$y)\$$	shift y
$\$(E + y$	$)\$$	reduce $F \to id$

stack	input	action$[X, a]$
$\$(E + F$	$)\$$	reduce $T \to F$
$\$(E + T$	$)\$$	reduce $E \to E + T$
$\$(E$	$)\$$	shift)
$\$(E)$	$\$$	reduce $F \to (E)$
$\$F$	$\$$	reduce $T \to F$
$\$T$	$\$$	reduce $E \to T$
$\$E$	$\$$	reduce $S \to E$
$\$S$	$\$$	accept!

Aside: shift and reduce construct trees

shift (
shift x
reduce $F \to id$
reduce $T \to F$
reduce $E \to T$
shift $+$
shift y
reduce $F \to id$
reduce $T \to F$
reduce $E \to E + T$
shift)
reduce $F \to (E)$
reduce $T \to F$
reduce $E \to T$
reduce $S \to E$



$S \to E\,\$$
$E \to E + T$
$E \to T$
$T \to T * F$
$T \to F$
$F \to (E)$
$F \to \text{id}$

How do we decide when to shift or reduce?

Suppose $A \to \beta\gamma$ is a production. In the configuration

$$\$\alpha\beta\gamma, x\$$$

we *might* want to reduce with $A \to \beta\gamma$.

However, if we have

$$\$\alpha\beta, x\$$$

we *might* want to continue parsing,
hoping to eventually have $\beta\gamma$ on top of the stack,
so that we can then reduce to A.

Items

LR(0) items

LR(0) items record how much of a production's RHS is already parsed.

For every grammar production

$$A \rightarrow \beta\gamma \qquad (\beta, \gamma \in (N \cup T)^*)$$

there is an LR(0) item

$$A \rightarrow \beta \bullet \gamma$$

$A \rightarrow \beta \bullet \gamma$ means: we've parsed input x derivable from β
we *might* next see input derivable from γ

LR(0) items for G_2'

$S \to \bullet E$
$S \to E \bullet$

$E \to \bullet E + T$
$E \to E \bullet + T$
$E \to E + \bullet T$
$E \to E + T \bullet$

$E \to \bullet T$
$E \to T \bullet$

$T \to \bullet T * F$
$T \to T \bullet * F$
$T \to T * \bullet F$
$T \to T * F \bullet$

$T \to \bullet F$
$T \to F \bullet$

$F \to \bullet (E)$
$F \to (\bullet E)$
$F \to (E \bullet)$
$F \to (E) \bullet$

$F \to \bullet \mathrm{id}$
$F \to \mathrm{id} \bullet$

Derivations
Formalisation
Shift & reduce
Items
Key idea

Definition: item $A \to \beta\bullet\gamma$ is **valid for** $\phi\beta$ if there exists a derivation

$$\begin{aligned} & S \\ \Rightarrow^*_{rm} \quad & \phi A w \\ \Rightarrow_{rm} \quad & \phi\beta\gamma w \end{aligned}$$

If

$$A \to \beta\bullet\gamma \text{ is valid for } \phi\beta$$

then

$$\text{parser can use } A \to \beta\bullet\gamma \text{ as a guide in configuration } \$\phi\beta, w\$$$

Using items as parsing guides

Suppose parser is in config $\$\phi\beta, cz\$$ and $A \to \beta \bullet c\gamma$ is valid for $\phi\beta$.
Then we *might* shift c onto the stack:

$$\$\phi\beta, cz\$ \xrightarrow{\text{shift } c} \$\phi\beta c, z\$$$

Suppose parser is in config $\$\phi\beta, z\$$ and $A \to \beta \bullet$ is valid for $\phi\beta$.
Then we *might* perform a reduction

$$\$\phi\beta, z\$ \xrightarrow[A \to \beta]{\text{reduce}} \$\phi A, z\$$$

Using items as parsing guides (continued)

Suppose parser is in valid config $\$\phi\beta, w\$$ (so $S \Rightarrow^*_{rm} \phi\beta w$).

Suppose $A \to \beta\bullet\gamma$ is valid for $\phi\beta$.

Then γ *might* capture the future of our parse (the past of the derivation).

That is, it *might* be that

$$\begin{array}{ll} & S \\ \Rightarrow^*_{rm} & \phi A x \\ \Rightarrow_{rm} & \phi\beta\gamma x \\ \Rightarrow^*_{rm} & \phi\beta y x \quad = \quad \phi\beta w \end{array}$$

If so, our parser *might* proceed like so:

$$\begin{array}{ll} & \$\phi\beta, yx\$ \quad = \quad \$\phi\beta, w\$ \\ \to^* & \$\phi\beta\gamma, x\$ \\ \xrightarrow{\text{reduce}} & \$\phi A, x\$ \end{array}$$

i.e. our parser could guess that γ will derive a prefix of the remaining input w.

Key idea

The key idea in LR parsing

Idea: Augment shift/reduce parser so that in every configuration $\$\alpha, w\$$ it can derive the set of items valid for α.

At each step parser can (non-deterministically) select an item to use as a guide.

NFA with LR(0) items as states

$$A \to \beta \bullet c\gamma \xrightarrow{\quad c \quad} A \to \beta c \bullet \gamma$$

$$A \to \beta \bullet B\gamma \xrightarrow{\quad B \quad} A \to \beta B \bullet \gamma$$

$$A \to \beta \bullet B\gamma \xrightarrow{\quad \epsilon \quad} B \to \bullet \alpha_i$$

Initial state is item constructed from unique starting production, e.g.:

$$q0 \quad = \quad S \to \bullet E$$

Let δ_G be the transition function of this NFA (and every state be accepting).

Theorem:

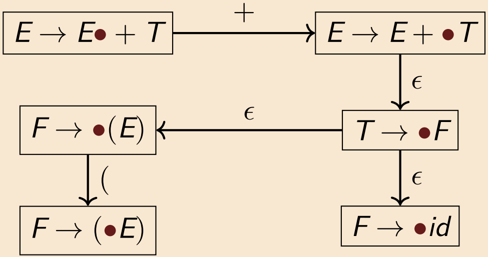
$$A \to \beta \bullet \gamma \in \delta_G(q_0, \phi\beta)$$

if and only if

$A \to \beta \bullet \gamma$ is valid for $\phi\beta$.

(NB: The set of words $\phi\beta$ is a *regular* language!)

A few NFA transitions for grammar G_2



A non-deterministic LR parsing algorithm

$c :=$ NextToken()

while true:

  $\alpha :=$ the stack

  if $A \to \beta\bullet c\gamma \in \delta_G(q_0, \alpha)$
    then SHIFT c; $c :=$ NextToken()

  if $A \to \beta\bullet \in \delta_G(q_0, \alpha)$
    then REDUCE via $A \to \beta$

  if $S \to \beta\bullet \in \delta_G(q_0, \alpha)$
    then ACCEPT (if no more input)

  if none of the above
    then ERROR

non-deterministic since
  multiple "if" conditions can be true
& multiple items can match any condition

Next time: SLR(1) & LR(1)