

# Type Systems

## Lecture 5: System F and Church Encodings

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# System F, The Girard-Reynolds Polymorphic Lambda Calculus

Types  $A ::= \alpha \mid A \rightarrow B \mid \forall \alpha. A$

Terms  $e ::= x \mid \lambda x : A. e \mid e e \mid \Lambda \alpha. e \mid e A$

Type Contexts  $\Theta ::= \cdot \mid \Theta, \alpha$

Term Contexts  $\Gamma ::= \cdot \mid \Gamma, x : A$

Judgement	Notation
Well-formedness of types	$\Theta \vdash A \text{ type}$
Well-formedness of term contexts	$\Theta \vdash \Gamma \text{ ctx}$
Term typing	$\Theta; \Gamma \vdash e : A$

# Well-formedness of Types

$$\frac{\alpha \in \Theta}{\Theta \vdash \alpha \text{ type}}$$

$$\frac{\Theta \vdash A \text{ type} \quad \Theta \vdash B \text{ type}}{\Theta \vdash A \rightarrow B \text{ type}}$$

$$\frac{\Theta, \alpha \vdash A \text{ type}}{\Theta \vdash \forall \alpha. A \text{ type}}$$

- Judgement  $\Theta \vdash A \text{ type}$  checks if a type is well-formed
- Because types can have free variables, we need to check if a type is well-scoped

# Well-formedness of Term Contexts

Term Variable Contexts  $\Gamma ::= \cdot \mid \Gamma, x : A$

$$\frac{}{\Theta \vdash \cdot \text{ ctx}} \qquad \frac{\Theta \vdash \Gamma \text{ ctx} \quad \Theta \vdash A \text{ type}}{\Theta \vdash \Gamma, x : A \text{ ctx}}$$

- Judgement  $\Theta \vdash \Gamma \text{ type}$  checks if a *term context* is well-formed
- We need this because contexts associate variables with types, and types now have a well-formedness condition

# Typing for System F

$$\frac{x : A \in \Gamma}{\Theta; \Gamma \vdash x : A}$$

$$\frac{\Theta \vdash A \text{ type} \quad \Theta; \Gamma, x : A \vdash e : B}{\Theta; \Gamma \vdash \lambda x : A. e : A \rightarrow B}$$

$$\frac{\Theta; \Gamma \vdash e : A \rightarrow B \quad \Theta; \Gamma \vdash e' : A}{\Theta; \Gamma \vdash e e' : B}$$

$$\frac{\Theta, \alpha; \Gamma \vdash e : B}{\Theta; \Gamma \vdash \Lambda \alpha. e : \forall \alpha. B}$$

$$\frac{\Theta; \Gamma \vdash e : \forall \alpha. B \quad \Theta \vdash A \text{ type}}{\Theta; \Gamma \vdash e A : \boxed{[A/\alpha]B}}$$

- Note the presence of substitution in the typing rules!

Values  $v ::= \lambda x : A. e \mid \Lambda \alpha. e$

$$\frac{e_0 \rightsquigarrow e'_0}{e_0 e_1 \rightsquigarrow e'_0 e_1} \text{ CONGFUN}$$

$$\frac{e_1 \rightsquigarrow e'_1}{v_0 e_1 \rightsquigarrow v_0 e'_1} \text{ CONGFUNARG}$$

$$\frac{}{(\lambda x : A. e) v \rightsquigarrow [v/x]e} \text{ FUNEVAL}$$

$$\frac{e \rightsquigarrow e'}{e A \rightsquigarrow e' A} \text{ CONGFORALL}$$

$$\frac{}{(\Lambda \alpha. e) A \rightsquigarrow [A/\alpha]e} \text{ FORALLEVAL}$$

# The Bookkeeping

- Ultimately, we want to prove type safety for System F
- However, the introduction of type variables means that a fair amount of additional administrative overhead is introduced
- This may look intimidating on first glance, BUT really it's all just about keeping track of the free variables in types
- As a result, none of these lemmas are hard – just a little tedious

# Structural Properties and Substitution for Types

1. (Type Weakening) If  $\Theta, \Theta' \vdash A$  type then  $\Theta, \beta, \Theta' \vdash A$  type.
2. (Type Exchange) If  $\Theta, \beta, \gamma, \Theta' \vdash A$  type then  $\Theta, \gamma, \beta, \Theta' \vdash A$  type
3. (Type Substitution) If  $\Theta \vdash A$  type and  $\Theta, \alpha \vdash B$  type then  $\Theta \vdash [A/\alpha]B$  type
  - These follow the pattern in lecture 1, except with fewer cases
  - Needed to handle the type application rule



# Structural Properties and Substitutions for Contexts

1. (Context Weakening) If  $\Theta, \Theta' \vdash \Gamma \text{ ctx}$  then  $\Theta, \alpha, \Theta' \vdash \Gamma \text{ ctx}$
2. (Context Exchange) If  $\Theta, \beta, \gamma, \Theta' \vdash \Gamma \text{ ctx}$  then  $\Theta, \gamma, \beta, \Theta' \vdash \Gamma \text{ ctx}$
3. (Context Substitution) If  $\Theta \vdash A \text{ type}$  and  $\Theta, \alpha \vdash \Gamma \text{ type}$  then  $\Theta \vdash [A/\alpha]\Gamma \text{ type}$ 
  - This just lifts the type-level structural properties to contexts
  - Proof via induction on derivations of  $\Theta \vdash \Gamma \text{ ctx}$

**Regularity:** If  $\Theta \vdash \Gamma$  ctx and  $\Theta; \Gamma \vdash e : A$  then  $\Theta \vdash A$  type

**Proof:** By induction on the derivation of  $\Theta; \Gamma \vdash e : A$

- This just says if typechecking succeeds, then it found a well-formed type

# Structural Properties and Substitution of Types into Terms

- (Type Weakening of Terms) If  $\Theta, \Theta' \vdash \Gamma \text{ ctx}$  and  $\Theta, \Theta'; \Gamma \vdash e : A$  then  $\Theta, \alpha, \Theta'; \Gamma \vdash e : A$ .
- (Type Exchange of Terms) If  $\Theta, \alpha, \beta, \Theta' \vdash \Gamma \text{ ctx}$  and  $\Theta, \alpha, \beta, \Theta'; \Gamma \vdash e : A$  then  $\Theta, \beta, \alpha, \Theta'; \Gamma \vdash e : A$ .
- (Type Substitution of Terms) If  $\Theta, \alpha \vdash \Gamma \text{ ctx}$  and  $\Theta \vdash A \text{ type}$  and  $\Theta, \alpha; \Gamma \vdash e : B$  then  $\Theta; [A/\alpha]\Gamma \vdash [A/\alpha]e : [A/\alpha]B$ .

# Structural Properties and Substitution for Term Variables

- (Weakening for Terms) If  $\Theta \vdash \Gamma, \Gamma' \text{ ctx}$  and  $\Theta \vdash B \text{ type}$  and  $\Theta; \Gamma, \Gamma' \vdash e : A$  then  $\Theta; \Gamma, y : B, \Gamma' \vdash e : A$
- (Exchange for Terms) If  $\Theta \vdash \Gamma, y : B, z : C, \Gamma' \text{ ctx}$  and  $\Theta; \Gamma, y : B, z : C, \Gamma' \vdash e : A$ , then  $\Theta; \Gamma, z : C, y : B, \Gamma' \vdash e : A$
- (Substitution of Terms) If  $\Theta \vdash \Gamma, x : A \text{ ctx}$  and  $\Theta; \Gamma \vdash e : A$  and  $\Theta; \Gamma, x : A \vdash e' : B$  then  $\Theta; \Gamma \vdash [e/x]e' : B$ .

- There are two sets of substitution theorems, since there are two contexts
- We also need to assume well-formedness conditions
- But proofs are all otherwise similar to the simply-typed case

**Progress:** If  $\cdot; \cdot \vdash e : A$  then either  $e$  is a value or  $e \rightsquigarrow e'$ .

**Type preservation:** If  $\cdot; \cdot \vdash e : A$  and  $e \rightsquigarrow e'$  then  $\cdot; \cdot \vdash e' : A$ .

# Progress: Big Lambdas

Proof by induction on derivations:

$$\frac{\overbrace{.; \cdot \vdash e : \forall \alpha. B}^{(2)} \quad \overbrace{\cdot \vdash A \text{ type}}^{(3)}}{\cdot; \cdot \vdash e A : [A/\alpha]B}$$

(1)  $\cdot; \cdot \vdash e A : [A/\alpha]B$

Assumption

(4)  $e \rightsquigarrow e'$  or  $e$  is a value

Induction on (2)

Case on (4)

(5) Case  $e \rightsquigarrow e'$  :

(6)  $e A \rightsquigarrow e' A$

by CONGFORALL on (5)

(7) Case  $e$  is a value:

(8)  $e = \Lambda \alpha. e'$

By canonical forms on (2)

(9)  $(\Lambda \alpha. e') A \rightsquigarrow [A/\alpha]e$

By FORALLEVAL

# Preservation: Big Lambdas

By induction on the derivation of  $e \rightsquigarrow e'$ :

$$(1) \quad \frac{}{(\Lambda\alpha. e) A \rightsquigarrow [A/\alpha]e} \text{FORALLEVAL} \quad \text{Assumption}$$

$$(2) \quad \frac{\overbrace{\alpha; \cdot \vdash e : B}^{(3)} \quad \overbrace{\cdot \vdash A \text{ type}}^{(4)}}{\cdot; \cdot \vdash \Lambda\alpha. e : \forall\alpha. B} \quad \text{Assumption}$$

$$(5) \quad \cdot; \cdot \vdash [A/\alpha]e : [A/\alpha]B \quad \text{Type subst. on (3), (4)}$$



# Church Encodings: Representing Data with Functions

- System has the types  $\forall\alpha. A$  and  $A \rightarrow B$
- No booleans, sums, numbers, tuples or anything else
- Seemingly, there is no data in this calculus
- Surprisingly, it is unnecessary!
- Discovered in 1941 by Alonzo Church
- The idea:
  1. Data is used to make choices
  2. Based on the choice, you perform different results
  3. So we can encode data as functions which take different possible results, and return the right one

# Church Encodings: Booleans

$$\frac{}{\Gamma \vdash \text{true} : \text{bool}} \quad \frac{}{\Gamma \vdash \text{false} : \text{bool}} \quad \frac{\Gamma \vdash e : \text{bool} \quad \Gamma \vdash e' : X \quad \Gamma \vdash e'' : X}{\Gamma \vdash \text{if } e \text{ then } e' \text{ else } e'' : X}$$

- Boolean type has two values, true and false
- Conditional switches between two  $X$ 's based on  $e$ 's value

Type		Encoding
bool	$\triangleq$	$\forall \alpha. \alpha \rightarrow \alpha \rightarrow \alpha$
True	$\triangleq$	$\Lambda \alpha. \lambda x : \alpha. \lambda y : \alpha. x$
False	$\triangleq$	$\Lambda \alpha. \lambda x : \alpha. \lambda y : \alpha. y$
if $e$ then $e'$ else $e'' : X$	$\triangleq$	$e X e' e''$

## Evaluating Church conditionals

$$\begin{aligned}\text{if true then } e' \text{ else } e'' : A &= \text{true } A \ e' \ e'' \\ &= (\Lambda \alpha. \lambda x : \alpha. \lambda y : \alpha. x) A \ e' \ e'' \\ &= (\lambda x : A. \lambda y : A. x) \ e' \ e'' \\ &= (\lambda y : A. e') \ e'' \\ &= e'\end{aligned}$$

$$\begin{aligned}\text{if false then } e' \text{ else } e'' : A &= \text{false } A \ e' \ e'' \\ &= (\Lambda \alpha. \lambda x : \alpha. \lambda y : \alpha. y) A \ e' \ e'' \\ &= (\lambda x : A. \lambda y : A. y) \ e' \ e'' \\ &= (\lambda y : A. y) \ e'' \\ &= e''\end{aligned}$$

Type		Encoding
$X \times Y$	$\triangleq$	$\forall \alpha. (X \rightarrow Y \rightarrow \alpha) \rightarrow \alpha$
$\langle e, e' \rangle$	$\triangleq$	$\Lambda \alpha. \lambda k : X \rightarrow Y \rightarrow \alpha. k e e'$
$\text{fst } e$	$\triangleq$	$e X (\lambda x : X. \lambda y : Y. x)$
$\text{snd } e$	$\triangleq$	$e Y (\lambda x : X. \lambda y : Y. y)$

## Evaluating Church Pairs

$$\begin{aligned}\text{fst } \langle e, e' \rangle &= \langle e, e' \rangle X (\lambda x : X. \lambda y : Y. x) \\ &= (\Lambda \alpha. \lambda k : X \rightarrow Y \rightarrow \alpha. k e e') X (\lambda x : X. \lambda y : Y. x) \\ &= (\lambda k : X \rightarrow Y \rightarrow X. k e e') (\lambda x : X. \lambda y : Y. x) \\ &= (\lambda x : X. \lambda y : Y. x) e e' \\ &= (\lambda y : Y. e) e' \\ &= e\end{aligned}$$

$$\begin{aligned}\text{snd } \langle e, e' \rangle &= \langle e, e' \rangle Y (\lambda x : X. \lambda y : Y. y) \\ &= (\Lambda \alpha. \lambda k : X \rightarrow Y \rightarrow \alpha. k e e') Y (\lambda x : X. \lambda y : Y. y) \\ &= (\lambda k : X \rightarrow Y \rightarrow Y. k e e') (\lambda x : X. \lambda y : Y. y) \\ &= (\lambda x : X. \lambda y : Y. y) e e' \\ &= (\lambda y : Y. y) e' \\ &= e'\end{aligned}$$

Type	Encoding
$X + Y$	$\forall \alpha. (X \rightarrow \alpha) \rightarrow (Y \rightarrow \alpha) \rightarrow \alpha$
$L e$	$\Lambda \alpha. \lambda f : X \rightarrow \alpha. \lambda g : Y \rightarrow \alpha. f e$
$R e$	$\Lambda \alpha. \lambda f : X \rightarrow \alpha. \lambda g : Y \rightarrow \alpha. g e$
$\text{case}(e, Lx \rightarrow e_1, Ry \rightarrow e_2) : Z$	$e Z (\lambda x : X. e_1) (\lambda y : Y. e_2)$

$$\begin{aligned} & \text{case}(\text{L } e, \text{L } x \rightarrow e_1, \text{R } y \rightarrow e_2) : Z \\ &= (\text{L } e) Z (\lambda x : X. e_1) (\lambda y : Y. e_2) \\ &= (\Lambda \alpha. \lambda f : X \rightarrow \alpha. \lambda g : Y \rightarrow \alpha. f e) \\ &\quad Z (\lambda x : X. e_1) (\lambda y : Y. e_2) \\ &= (\lambda f : X. \lambda g : Y. f e) \\ &\quad (\lambda x : X. e_1) (\lambda y : Y. e_2) \\ &= (\lambda g : Y. (\lambda x : X. e_1) e) \\ &\quad (\lambda y : Y. e_2) \\ &= (\lambda x : X. e_1) e \\ &= [e/x]e_1 \end{aligned}$$

## Church Encodings: Natural Numbers

Type	Encoding
$\mathbb{N}$	$\forall \alpha. \alpha \rightarrow (\alpha \rightarrow \alpha) \rightarrow \alpha$
$z$	$\Lambda \alpha. \lambda Z : \alpha. \lambda S : \alpha \rightarrow \alpha. Z$
$s(e)$	$\Lambda \alpha. \lambda Z : \alpha. \lambda S : \alpha \rightarrow \alpha. S (e \alpha Z S)$
$\text{iter}(e, z \rightarrow e_z, s(x) \rightarrow e_s) : X$	$e X e_z (\lambda x : X. e_s)$



$$\begin{aligned} & \text{iter}(z, z \rightarrow e_z, s(x) \rightarrow e_s) \\ &= z X e_z (\lambda x : X. e_s) \\ &= (\Lambda \alpha. \lambda z : \alpha. \lambda s : \alpha \rightarrow \alpha. z) X e_z (\lambda x : X. e_s) \\ &= (\lambda z : X. \lambda s : X \rightarrow X. z) e_z (\lambda x : X. e_s) \\ &= (\lambda s : X \rightarrow X. e_z) (\lambda x : X. e_s) \\ &= e_z \end{aligned}$$

$$\begin{aligned} & \text{iter}(s(e), z \rightarrow e_z, s(x) \rightarrow e_s) \\ &= (s(e)) X e_z (\lambda x : X. e_s) \\ &= (\Lambda \alpha. \lambda z : \alpha. \lambda s : \alpha \rightarrow \alpha. s (e \alpha z s)) X e_z (\lambda x : X. e_s) \\ &= (\lambda z : X. \lambda s : X \rightarrow X. s (e X z s)) e_z (\lambda x : X. e_s) \\ &= (\lambda s : X \rightarrow X. s (e X e_z s)) (\lambda x : X. e_s) \\ &= (\lambda x : X. e_s) (e X e_z (\lambda x : X. e_s)) \\ &= (\lambda x : X. e_s) \text{iter}(e, z \rightarrow e_z, s(x) \rightarrow e_s) \\ &= [\text{iter}(e, z \rightarrow e_z, s(x) \rightarrow e_s) / x] e_s \end{aligned}$$

Type	Encoding
$\text{list } X$	$\forall \alpha. \alpha \rightarrow (X \rightarrow \alpha \rightarrow \alpha) \rightarrow \alpha$
$[]$	$\Lambda \alpha. \lambda n : \alpha. \lambda c : X \rightarrow \alpha \rightarrow \alpha. n$
$e :: e'$	$\Lambda \alpha. \lambda n : \alpha. \lambda c : X \rightarrow \alpha \rightarrow \alpha. c \ e \ (e' \ \alpha \ n \ c)$

$\text{fold}(e, [] \rightarrow e_n, x :: r \rightarrow e_c) : Z = e \ Z \ e_n \ (\lambda x : X. \lambda r : Z. e_c)$

- System F is very simple, and very expressive
- Formal basis of polymorphism in ML, Java, Haskell, etc.
- Surprise: from polymorphism and functions, data is definable

1. Prove the regularity lemma.
2. Define a Church encoding for the unit type.
3. Define a Church encoding for the empty type.
4. Define a Church encoding for binary trees, corresponding to the ML datatype  
`type tree = Leaf | Node of tree * X * tree.`