Randomised Algorithms

Lecture 5: Random Walks, Hitting Times and Application to 2-SAT

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UNIVERSITY OF CAMBRIDGE

Outline

Application 3: Ehrenfest Chain and Hypercubes

Random Walks on Graphs, Hitting Times and Cover Times

Random Walks on Paths and Grids

SAT and a Randomised Algorithm for 2-SAT

The Ehrenfest Markov Chain

Ehrenfest Model

- A simple model for the exchange of molecules between two boxes

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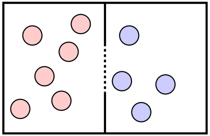

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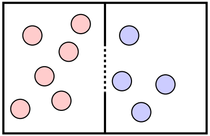

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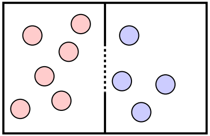

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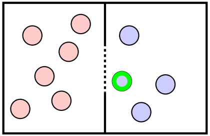

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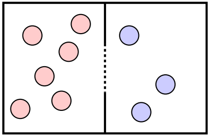

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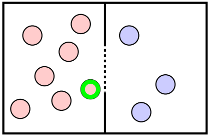

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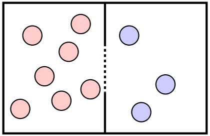
- At each step a particle is selected uniformly at random and switches to the other box
- If $\Omega = \{0, 1, \ldots, d\}$ denotes the number of particles in the red box, then:

$$P_{x,x-1} = \frac{x}{d} \qquad \text{and} \qquad P_{x,x+1} = \frac{d-x}{d}.$$



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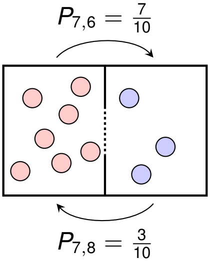

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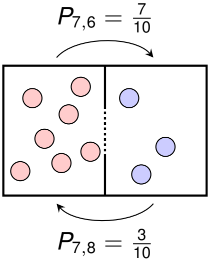

Let us now enlarge the state space by looking at each particle individually!

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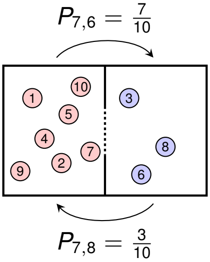

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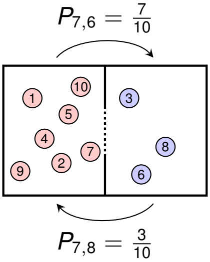

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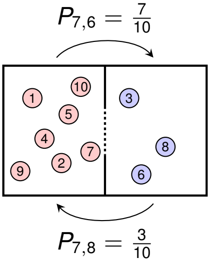

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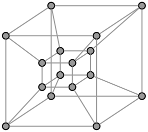

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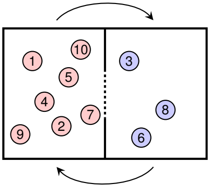
$P_{7,6} = \frac{7}{10}$



$P_{7,8} = \frac{3}{10}$

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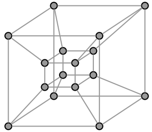

Analysis of the Mixing Time

(Non-Lazy) Random Walk on the Hypercube

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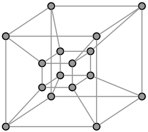

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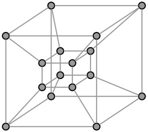

Problem: This Markov Chain is periodic, as the number of ones always switches between odd to even!

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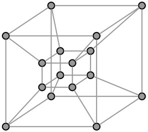

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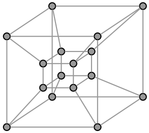

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Lazy Random Walk (1st Version)

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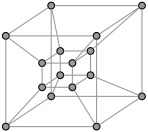

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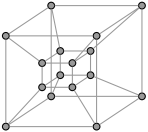

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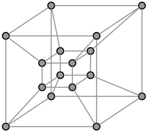

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Lazy Random Walk (1st Version)

- At each step $t = 0, 1, 2 \ldots$
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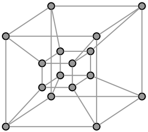

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Lazy Random Walk (2nd Version)

- At each step $t = 0, 1, 2 \ldots$
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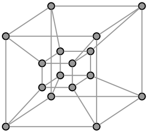

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Lazy Random Walk (2nd Version)

- At each step $t = 0, 1, 2 \ldots$
  - Pick a random coordinate in $[d]$
  - Set coordinate to $\{0, 1\}$ uniformly.

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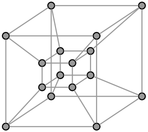

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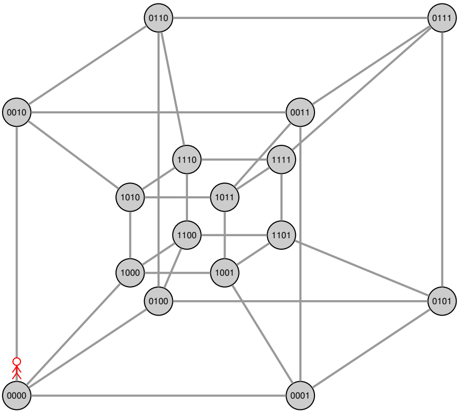
These two chains are equivalent!

Example of a Random Walk on a 4-Dimensional Hypercube



t	Coord.	X_t
0		0 0 0 0

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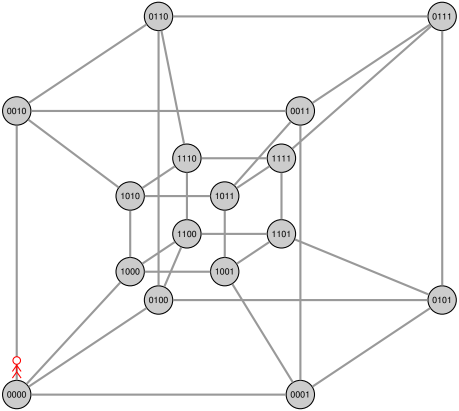

t	Coord.	X_t			
0	2	0	0	0	0
1		0	?	0	0

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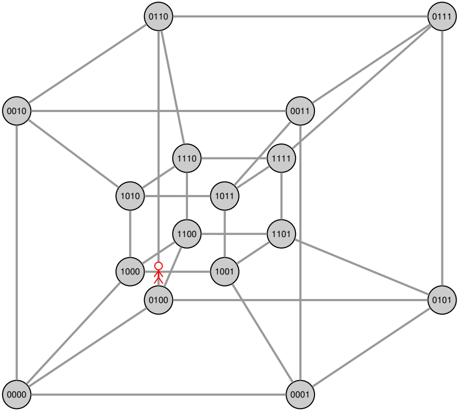

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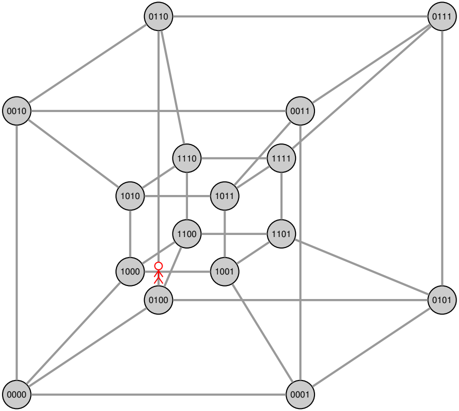

t	Coord.	X_t			
0	2	0	0	0	0
1	3	0	1	0	0
2		0	1	?	0

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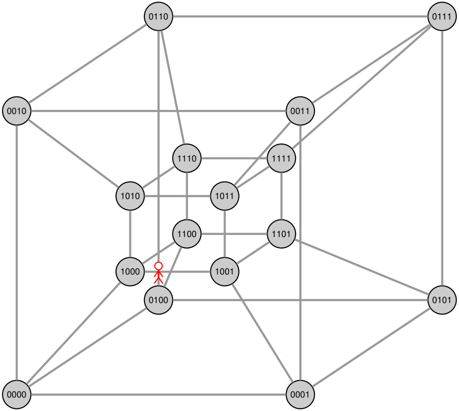

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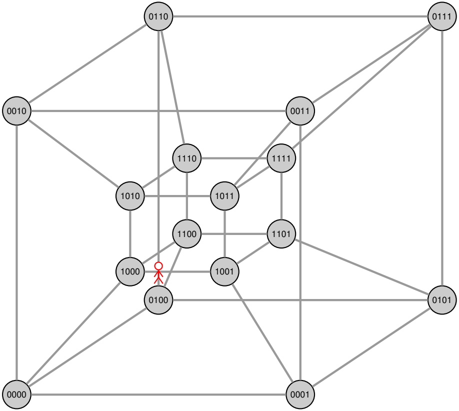

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2	3	0	1	0	0
3		0	1	?	0

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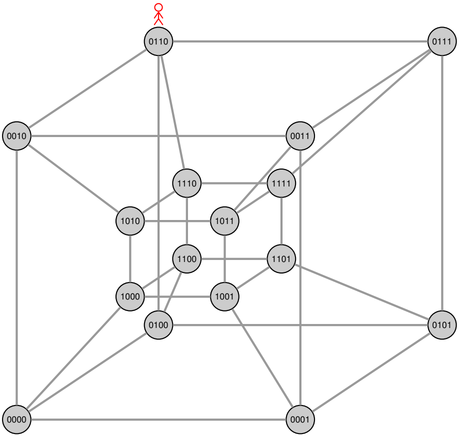

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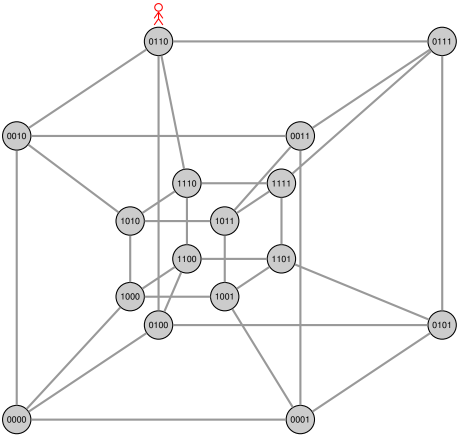

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3	4	0	1	1	0
4		0	1	1	?

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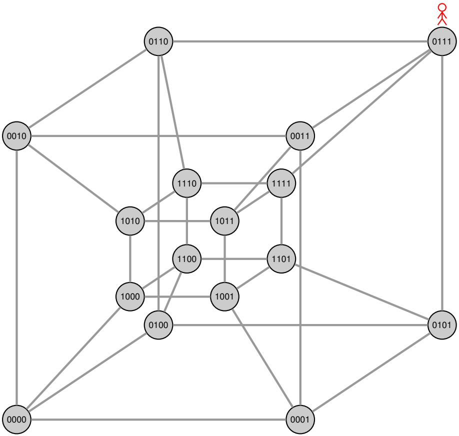

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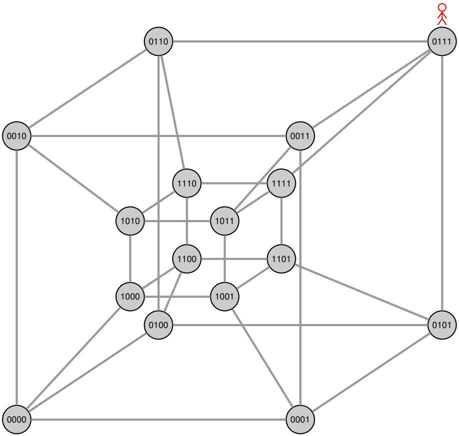

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2	3	0	1	0	0
3	4	0	1	1	0
4	2	0	1	1	1
5		0	?	1	1

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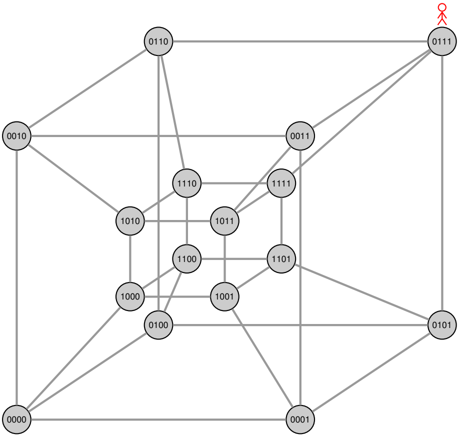

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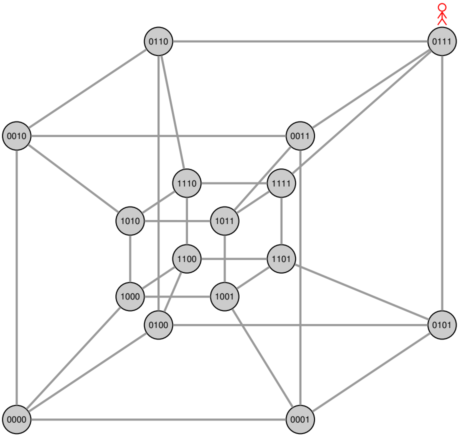

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3	4	0	1	1	0
4	2	0	1	1	1
5	4	0	1	1	1
6		0	1	1	?

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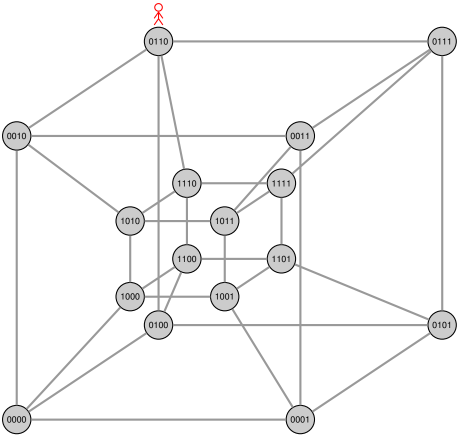

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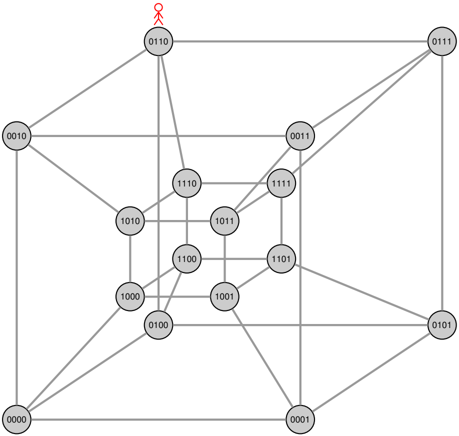

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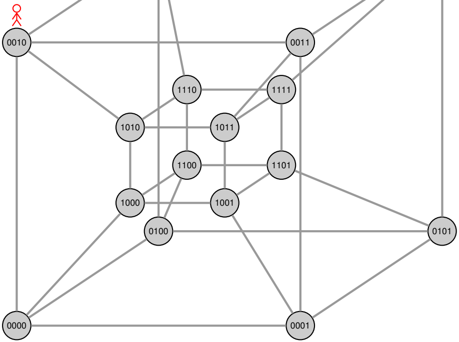
4	2	0	1	1	1
5	4	0	1	1	1
6	2	0	1	1	0
7		0	?	1	0

Example of a Random Walk on a 4-Dimensional Hypercube



t	Coord.	X_t			
0	2	0	0	0	0
1	3	0	1	0	0
2	3	0	1	0	0
3	4	0	1	1	0
4	2	0	1	1	1
5	4	0	1	1	1
6	2	0	1	1	0
7		0	0	1	0

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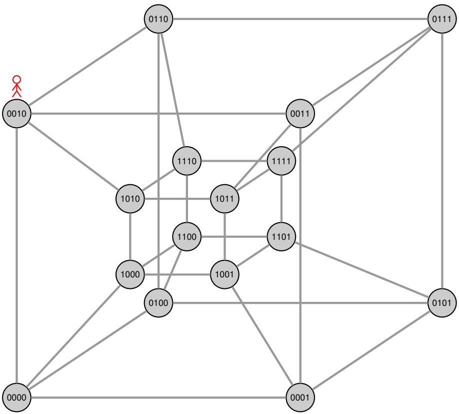

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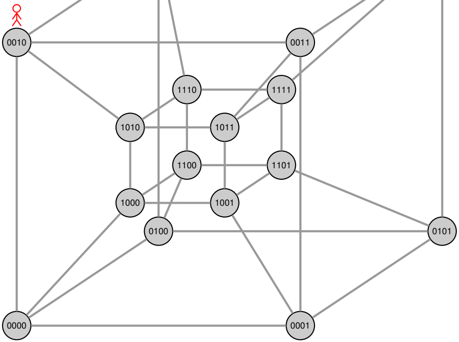

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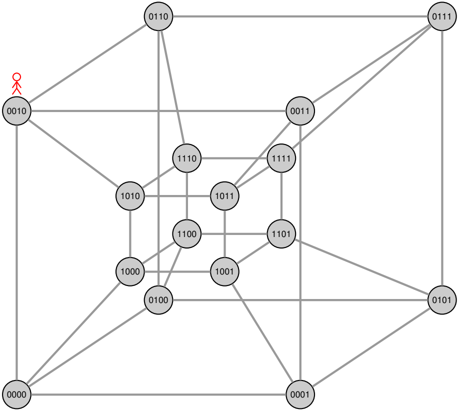

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4	2	0	1	1	1
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9		0	0	?	0

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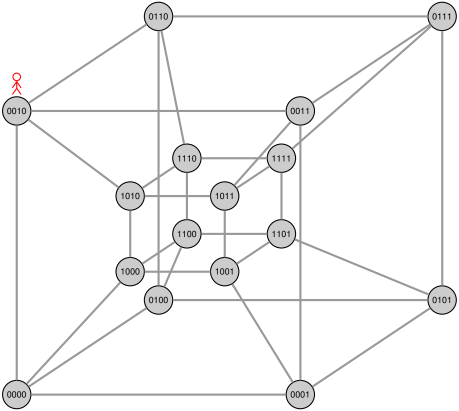

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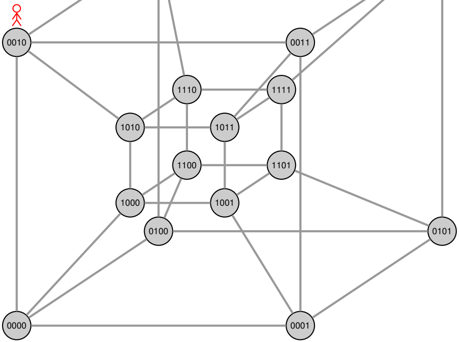

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9	1	0	0	1	0
10	done!	?	0	1	0

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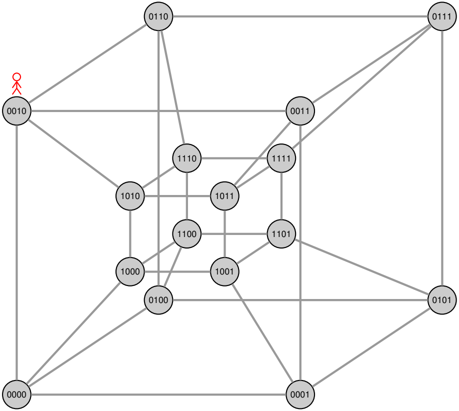

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Once all coordinates have been picked at least once, the state is uniformly at random in $\{0,1\}^d$.

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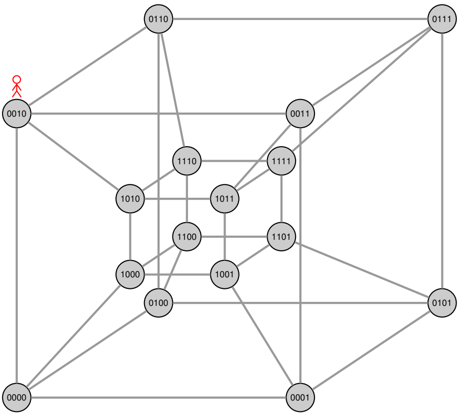

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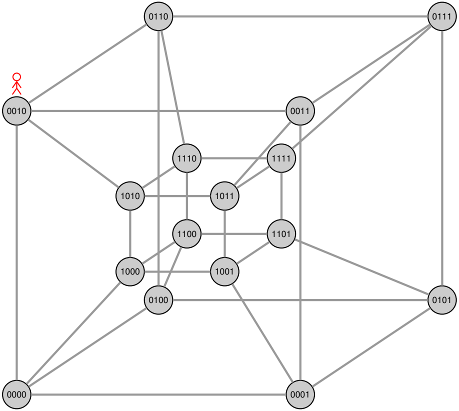

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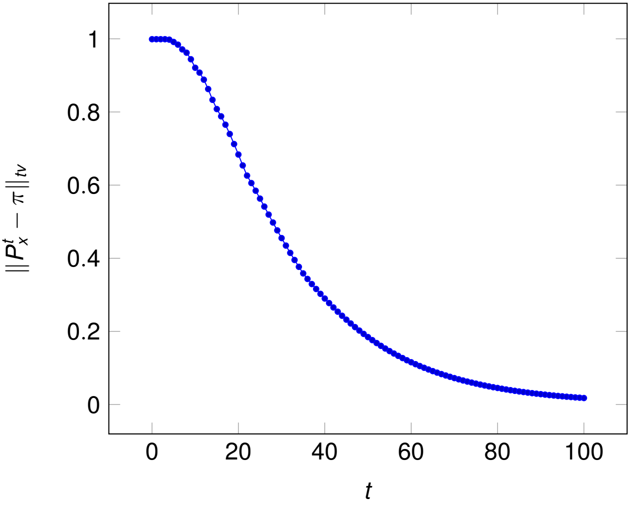
We won't formalise this argument here (see *[Ex. 4/5.11]*)

Total Variation Distance of Random Walk on Hypercube ($d = 22$)



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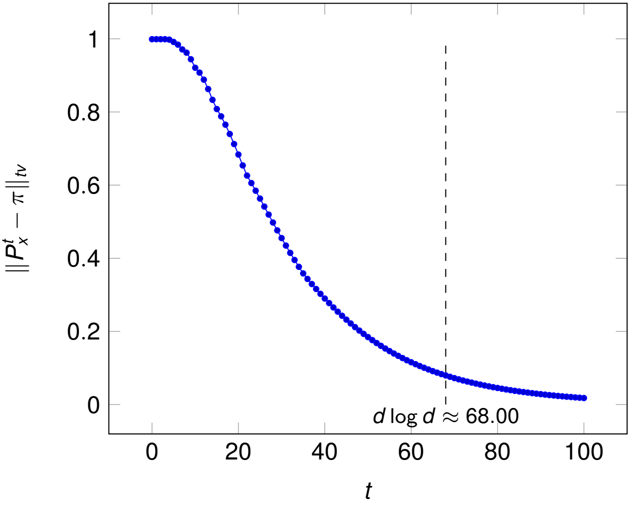

Theoretical Results (by Diaconis, Graham and Morrison)

RANDOM WALK ON A HYPERCUBE **53**

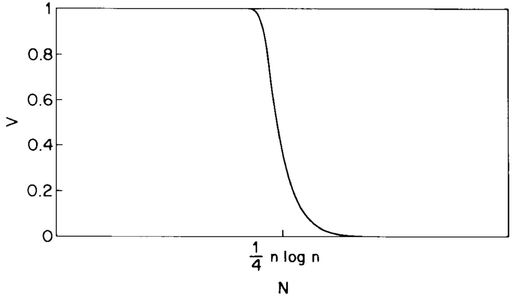

Fig. 1. The variation distance V as a function of N, for $n = 10^{12}$.

Source: “Asymptotic analysis of a random walk on a hypercube with many dimensions”, P. Diaconis, R.L. Graham, J.A. Morrison; Random Structures & Algorithms, 1990.

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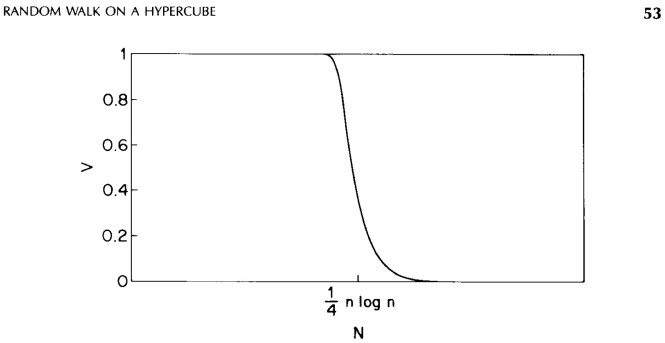

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  (Minor Remark: This random walk is with a loop probability of $1/(d+1)$)
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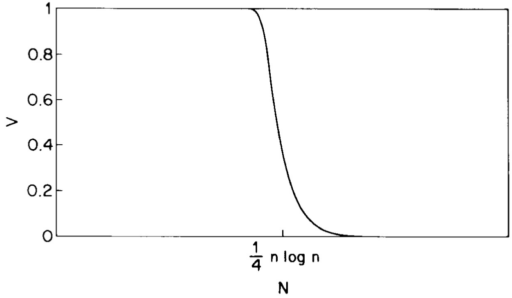
RANDOM WALK ON A HYPERCUBE 53



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- This is a numerical plot of a theoretical bound, where $d = 10^{12}$ (Minor Remark: This random walk is with a loop probability of $1/(d+1)$)
- The variation distance exhibits a so-called cut-off phenomena:
  - Distance remains close to its maximum value 1 until step $\frac{1}{4} n \log n - \Theta(n)$
  - Then distance moves close to 0 before step $\frac{1}{4} n \log n + \Theta(n)$

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Random Walks on Graphs, Hitting Times and Cover Times

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For two states $x, y \in \Omega$ we call $h(x, y)$ the hitting time of y from x:

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Some distinguish between $\tau_y^+ = \min\{t \geq 1 \colon X_t = y\}$ and $\tau_y = \min\{t \geq 0 \colon X_t = y\}$

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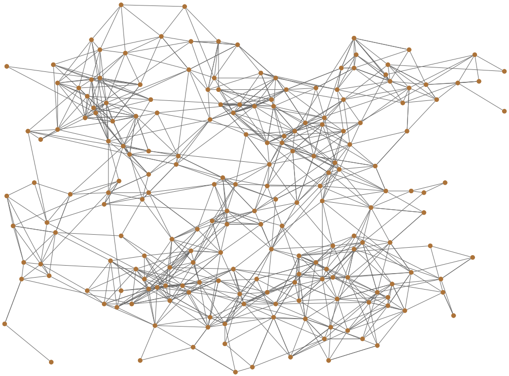
A Useful Identity

Hitting times are the solution to a set of linear equations:

$$h(x, y) \overset{\text{Markov Prop.}}{=} 1 + \sum_{z \in \Omega \setminus \{y\}} P(x, z) \cdot h(z, y) \qquad \forall x, y \in \Omega.$$

A Simple Random Walk (SRW) on a graph G is a Markov chain on $V(G)$ with

$$P(u,v) = \begin{cases} \frac{1}{\deg(u)} & \text{if } \{u,v\} \in E, \\ 0 & \text{if } \{u,v\} \notin E. \end{cases}, \qquad \text{and} \qquad \pi(u) = \frac{\deg(u)}{2|E|}$$



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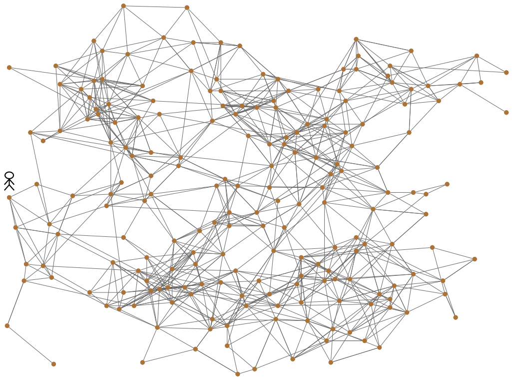

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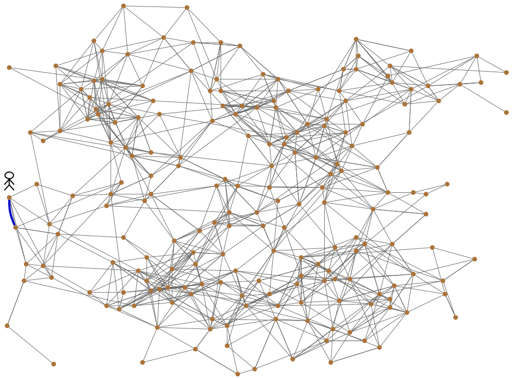

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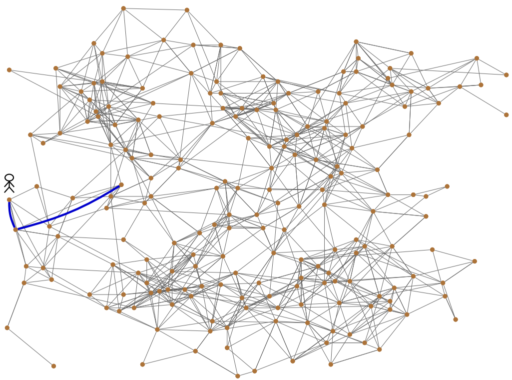

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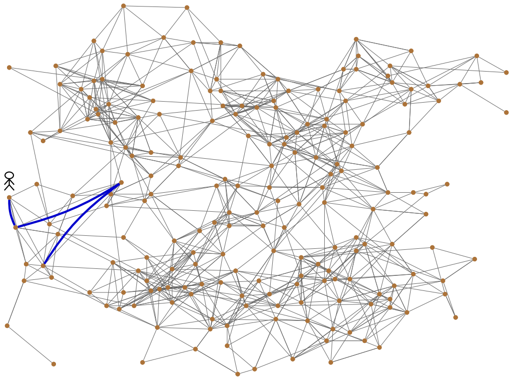

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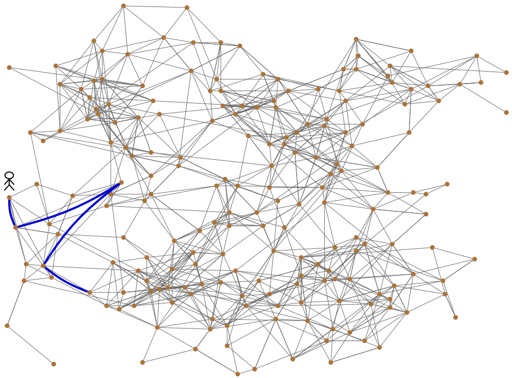

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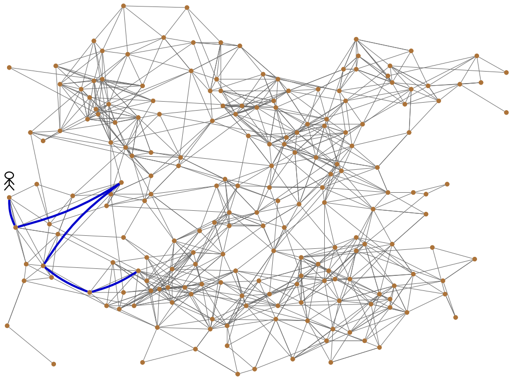

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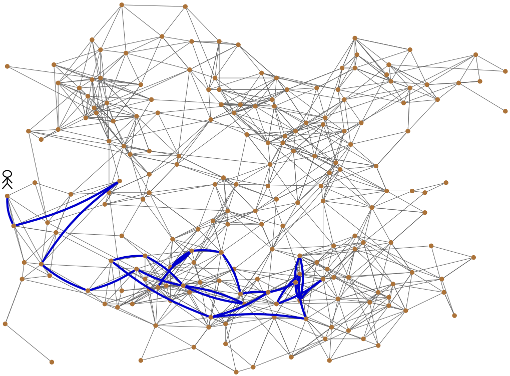

Lazy Random Walks and Periodicity

The Lazy Random Walk (LRW) on G given by $\widetilde{P} = (P + I)/2$,

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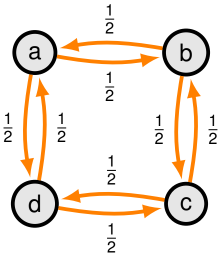

SRW on C_4, *Periodic*

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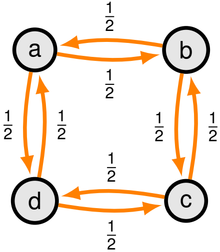

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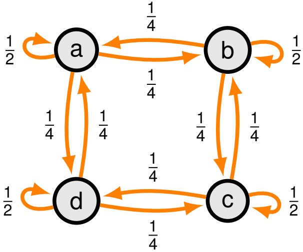

LRW on C_4, *Aperiodic*

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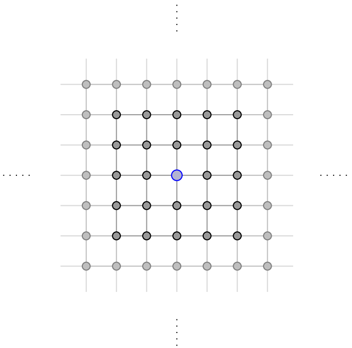

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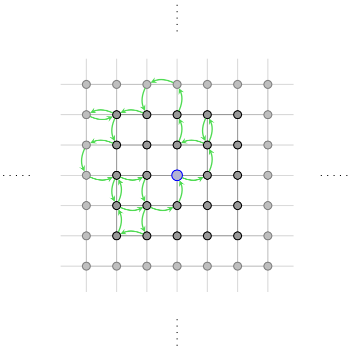

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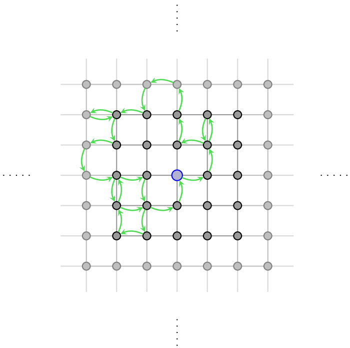
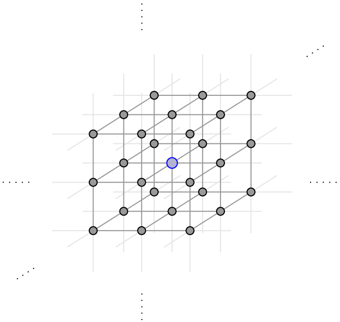


Infinite 3D Grid



1921: The Birth of Random Walks on (Infinite) Graphs (Polyá)

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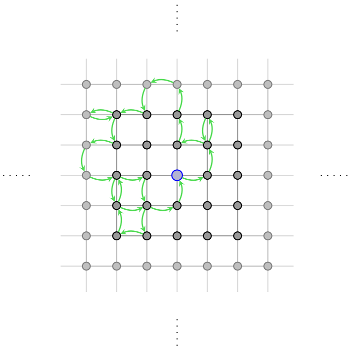

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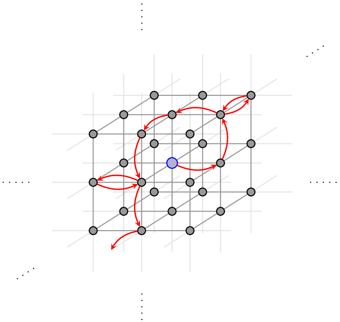

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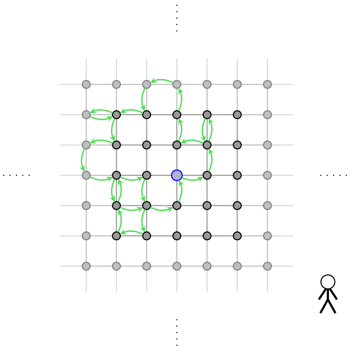

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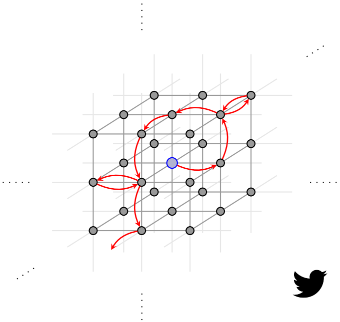

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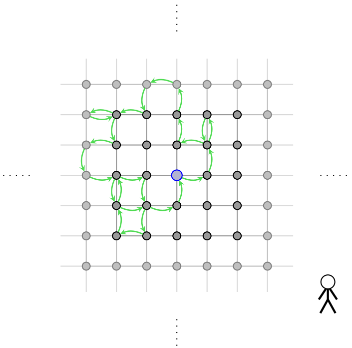

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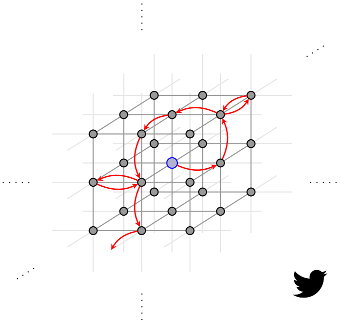

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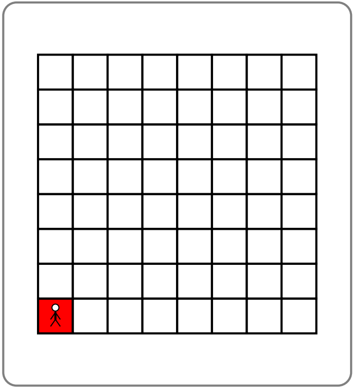
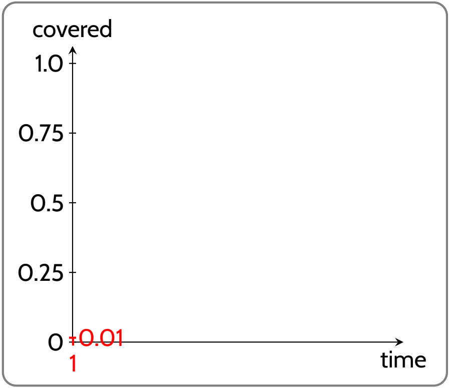
But for any regular (finite) graph, the expected return time to u is $1/\pi(u) = n$

SRW Random Walk on Two-Dimensional Grids: Animation





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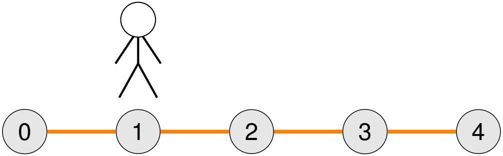

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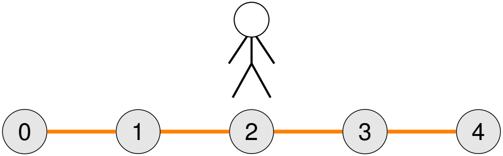

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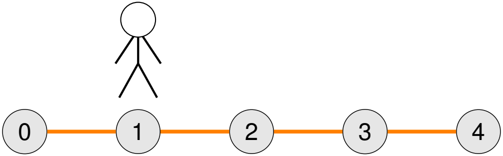

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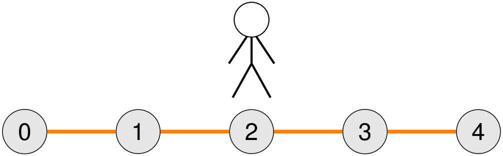

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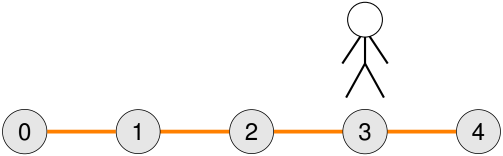

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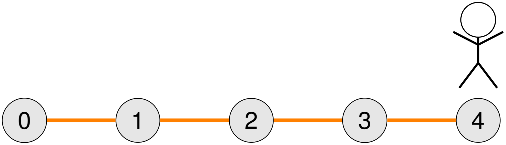

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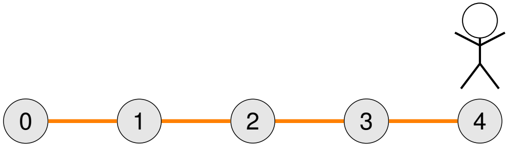

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Exercise: *[Exercise 4/5.15]* What happens for the LRW on P_n?

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Outline

Application 3: Ehrenfest Chain and Hypercubes

Random Walks on Graphs, Hitting Times and Cover Times

Random Walks on Paths and Grids

SAT and a Randomised Algorithm for 2-SAT

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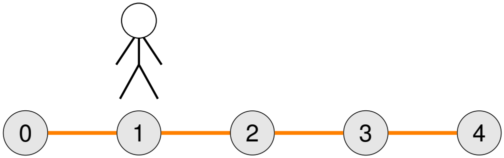

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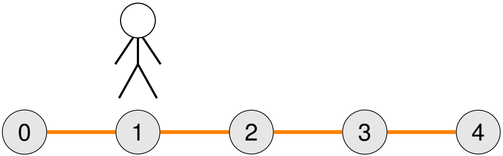

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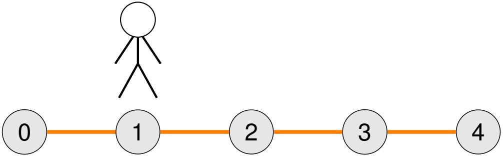

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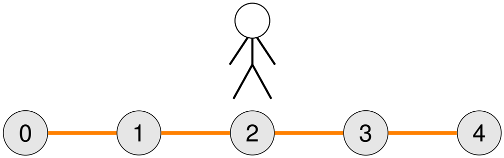
F F T T F T F T F T

$$\alpha = (\mathtt{T}, \mathtt{T}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	T	F	F

2-SAT

RANDOMISED-2-SAT (Input: a 2-SAT-Formula)

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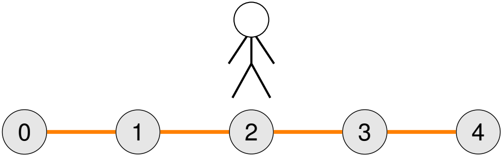

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1	F	T	F	F

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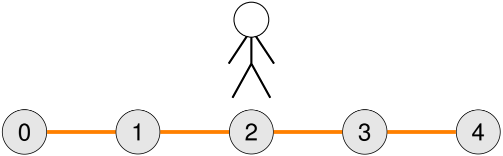

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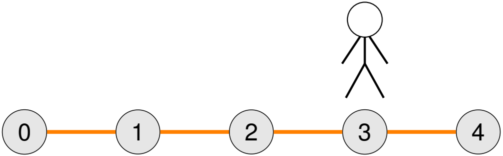

t	x_1	x_2	x_3	x_4
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1	F	T	F	F
2	T	T	F	F

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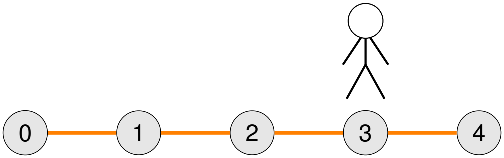

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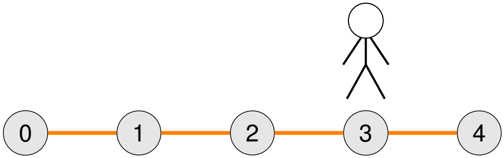

t	x_1	x_2	x_3	x_4
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1	F	T	F	F
2	T	T	F	F

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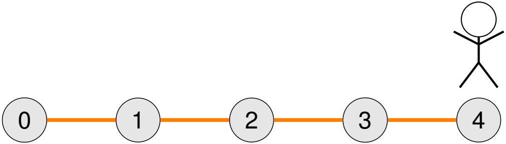
0	F	F	F	F
1	F	T	F	F
2	T	T	F	F
3	T	T	F	T



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Example 1 : Solution Found

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T F F T T T T T T F

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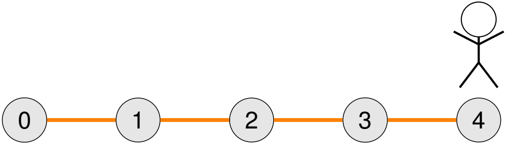
t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	T	F	F
2	T	T	F	F
3	T	T	F	T



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$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (x_4 \vee x_3) \wedge (x_4 \vee \overline{x_1})$$

F T T T F F F F F T

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t	x_1	x_2	x_3	x_4
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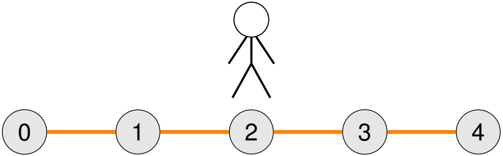

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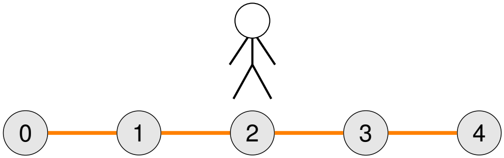

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F T T T F F F F F T

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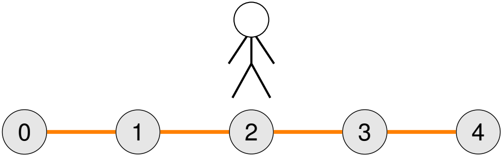

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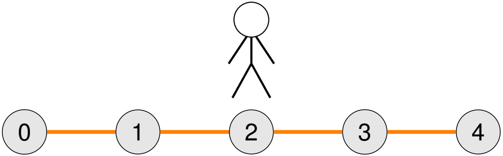

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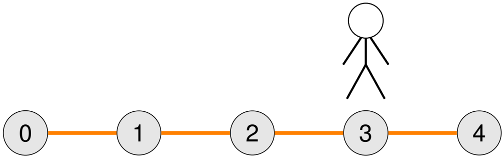

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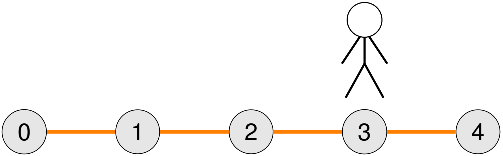

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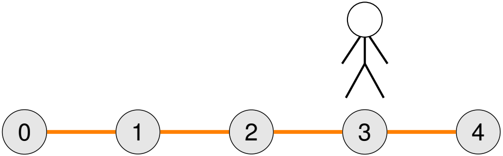

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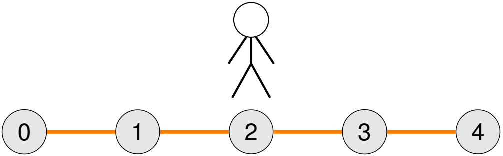

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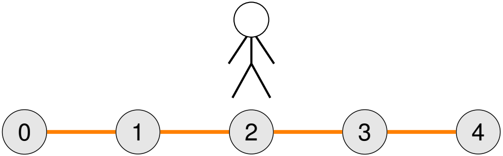

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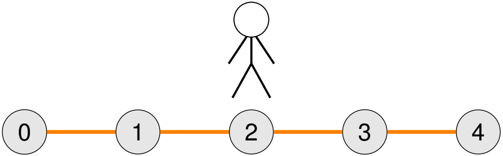
2	F	T	F	T



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T F F T T T T F T F

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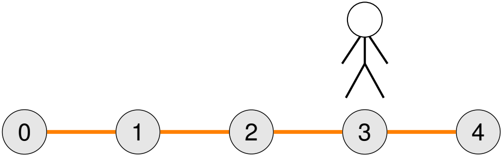

t	x_1	x_2	x_3	x_4
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1: Start with an arbitrary truth assignment
2: **Repeat up to** $2n^2$ **times**
3: Pick an arbitrary unsatisfied clause
4: Choose a random literal and switch its value
5: **If** formula is satisfied **then return** "Satisfiable"
6: **return** "Unsatisfiable"

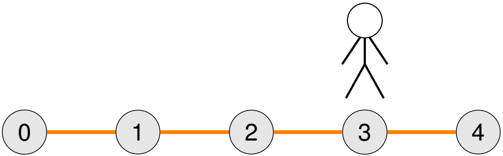
- Call each loop of (2) a step. Let A_i be the variable assignment at step i.
- Let α be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

Example 2 : (Another) Solution Found

$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (x_4 \vee x_3) \wedge (x_4 \vee \overline{x_1})$$

T F F T T T T F T F

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	F	F	T
2	F	T	F	T
3	T	T	F	T

2-SAT and the SRW on the Path

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Proposition

If the formula is satisfiable, RANDOMISED-2-SAT will return a valid solution in $O(n^2)$ steps with probability at least $1/2$.

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Exercise: (difficult, beyond this course) What happens to the above analysis if we apply the same algorithm to 3-SAT?

Boosting Success Probabilities

Boosting Lemma

Suppose a randomised algorithm succeeds with probability (at least) p. Then for any $C \geq 1$, $\lceil \frac{C}{p} \cdot \log n \rceil$ repetitions are sufficient to succeed (in at least one repetition) with probability at least $1 - n^{-C}$.

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Proof: Recall that $1 - p \leq e^{-p}$ for all real p. Let $t = \lceil \frac{C}{p} \log n \rceil$ and observe

$$\begin{aligned} \mathbf{P}[\, t \text{ runs all fail} \,] &\leq (1-p)^t \\ &\leq e^{-pt} \\ &\leq n^{-C}, \end{aligned}$$

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RANDOMISED-2-SAT

There is a $O(n^2 \log n)$-step algorithm for 2-SAT which succeeds w.h.p.