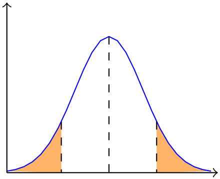

Randomised Algorithms

Lecture 2: Concentration Inequalities, Application to Balls-into-Bins

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UNIVERSITY OF CAMBRIDGE

Outline

General Recipe for Deriving Chernoff Bounds

Recipe

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2. Compute an upper bound for $\mathbf{E}\left[\, e^{\lambda X} \,\right]$ (using independence)
3. Optimise value of λ to obtain best tail bound

Chernoff Bound: Proof

Chernoff Bound (General Form, Upper Tail)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum_{i=1}^n p_i$. Then, for any $\delta > 0$ it holds that

$$\mathbf{P}[X \geq (1+\delta)\mu] \leq \left[\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right]^\mu .$$

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5. Choose $\lambda = \log(1+\delta) > 0$ to get the result.

Chernoff Bounds: Lower Tails

We can also use Chernoff Bounds to show a random variable is **not too small** compared to its mean:

Chernoff Bounds (General Form, Lower Tail)

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$$\mathbf{P}[X \leq (1-\delta)\mu] \leq \left[\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right]^{\mu},$$

and thus, by substitution, for any $t < \mu$,

$$\mathbf{P}[X \leq t] \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^{t}.$$

Exercise on Supervision Sheet

Hint: multiply both sides by -1 and repeat the proof of the Chernoff Bound

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- For all $t > 0$,

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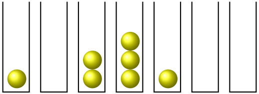
All upper tail bounds hold even under a relaxed independence assumption: For all $1 \leq i \leq n$ and $x_1, x_2, \ldots, x_{i-1} \in \{0, 1\}$,

$$\mathbf{P}[X_i = 1 \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \leq p_i.$$

Outline

Balls into Bins



Balls into Bins Model

You have m balls and n bins. Each ball is allocated in a bin picked independently and uniformly at random.

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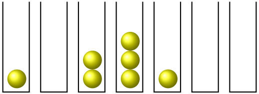

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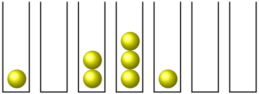

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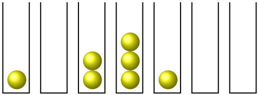

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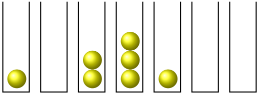

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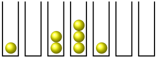
Exercise: Think about the relation between the Balls into Bins Model and the Coupon Collector Problem.

Balls into Bins: Bounding the Maximum Load (1/4)



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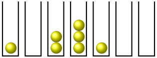

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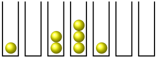

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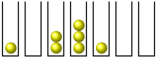

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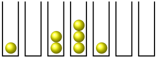

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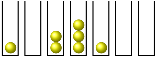

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$$\mathbf{P}[X \geq t] \leq e^{-\mu}(e\mu/t)^t$$

- By the Chernoff Bound,

$$\mathbf{P}[X \geq 6 \log n] \leq e^{-2\log n}\left(\frac{2e \log n}{6 \log n}\right)^{6 \log n} \leq e^{-2\log n} = n^{-2}$$

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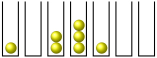

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here we could have used the “nicer” bounds as well!

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whp stands for *with high probability*:
An event $\mathcal{E}$ (that implicitly depends on an input parameter n) occurs whp if
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This is a very standard notation in randomised algorithms
but it may vary from author to author. Be careful!

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- Therefore whp, no bin receives at least $6\log n$ balls
- By pigeonhole principle, the max loaded bin receives at least $2\log n$ balls. Hence our bound is pretty sharp.

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- By setting $t = 4\log n / \log\log n$, we claim to obtain $\mathbf{P}[X \geq t] \leq n^{-2}$.

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$$\mathbf{P}\left[\, X \geq 4 \log n / \log \log n \,\right] \leq n^{-2},$$

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- One can prove that whp at least one bin receives at least $c \log n / \log \log n$ balls, for some constant $c > 0$.

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$\Rightarrow$ for $m = n$ this gives a maximum load of $\log_2 \log n + \Theta(1)$ w.p. $1 - 1/n$.

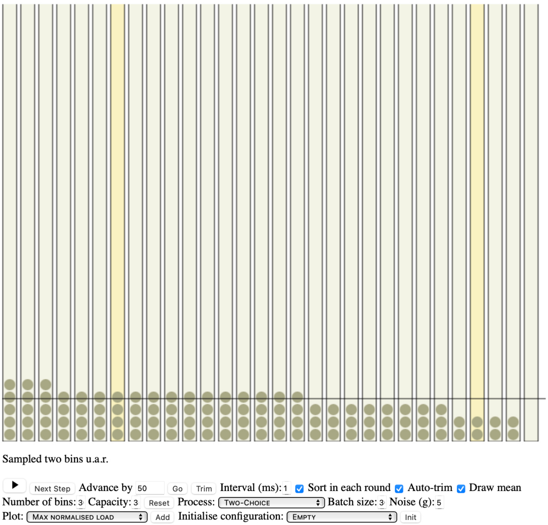
This is called the **power of two choices**: It is a common technique to improve the performance of randomised algorithms (covered in Chapter 17 of the textbook by Mitzenmacher and Upfal)

ACM Paris Kanellakis Theory and Practice Award 2020

For “the discovery and analysis of balanced allocations, known as the power of two choices, and their extensive applications to practice.”

“These include i-Google’s web index, Akamai’s overlay routing network, and highly reliable distributed data storage systems used by Microsoft and Dropbox, which are all based on variants of the power of two choices paradigm. There are many other software systems that use balanced allocations as an important ingredient.”

Simulation



https://www.dimitrioslos.com/balls_and_bins/visualiser.html

Outline

How to Derive Chernoff Bounds

Application 1: Balls into Bins

Appendix: More on Moment Generating Functions (non-examinable)

Moment Generating Functions (non-examinable)

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Lemma

1. If X and Y are two r.v.'s with $M_X(t) = M_Y(t)$ for all $t \in (-\delta, +\delta)$ for some $\delta > 0$, then the distributions X and Y are identical.
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Proof of 2:

$$M_{X+Y}(t) = \mathbf{E}\left[e^{t(X+Y)}\right] = \mathbf{E}\left[e^{tX} \cdot e^{tY}\right] \stackrel{(!)}{=} \mathbf{E}\left[e^{tX}\right] \cdot \mathbf{E}\left[e^{tY}\right] = M_X(t)M_Y(t) \qquad \square$$