Randomised Algorithms

Lecture 1: Introduction to Course & Introduction to Chernoff Bounds

Thomas Sauerwald (tms41@cam.ac.uk)

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Outline

Randomised Algorithms

What? Randomised Algorithms utilise random bits to compute their output.

Why? Randomised Algorithms often provide an efficient (and elegant!) solution or approximation to a problem that is costly (or impossible) to solve deterministically.

But often: simple algorithm at the cost of a sophisticated analysis!

"... If somebody would ask me, what in the last 10 years, what was the most important change in the study of algorithms I would have to say that people getting really familiar with randomised algorithms had to be the winner."
- Donald E. Knuth (in *Randomization and Religion*)

How? This course aims to strengthen your knowledge of probability theory and apply this to analyse examples of randomised algorithms.

What if I (initially) don't care about randomised algorithms?
Many of the techniques in this course (Markov Chains, Concentration of Measure, Spectral Theory) are very relevant to other popular areas of research and employment such as Data Science and Machine Learning.

Some stuff you should know...

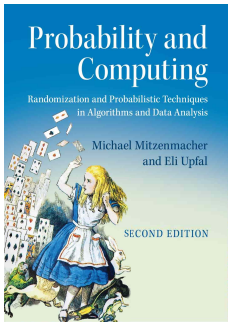
In this course we will assume some basic knowledge of probability:

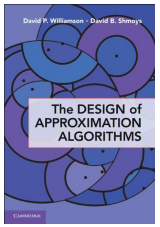
- random variable
- computing expectations and variances
- notions of independence and conditional probabilities
- "general" idea of how to compute probabilities (manipulating, counting and **estimating**)

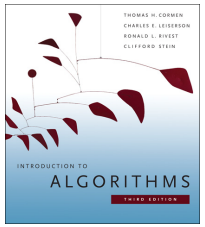
You should also be familiar with basic computer science, mathematics knowledge such as:

- graphs
- basic algorithms (sorting, graph algorithms etc.)
- matrices, norms and vectors

Textbooks







- **($\star$) Michael Mitzenmacher and Eli Upfal. Probability and Computing: Randomized Algorithms and Probabilistic Analysis, Cambridge University Press, 2nd edition, 2017**
- David P. Williamson and David B. Shmoys. The Design of Approximation Algorithms, Cambridge University Press, 2011
- Cormen, T.H., Leiserson, C.D., Rivest, R.L. and Stein, C. Introduction to Algorithms. MIT Press (3rd ed.), 2009
  (We will adopt some of the labels (e.g., Theorem 35.6) from this book in Lectures 6-10)

Outline

1 **Introduction** (Lecture)
- Intro to Randomised Algorithms; Logistics; Recap of Probability; Examples.

Lectures 2-5 focus on probabilistic tools and techniques.

2–3 **Concentration** (Lectures)
- Concept of Concentration; Recap of Markov and Chebyshev; Chernoff Bounds and Applications; Extensions: Hoeffding's Inequality and Method of Bounded Differences; Applications.

4 **Markov Chains and Mixing Times** (Lecture)
- Recap; Stopping and Hitting Times; Properties of Markov Chains; Convergence to Stationary Distribution; Variation Distance and Mixing Time

5 **Hitting Times and Application to 2-SAT** (Lecture)
- Reversible Markov Chains and Random Walks on Graphs; Cover Times and Hitting Times on Graphs (Example: Paths and Grids); A Randomised Algorithm for 2-SAT Algorithm

Lectures 6-8 introduce linear programming, a (mostly) deterministic but very powerful technique to solve various optimisation problems.

6–7 **Linear Programming** (Lectures)
- Introduction to Linear Programming, Applications, Standard and Slack Forms, Simplex Algorithm, Finding an Initial Solution, Fundamental Theorem of Linear Programming

8 **Travelling Salesman Problem** (Interactive Demo)
- Hardness of the general TSP problem, Formulating TSP as an integer program; Classical TSP instance from 1954; Branch & Bound Technique to solve integer programs using linear programs

We then see how we can efficiently combine linear programming with randomised techniques, in particular, rounding:

9–10 **Randomised Approximation Algorithms** (Lectures)
- MAX-3-CNF and Guessing, Vertex-Cover and Deterministic Rounding of Linear Program, Set-Cover and Randomised Rounding, Concluding Example: MAX-CNF and Hybrid Algorithm

Lectures 11-12 cover a more advanced topic with ML flavour:

11–12 **Spectral Graph Theory and Spectral Clustering** (Lectures)
- Eigenvalues, Eigenvectors and Spectrum; Visualising Graphs; Expansion; Cheeger's Inequality; Clustering and Examples; Analysing Mixing Times

Outline

Recap: Probability Space

In probability theory we wish to evaluate the likelihood of certain results from an experiment. The setting of this is the probability space $(\Omega, \Sigma, \mathbf{P})$.

Components of the Probability Space $(\Omega, \Sigma, \mathbf{P})$

- The Sample Space Ω contains all the possible outcomes $\omega_1, \omega_2, \ldots$ of the experiment.
- The Event Space Σ is the power-set of Ω containing events, which are combinations of outcomes (subsets of Ω including $\emptyset$ and Ω).
- The Probability Measure $\mathbf{P}$ is a function from Σ to $\mathbb{R}$ satisfying
  - (i) $0 \leq \mathbf{P}[\mathcal{E}] \leq 1$, for all $\mathcal{E} \in \Sigma$
  - (ii) $\mathbf{P}[\Omega] = 1$
  - (iii) If $\mathcal{E}_1, \mathcal{E}_2, \ldots \in \Sigma$ are pairwise disjoint ($\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for all $i \neq j$) then

$$\mathbf{P}\left[\bigcup_{i=1}^{\infty} \mathcal{E}_i\right] = \sum_{i=1}^{\infty} \mathbf{P}[\mathcal{E}_i].$$

Recap: Random Variables

A random variable X on $(\Omega, \Sigma, \mathbf{P})$ is a function $X : \Omega \to \mathbb{R}$ mapping each sample "outcome" to a real number.

Intuitively, random variables are the "observables" in our experiment.

Examples of random variables

- The number of heads in three coin flips $X_1, X_2, X_3 \in \{0, 1\}$ is:

$$X_1 + X_2 + X_3$$

- The indicator random variable $\mathbf{1}_{\mathcal{E}}$ of an event $\mathcal{E} \in \Sigma$ given by

$$\mathbf{1}_{\mathcal{E}}(\omega) = \begin{cases} 1 & \text{if } \omega \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

  For the indicator random variable $\mathbf{1}_{\mathcal{E}}$ we have $\mathbf{E}[\mathbf{1}_{\mathcal{E}}] = \mathbf{P}[\mathcal{E}]$.
- The number of sixes of two dice throws $X_1, X_2 \in \{1, 2, \ldots, 6\}$ is

$$\mathbf{1}_{X_1=6} + \mathbf{1}_{X_2=6}$$

Recap: Boole's Inequality (Union Bound)

Union Bound is one of the most basic probability inequalities, yet it is extremely useful and easy to apply!

Union Bound

Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be a collection of events in Σ. Then

$$\mathbf{P}\left[\bigcup_{i=1}^{n} \mathcal{E}_i\right] \leq \sum_{i=1}^{n} \mathbf{P}[\mathcal{E}_i].$$

A Proof using Indicator Random Variables:

1. Let $\mathbf{1}_{\mathcal{E}_i}$ be the random variable that takes value 1 if $\mathcal{E}_i$ holds, 0 otherwise
2. $\mathbf{E}[\mathbf{1}_{\mathcal{E}_i}] = \mathbf{P}[\mathcal{E}_i]$ (**Check this**)
3. It is clear that $\mathbf{1}_{\bigcup_{i=1}^{n} \mathcal{E}_i} \leq \sum_{i=1}^{n} \mathbf{1}_{\mathcal{E}_i}$ (**Check this**)
4. Taking expectation completes the proof.

Outline

A Randomised Algorithm for MAX-CUT (1/2)

$E(A, B)$: set of edges with one endpoint in $A \subseteq V$ and the other in $B \subseteq V$.

MAX-CUT Problem

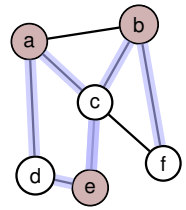
- Given: Undirected graph $G = (V, E)$
- Goal: Find $S \subseteq V$ such that $e(S, S^c) := |E(S, S^c)|$ is maximised.

Applications:

- network or chip design
- machine learning
- statistical physics

Comments:

- MAX-CUT is NP-hard
- It is different from the clustering problem, where we want to find a sparse cut
- Note that the MIN-CUT problem is solvable in polynomial time!



$S = \{a, b, e\}$
$e(S, S^c) = 6$

A Randomised Algorithm for MAX-CUT (2/2)

RandMaxCut(G)
1: Start with $S \leftarrow \emptyset$
2: **For** each $v \in V$, add v to S with probability $1/2$
3: **Return** S

This kind of "random guessing" will appear often in this course!

Ratio between optimal and expected value of our solution is ≤ 2 (more on this in Lecture 9)

RandMaxCut(G) gives a 2-approximation using time $O(n)$.

Later: learn stronger tools that imply concentration around the expectation!

Proof:

- We need to analyse the expectation of $e(S, S^c)$:

$$\begin{aligned}
\mathbf{E}\left[e(S,S^c)\right] &= \mathbf{E}\left[\sum_{\{u,v\}\in E} \mathbf{1}_{\{u\in S, v\in S^c\}\cup\{u\in S^c, v\in S\}}\right] \\
&= \sum_{\{u,v\}\in E} \mathbf{E}\left[\mathbf{1}_{\{u\in S, v\in S^c\}\cup\{u\in S^c, v\in S\}}\right] \\
&= \sum_{\{u,v\}\in E} \mathbf{P}\left[\{u\in S, v\in S^c\}\cup\{u\in S^c, v\in S\}\right] \\
&= 2\sum_{\{u,v\}\in E} \mathbf{P}\left[u\in S, v\in S^c\right] = 2\sum_{\{u,v\}\in E} \mathbf{P}\left[u\in S\right]\cdot\mathbf{P}\left[v\in S^c\right] = |E|/2.
\end{aligned}$$

- Since for any $S \subseteq V$, we have $e(S, S^c) \leq |E|$, the proof is complete.

Example: Coupon Collector

Source: https://www.express.co.uk/life-style/life/567954/Discount-codes-money-saving-vouchers-coupons-mum

This is a very important example in the design and analysis of randomised algorithms.

Coupon Collector Problem

Suppose that there are n coupons to be collected from the cereal box. Every morning you open a new cereal box and get one coupon. We assume that each coupon appears with the same probability in the box.

Example Sequence for $n = 8$: $7, 6, 3, 3, 3, 2, 5, 4, 2, 4, 1, 4, 2, 1, 4, 3, 1, 4, 8$ ✓

Exercise (*[Ex. 1.11]*)

In this course: $\log n = \ln n$

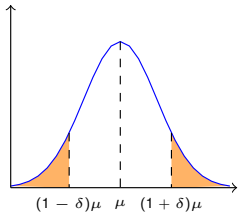
1. Prove it takes $n\sum_{k=1}^{n} \frac{1}{k} \approx n \log n$ expected boxes to collect all coupons
2. Use Union Bound to prove that the probability it takes more than $n \log n + cn$ boxes to collect all n coupons is $\leq e^{-c}$.

   Hint: It is useful to remember that $1 - x \leq e^{-x}$ for all x

Outline

Concentration Inequalities

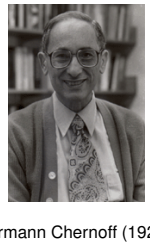
- Concentration refers to the phenomena where random variables are very close to their mean
- This is very useful in randomised algorithms as it ensures an **almost** deterministic behaviour
- It gives us the best of two worlds:
  1. **Randomised Algorithms:** Easy to Design and Implement
  2. **Deterministic Algorithms:** They do what they claim



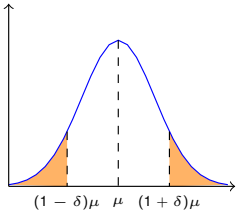
Chernoff Bounds: A Tool for Concentration (1952)

- Chernoffs bounds are "strong" bounds on the tail probabilities of sums of independent random variables
- random variables can be discrete (or continuous)
- usually these bounds decrease exponentially as opposed to a polynomial decrease in Markov's or Chebyshev's inequality (see example)
- easy to apply, but requires independence
- have found various applications in:
  - Randomised Algorithms and Statstics
  - Random Projections and Dimensionality Reduction
  - Complexity Theory and Learning Theory (e.g., PAC-learning)

  ⋮



Hermann Chernoff (1923-)



Recap: Markov and Chebyshev

Markov's Inequality

If X is a non-negative random variable, then for any $a > 0$,

$$\mathbf{P}[X \geq a] \leq \mathbf{E}[X]/a.$$

Chebyshev's Inequality

If X is a random variable, then for any $a > 0$,

$$\mathbf{P}[|X - \mathbf{E}[X]| \geq a] \leq \mathbf{V}[X]/a^2.$$

- Let $f : \mathbb{R} \to [0, \infty)$ and increasing, then $f(X) \geq 0$, and thus

$$\mathbf{P}[X \geq a] = \mathbf{P}[f(X) \geq f(a)] \leq \mathbf{E}[f(X)]/f(a).$$

- Similarly, if $g : \mathbb{R} \to [0, \infty)$ and decreasing, then $g(X) \geq 0$, and thus

$$\mathbf{P}[X \leq a] = \mathbf{P}[g(X) \geq g(a)] \leq \mathbf{E}[g(X)]/g(a).$$

Chebyshev's inequality (or Markov) can be obtained by chosing $f(X) := (X - \mu)^2$ (or $f(X) := X$, respectively).

From Markov and Chebyshev to Chernoff

Markov and Chebyshev use the first and second moment of the random variable. Can we keep going?

- Yes!

We can consider the first, second, third and more moments! That is the basic idea behind the Chernoff Bounds

Our First Chernoff Bound

Chernoff Bounds (General Form, Upper Tail)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum_{i=1}^{n} p_i$. Then, for any $\delta > 0$ it holds that

$$\mathbf{P}[X \geq (1+\delta)\mu] \leq \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{\mu}. \qquad (\bigstar)$$

This implies that for any $t > \mu$,

$$\mathbf{P}[X \geq t] \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^{t}.$$

While ($\bigstar$) is one of the easiest (and most generic) Chernoff bounds to derive, the bound is complicated and hard to apply...

Example: Coin Flips (1/3)

- Consider throwing a fair coin n times and count the total number of heads
- $X_i \in \{0,1\}$, $X = \sum_{i=1}^{n} X_i$ and $\mathbf{E}[X] = n \cdot 1/2 = n/2$
- The Chernoff Bound gives for any $\delta > 0$,

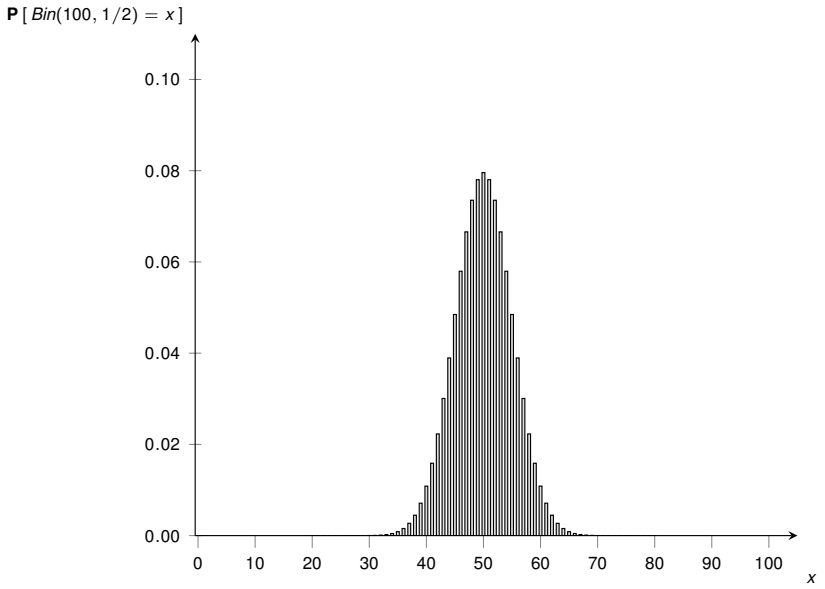
$$\mathbf{P}[X \geq (1+\delta)(n/2)] \leq \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{n/2}.$$

- The above expression equals 1 only for $\delta = 0$, and then it gives a value strictly less than 1 (check this!)

$\Rightarrow$ The inequality is **exponential in** n, (for fixed δ) which is much better than Chebyshev's inequality.

What about a concrete value of n, say $n = 100$?

Example: Coin Flips (2/3)



Example: Coin Flips (3/3)

Consider $n = 100$ independent coin flips. We wish to find an upper bound on the probability that the number of heads is greater or equal than 75.

- Markov's inequality: $\mathbf{E}[X] = 100/2 = 50$.

$$\mathbf{P}[X \geq 3/2 \cdot \mathbf{E}[X]] \leq 2/3 = \mathbf{0.666}.$$

- Chebyshev's inequality: $\mathbf{V}[X] = \sum_{i=1}^{100} \mathbf{V}[X_i] = 100 \cdot (1/2)^2 = 25$.

$$\mathbf{P}[|X - \mu| \geq t] \leq \frac{\mathbf{V}[X]}{t^2},$$

  and plugging in $t = 25$ gives an upper bound of $25/25^2 = 1/25 = \mathbf{0.04}$, much better than what we obtained by Markov's inequality.

- Chernoff bound: setting $\delta = 1/2$ gives

$$\mathbf{P}[X \geq 3/2 \cdot \mathbf{E}[X]] \leq \left(\frac{e^{1/2}}{(3/2)^{3/2}}\right)^{50} = \mathbf{0.004472}.$$

- Remark: The exact probability is **0.00000028 . . .**

Chernoff bound yields a much better result (but needs independence!)