Proof Assistants

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Part I

Isabelle

Material

- Isabelle part of this course based on book "Concrete Semantics with Isabelle/HOL" (2014) by Tobias Nipkow and Gerwin Klein

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- Slides shamelessly copied from Tobias Nipkow (errors are my own)

Chapter 1

Programming and Proving in Isabelle/HOL

1. Overview of Isabelle/HOL

2. Type and function definitions

3. Induction Heuristics

4. Simplification

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HOL Formulas:

- Equalities ($term = term$), e.g. $1 + 2 = 4$
- Later: $\wedge$, $\vee$, $\longrightarrow$, $\forall$, $\ldots$

1. Overview of Isabelle/HOL
   - Types and terms
   - By example: types $bool$, nat and $list$

Types

Basic syntax:

$$\tau \quad ::=$$

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\begin{array}{lcll}
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Convention: $\tau_1 \Rightarrow \tau_2 \Rightarrow \tau_3 \equiv \tau_1 \Rightarrow (\tau_2 \Rightarrow \tau_3)$

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Examples: $f\ (g\ x)\ y$
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Convention: $f\ t_1\ t_2\ t_3\ \equiv\ ((f\ t_1)\ t_2)\ t_3$

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This language of terms is known as the *λ-calculus*.

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$$(\lambda x.\ t)\ u\ =\ t[u/x]$$

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- The step from $(\lambda x.\ t)\ u$ to $t[u/x]$ is called *β-reduction*.
- Isabelle performs β-reduction automatically.

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$t\ ::\ \tau$ means "t is a well-typed term of type τ".

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User can help with *type annotations* inside the term.
Example: $f\ (x{::}nat)$

`Overview_Demo.thy`

(including an example of how to define a simple function and prove a lemma about it)

1 Overview of Isabelle/HOL

Types and terms

By example: types $bool$, nat and $list$

Type $bool$

datatype $bool \;=\; True \;\mid\; False$

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Predefined functions:

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if-and-only-if: $=$ or $\longleftrightarrow$

Type nat

datatype $nat = 0 \mid Suc\ nat$

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Values of type nat: $0,\ \ Suc\ 0,\ \ Suc\ (Suc\ 0),\ \ldots$

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! Numbers and arithmetic operations are overloaded:
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Values of type nat: 0, $Suc\ 0$, $Suc\ (Suc\ 0)$, $\ldots$

Predefined functions: $+$, $*$, ... $::\ nat \Rightarrow nat \Rightarrow nat$

! Numbers and arithmetic operations are overloaded:
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You need type annotations: $1 :: nat$, $x + (y::nat)$

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Values of type nat: $0,\ \ Suc\ 0,\ \ Suc\ (Suc\ 0),\ \ldots$

Predefined functions: $+$, $*$, ... :: $nat \Rightarrow nat \Rightarrow nat$

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You need type annotations: $1\ ::\ nat$, $x + (y{::}nat)$
unless the context is unambiguous: $Suc\ z$

Nat_Demo.thy

An informal proof

Lemma $add\ m\ 0 = m$

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Proof by induction on m.

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  $$\begin{array}{lcll} add\ (Suc\ m)\ 0 & = & Suc\ (add\ m\ 0) & \text{by def. of } add \\ & = & Suc\ m & \text{by IH} \end{array}$$

Induction on natural numbers

To prove $P(n)$ for all natural numbers n, prove

- $P(0)$ and
- for arbitrary but fixed n,
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$$\frac{P(0) \qquad \bigwedge n.\ P(n) \Longrightarrow P(Suc(n))}{P(n)}$$

Type $'a\ list$

Lists of elements of type $'a$

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Syntactic sugar:

- $[] = Nil$: empty list

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- $x\ \#\ xs = Cons\ x\ xs$:
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Syntactic sugar:

- $[] = Nil$: empty list
- $x\ \#\ xs = Cons\ x\ xs$:
  list with first element x (*"head"*) and rest xs (*"tail"*)
- $[x_1, \ldots, x_n] = x_1\ \#\ \ldots\ x_n\ \#\ []$

Structural Induction for lists

To prove that $P(xs)$ for all lists xs, prove

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List_Demo.thy

An informal proof

Lemma $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$
Proof by induction on xs.

- Case Nil: $app\ (app\ Nil\ ys)\ zs = app\ ys\ zs = app\ Nil\ (app\ ys\ zs)$ holds by definition of app.
- Case $Cons\ x\ xs$: We assume $app\ (app\ xs\ ys)\ zs = app\ xs\ (app\ ys\ zs)$ (IH), and we need to show $app\ (app\ (Cons\ x\ xs)\ ys)\ zs = app\ (Cons\ x\ xs)\ (app\ ys\ zs)$.
  The proof is as follows:
  $app\ (app\ (Cons\ x\ xs)\ ys)\ zs$
  $= Cons\ x\ (app\ (app\ xs\ ys)\ zs)$ by definition of app
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Large library: `HOL/List.thy`

Included in `Main`.

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Don't reinvent, reuse!

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Large library: `HOL/List.thy`

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Predefined: $xs\ @\ ys$ (append), $length$,

Large library: `HOL/List.thy`

Included in `Main`.

Don't reinvent, reuse!

Predefined: $xs\ @\ ys$ (append), $length$, and map

2 Type and function definitions

Type definitions

Function definitions

Type synonyms

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Introduces a *synonym* $name$ for type τ

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Introduces a *synonym* $name$ for type τ

Examples

type_synonym $string = char\ list$

type_synonym $('a,'b)foo = 'a\ list \times\ 'b\ list$

Type synonyms are expanded after parsing
and are not present in internal representation and output

datatype — the general case

$$\textbf{datatype}\ (\alpha_1, \ldots, \alpha_n)t \quad = \quad C_1\ \tau_{1,1} \ldots \tau_{1,n_1}$$
$$\quad | \quad \ldots$$
$$\quad | \quad C_k\ \tau_{k,1} \ldots \tau_{k,n_k}$$

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- *Types:* $C_i :: \tau_{i,1} \Rightarrow \dots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1,\dots,\alpha_n)t$

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$$\mid \quad C_k\ \tau_{k,1} \dots \tau_{k,n_k}$$

- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1,\dots,\alpha_n)t$
- *Distinctness:* $C_i\ \dots \neq C_j\ \dots \qquad$ if $i \neq j$

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$$\quad | \quad \ldots$$
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- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)t$
- *Distinctness:* $C_i\ \ldots \neq C_j\ \ldots$ if $i \neq j$
- *Injectivity:* $(C_i\ x_1 \ldots x_{n_i} = C_i\ y_1 \ldots y_{n_i}) =$
  $(x_1 = y_1 \wedge \cdots \wedge x_{n_i} = y_{n_i})$

datatype — the general case

$$\textbf{datatype } (\alpha_1, \ldots, \alpha_n)t \;=\; C_1\ \tau_{1,1} \ldots \tau_{1,n_1} \mid \ldots \mid C_k\ \tau_{k,1} \ldots \tau_{k,n_k}$$

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Distinctness and injectivity are applied automatically

Induction must be applied explicitly

Case expressions

Datatype values can be taken apart with *case*:

$$(\textit{case}\ xs\ \textit{of}\ \ [] \Rightarrow \ldots \quad | \quad y\#ys \Rightarrow \ldots\ y\ \ldots\ ys\ \ldots)$$

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Need ( ) in context

Tree_Demo.thy

The *option* type

datatype $'a\ option = None \mid Some\ 'a$

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If $'a$ has values a_1, a_2, ...
then $'a\ option$ has values $None$, $Some\ a_1$, $Some\ a_2$, ...

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$\quad (\textsf{\textit{if}}\ a = x\ \textsf{\textit{then}}\ Some\ b\ \textsf{\textit{else}}\ lookup\ ps\ x)$

2 Type and function definitions

Type definitions

Function definitions

Non-recursive definitions

Example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n\ =\ n{*}n$

Non-recursive definitions

Example

definition $sq :: nat \Rightarrow nat$ **where** $sq\ n\ =\ n{*}n$

No pattern matching, just $f\,x_1\ \ldots\ x_n\ =\ \ldots$

The danger of nontermination

How about $f\,x = f\,x + 1$?

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! All functions in HOL must be total **!**

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- Termination must be provable automatically by size measures
- Proves customized induction schema

Example: separation

fun $sep :: {}'a \Rightarrow {}'a\ list \Rightarrow {}'a\ list$ **where**
$sep\ a\ (x\#y\#zs) = x\ \#\ a\ \#\ sep\ a\ (y\#zs)\ \ |$
$sep\ a\ xs = xs$

Example: Ackermann

fun $ack :: nat \Rightarrow nat \Rightarrow nat$ **where**

$$
\begin{array}{lll}
ack\ 0\ \ n & = Suc\ n & | \\
ack\ (Suc\ m)\ 0 & = ack\ m\ (Suc\ 0) & | \\
ack\ (Suc\ m)\ (Suc\ n) & = ack\ m\ (ack\ (Suc\ m)\ n) &
\end{array}
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ack\ (Suc\ m)\ (Suc\ n) & & = ack\ m\ (ack\ (Suc\ m)\ n)
\end{array}
$$

Terminates because the arguments decrease *lexicographically* with each recursive call:

- $(Suc\ m,\ 0) > (m,\ Suc\ 0)$
- $(Suc\ m,\ Suc\ n) > (Suc\ m,\ n)$
- $(Suc\ m,\ Suc\ n) > (m,\ \_)$

Basic induction heuristics

Theorems about recursive functions
are proved by induction

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Induction on argument number i of f
if f is defined by recursion on argument number i

A tail recursive reverse

Our initial reverse:

fun $rev :: {}'a\ list \Rightarrow {}'a\ list$ **where**

$rev\ [] \quad = [] \quad |$

$rev\ (x\#xs) \quad = rev\ xs\ @\ [x]$

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fun $itrev :: 'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list$ **where**

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lemma $itrev\ xs\ [] = rev\ xs$

`Induction_Demo.thy`

Generalisation

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- Replace constants by variables

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- Replace constants by variables
- Generalize free variables
  - by $arbitrary$ in induction proof
  - (or by universal quantifier in formula)

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Now: induction for complex recursion patterns.

Computation Induction

Example

fun $div2 :: nat \Rightarrow nat$ **where**
$div2\ 0 = 0 \ \ |$
$div2\ (Suc\ 0) = 0 \ \ |$
$div2\ (Suc(Suc\ n)) = Suc(div2\ n)$

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$\leadsto$ induction rule `div2.induct`:

$$\frac{P(0) \quad P(Suc\ 0) \qquad\qquad P(n) \Longrightarrow P(Suc(Suc\ n))}{P(m)}$$

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$\rightsquigarrow$ induction rule `div2.induct`:

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prove $P(e)$ *assuming* $P(r_1), \ldots, P(r_k)$.

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Induction follows course of (terminating!) computation

Motto: properties of f are best proved by rule $f.induct$

How to apply $f.induct$

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Heuristic:

- there should be a call $f\ a_1\ \ldots\ a_n$ in your goal

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Heuristic:

- there should be a call $f\ a_1\ \ldots\ a_n$ in your goal
- ideally the a_i should be variables.

Induction_Demo.thy

Computation Induction

1. Overview of Isabelle/HOL
2. Type and function definitions
3. Induction Heuristics
4. Simplification

Simplification means . . .

Using equations $l = r$ from left to right

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Terminology: equation $\leadsto$ *simplification rule*

Simplification means ...

Using equations $l = r$ from left to right

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Terminology: equation $\rightsquigarrow$ *simplification rule*

Simplification = (Term) Rewriting

An example

Equations:

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\begin{aligned}
0 + n &= n && (1)\\
(Suc\ m) + n &= Suc\ (m + n) && (2)\\
(Suc\ m \leq Suc\ n) &= (m \leq n) && (3)\\
(0 \leq m) &= True && (4)
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Simplification rules can be conditional:

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Example

$$\begin{array}{rcl} p(0) & = & True \\ p(x) \Longrightarrow\ f(x) & = & g(x) \end{array}$$

We can simplify $f(0)$ to $g(0)$ but we cannot simplify $f(1)$ because $p(1)$ is not provable.

Termination

Simplification may not terminate.
Isabelle uses $simp$-rules (almost) blindly from left to right.

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Principle:

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is suitable as a $simp$-rule only
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if l is "bigger" than r and each P_i

$$n < m \Longrightarrow (n < Suc\ m) = True \quad \text{YES}$$
$$Suc\ n < m \Longrightarrow (n < m) = True \quad \text{NO}$$

Proof method *simp*

Goal: 1. $\llbracket\ P_1; \ldots; P_m\ \rrbracket \Longrightarrow C$

apply$(simp\ add\colon\ eq_1\ \ldots\ \ eq_n)$

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Proof method $simp$

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Simplify $P_1\ \ldots\ P_m$ and C using

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- additional lemmas $eq_1\ \ldots\ eq_n$

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- assumptions $P_1 \ \ldots \ P_m$

Variations:

- $(simp \ \ldots \ del\!: \ \ldots)$ removes $simp$-lemmas
- add and del are optional

$auto$ versus $simp$

- $auto$ acts on all subgoals
- $simp$ acts only on subgoal 1

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- $auto$ applies $simp$ and more

- $auto$ can also be modified:
  $(auto\ simp\ add\colon \ldots\ simp\ del\colon \ldots)$

Rewriting with definitions

Definitions (**definition**) must be used explicitly:

$$(simp\ add\colon f\_def \ldots)$$

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f is the function whose definition is to be unfolded.

Case splitting with $simp/auto$

Automatic:

$$P\ (\textit{if}\ A\ \textit{then}\ s\ \textit{else}\ t)$$
$$=$$
$$(A \longrightarrow P(s)) \land (\neg A \longrightarrow P(t))$$

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By hand:

$$P\ (\textit{case}\ e\ \textit{of}\ 0 \Rightarrow a \mid Suc\ n \Rightarrow b)$$
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Proof method: $(simp\ split\colon nat.split)$

Case splitting with $simp/auto$

Automatic:

$$P\ (\textit{if}\ A\ \textit{then}\ s\ \textit{else}\ t)$$
$$=$$
$$(A \longrightarrow P(s)) \wedge (\neg A \longrightarrow P(t))$$

By hand:

$$P\ (\textit{case}\ e\ \textit{of}\ 0 \Rightarrow a \mid Suc\ n \Rightarrow b)$$
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Or $auto$. Similar for any datatype t: $t.split$

Simp_Demo.thy

Chapter 2

Case Study: IMP Expressions

5 Case Study: IMP Expressions

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This section introduces

arithmetic and boolean expressions

of our imperative language IMP.

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of our imperative language IMP.

IMP *commands* are introduced later.

5 Case Study: IMP Expressions

Arithmetic Expressions

Boolean Expressions

Concrete and abstract syntax

Concrete syntax: strings, eg `"a+5*b"`

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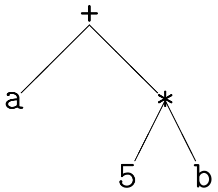
Abstract syntax: trees, eg



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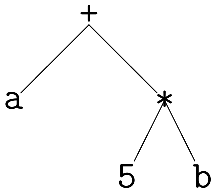

Parser: function from strings to trees

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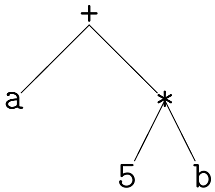

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Linear view of trees: terms, eg $Plus\ a\ (Times\ 5\ b)$

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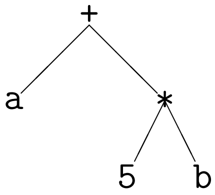

Parser: function from strings to trees

Linear view of trees: terms, eg $Plus\ a\ (Times\ 5\ b)$

Abstract syntax trees/terms are datatype values!

Concrete syntax is defined by a context-free grammar, eg

$$a ::= n \mid x \mid (a) \mid a + a \mid a * a \mid \ldots$$

where n can be any natural number and x any variable.

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We focus on *abstract* syntax
which we introduce via datatypes.

Datatype $aexp$

Variable names are strings, values are integers:

type_synonym $vname = string$
datatype $aexp = N\ int \mid V\ vname \mid Plus\ aexp\ aexp$

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Concrete	Abstract
5	$N\ 5$
`x`	$V\ ''x''$
`x+y`	$Plus\ (V\ ''x'')\ (V\ ''y'')$
`2+(z+3)`	$Plus\ (N\ 2)\ (Plus\ (V\ ''z'')\ (N\ 3))$

Warning

This is syntax, not (yet) semantics!

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$$N\,0 \;\neq\; Plus\;(N\,0)\;(N\,0)$$

The (program) state

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The (program) state

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- The value of all variables is recorded in the *state*.
- The state is a function from variable names to values:

  type_synonym $val = int$
  type_synonym $state = vname \Rightarrow val$

Function update notation

If $f :: \tau_1 \Rightarrow \tau_2$ and $a :: \tau_1$ and $b :: \tau_2$ then

$$f(a := b)$$

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is the function that behaves like f
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$$f(a := b) = (\lambda x.\ \mathit{if}\ x = a\ \mathit{then}\ b\ \mathit{else}\ f\ x)$$

How to write down a state

Some states:

- $\lambda x.\ 0$

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Nicer notation defined in `AExp.thy`:

$$<''a'' := 5,\ ''x'' := 3,\ ''y'' := 7>$$

How to write down a state

Some states:

- $\lambda x.\ 0$
- $(\lambda x.\ 0)(''a'' := 3)$
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Nicer notation defined in `AExp.thy`:

$$<''a'' := 5,\ ''x'' := 3,\ ''y'' := 7>$$

Maps everything to 0, but $''a''$ to 5, $''x''$ to 3, etc.

AExp.thy

5. Case Study: IMP Expressions
   - Arithmetic Expressions
   - Boolean Expressions

`BExp.thy`

This was easy.

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Because evaluation of expressions always terminates.
But execution of programs may *not* terminate.
Hence we cannot define it by a total recursive function.

We need more logical machinery
to define program execution and reason about it.

Chapter 3

Logic and Proof Beyond Equality

6 Logical Formulas

7 Proof Automation

8 Single Step Proofs

9 Inductive Definitions

6 Logical Formulas

7 Proof Automation

8 Single Step Proofs

9 Inductive Definitions

Syntax (in decreasing precedence):

$$
\begin{array}{lcllll}
\mathit{form} & ::= & (\mathit{form}) & | & \mathit{term} = \mathit{term} & | & \neg \mathit{form} \\
& | & \mathit{form} \wedge \mathit{form} & | & \mathit{form} \vee \mathit{form} & | & \mathit{form} \longrightarrow \mathit{form} \\
& | & \forall x.\ \mathit{form} & | & \exists x.\ \mathit{form} & &
\end{array}
$$

Syntax (in decreasing precedence):

$$\begin{array}{lclclcl} \mathit{form} & ::= & (\mathit{form}) & \mid & \mathit{term} = \mathit{term} & \mid & \neg \mathit{form} \\ & \mid & \mathit{form} \wedge \mathit{form} & \mid & \mathit{form} \vee \mathit{form} & \mid & \mathit{form} \longrightarrow \mathit{form} \\ & \mid & \forall x.\ \mathit{form} & \mid & \exists x.\ \mathit{form} & & \end{array}$$

Examples:

$$\neg\ A\ \wedge\ B \vee\ C\ \ \equiv\ \ ((\neg\ A)\ \wedge\ B)\ \vee\ C$$

Syntax (in decreasing precedence):

$$\begin{array}{rclclcl} form & ::= & (form) & \mid & term = term & \mid & \neg form \\ & \mid & form \wedge form & \mid & form \vee form & \mid & form \longrightarrow form \\ & \mid & \forall x.\ form & \mid & \exists x.\ form & & \end{array}$$

Examples:

$$\begin{array}{rcl} \neg\ A \wedge B \vee C & \equiv & ((\neg\ A) \wedge B) \vee C \\ s = t \wedge C & \equiv & (s = t) \wedge C \end{array}$$

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\end{array}
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Examples:

$$
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s = t \wedge\ C & \equiv & (s = t)\ \wedge\ C \\
A\ \wedge\ B = B\ \wedge\ A & \equiv & A\ \wedge\ (B = B)\ \wedge\ A
\end{array}
$$

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Examples:

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Input syntax: $\longleftrightarrow$ (same precedence as $\longrightarrow$)

Variable binding convention:

$$\forall\, x\ y.\ P\ x\ y\ \ \equiv\ \ \forall\, x.\ \forall\, y.\ P\ x\ y$$

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Similarly for $\exists$ and λ.

Warning

Quantifiers have low precedence
and need to be parenthesized (if in some context)

$$! \quad P \wedge \forall\, x.\; Q\; x \;\rightsquigarrow\; P \wedge (\forall\, x.\; Q\; x) \quad !$$

Mathematical symbols

and their ascii representations

$\forall$	\<forall>	ALL
$\exists$	\<exists>	EX
λ	\<lambda>	%
$\longrightarrow$	-->	
$\longleftrightarrow$	<->	
$\wedge$	/\	&
$\vee$	\/	|
$\neg$	\<not>	~
$\neq$	\<noteq>	~=

Sets over type $'a$

$'a\ set$

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- $\{\}, \quad \{e_1,\ldots,e_n\}$

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$\in$	`\<in>`	`:`
$\subseteq$	`\<subseteq>`	`<=`
$\cup$	`\<union>`	`Un`
$\cap$	`\<inter>`	`Int`

Set comprehension

- $\{x.\ P\}$ where x is a variable

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Set comprehension

- $\{x.\ P\}$ where x is a variable
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- Instead: $\{t \mid x\ y\ z.\ P\}$
  is short for $\{v.\ \exists\, x\ y\ z.\ v = t \wedge P\}$
  where x, y, z are the free variables in t

simp and *auto*

simp: rewriting and a bit of arithmetic

auto: rewriting and a bit of arithmetic, logic and sets

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Exception: $auto$ acts on all subgoals

fastforce

- rewriting, logic, sets, relations and a bit of arithmetic.

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blast

- A complete proof search procedure for FOL ...

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Automating arithmetic

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- proves linear formulas (no "$*$")
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- complete for first-order theory of nat and int (Presburger arithmetic)

Sledgehammer

Architecture:

$\boxed{\textbf{Isabelle}}$

external
ATPs[1]

[1] Automatic Theorem Provers

Architecture:

Isabelle

Goal
& filtered library $\downarrow$

external
ATPs[1]

---

[1] Automatic Theorem Provers

Architecture:

$$\begin{array}{c} \boxed{\textbf{Isabelle}} \\ \text{Goal \& filtered library} \downarrow \quad \uparrow \text{Proof} \\ \boxed{\begin{array}{c}\text{external}\\ \textbf{ATPs}^{1}\end{array}} \end{array}$$

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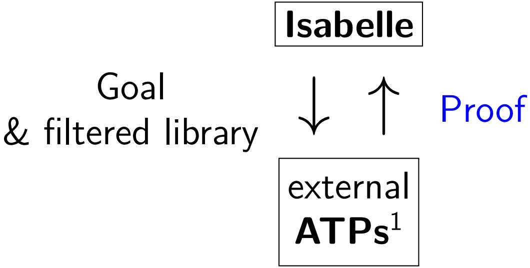

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Goal & filtered library $\downarrow\ \uparrow$ Proof

$$\boxed{\text{external } \textbf{ATPs}^{1}}$$

Characteristics:

- Sometimes it works,
- sometimes it doesn't.

Do you feel lucky?

[1] Automatic Theorem Provers

by($proof\text{-}method$)

$$\approx$$

apply($proof\text{-}method$)
done

Auto_Proof_Demo.thy

Step-by-step proofs can be necessary if automation fails and you have to explore where and why it failed by taking the goal apart.

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- By unification:
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- By unification:
  unifying $?P \wedge ?Q$ with $a{=}b \wedge False$
  sets $?P$ to $a{=}b$ and $?Q$ to $False$.

Rule application

Rule application

Example: rule: $[\![?P;\ ?Q]\!] \Longrightarrow ?P \land ?Q$

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$2.\ \ldots \Longrightarrow B$

Rule application

Example: rule: $\llbracket ?P;\ ?Q \rrbracket \Longrightarrow ?P \wedge ?Q$

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The general case: applying rule $\llbracket\ A_1;\ \ldots\ ;\ A_n\ \rrbracket \Longrightarrow A$ to subgoal $\ldots \Longrightarrow C$:

Rule application

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- Unify A and C

Rule application

Example: rule: $[\![?P; ?Q]\!] \Longrightarrow ?P \wedge ?Q$

subgoal: 1. $\ldots \Longrightarrow A \wedge B$

Result: 1. $\ldots \Longrightarrow A$

2. $\ldots \Longrightarrow B$

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- Replace C with n new subgoals $A_1 \ldots A_n$

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apply($rule\ xyz$)

"Backchaining"

Typical backwards rules

$$\frac{?P \quad ?Q}{?P \wedge\ ?Q}\ \texttt{conjI}$$

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$$\frac{?P \Longrightarrow ?Q}{?P \longrightarrow ?Q}\ \texttt{impI}$$

Typical backwards rules

$$\frac{?P \quad ?Q}{?P \wedge ?Q}\ \texttt{conjI}$$

$$\frac{?P \Longrightarrow ?Q}{?P \longrightarrow ?Q}\ \texttt{impI} \qquad \frac{\bigwedge x.\ ?P\ x}{\forall x.\ ?P\ x}\ \texttt{allI}$$

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They are known as introduction rules
because they *introduce* a particular connective.

Automating intro rules

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Can greatly increase the search space!

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$$\leadsto$$

$$?Q \Longrightarrow a = a \land ?Q$$

The general case:

If r is a theorem $[\![\ A_1; \ldots; A_n\ ]\!] \Longrightarrow A$
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$$\rightsquigarrow$$

$$a = a \land b = b$$

From now on: $?$ mostly suppressed on slides

`Single_Step_Demo.thy`

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Phrase theorems like this $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$
not like this $A_1 \wedge \ldots \wedge A_n \longrightarrow A$

Example: even numbers

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inductive $ev :: nat \Rightarrow bool$
where

$$ev\ 0 \quad |$$
$$ev\ n \Longrightarrow ev\ (n + 2)$$

An easy proof: $ev\ 4$

$$ev\ 0 \Longrightarrow ev\ 2 \Longrightarrow ev\ 4$$

Consider

fun $evn :: nat \Rightarrow bool$ **where**
$evn\ 0 = True\ |$
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To prove

$$ev\ n \Longrightarrow P\ n$$

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Rule `ev.induct`:

$$\frac{ev\ n \quad P\ 0 \quad \bigwedge n.\ [\![\ ev\ n;\ P\ n\ ]\!] \Longrightarrow P(n{+}2)}{P\ n}$$

Format of inductive definitions

inductive $I :: \tau \Rightarrow bool$ **where**

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Note:

- I may have multiple arguments.

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- Each rule may also contain *side conditions* not involving I.

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! Rule induction is absolutely central
to (operational) semantics
and the rest of this lecture course !

Inductive_Demo.thy

Inductively defined sets

inductive_set $I :: \tau\ set$ **where**

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Difference to **inductive**: I can later be used with set theoretic operators, eg $I \cup \ldots$

Chapter 4

Isar: A Language for Structured Proofs

Apply scripts

- unreadable

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- unreadable
- hard to maintain

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No structure!

Apply scripts versus Isar proofs

Apply script = assembly language program

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Apply scripts versus Isar proofs

Apply script = assembly language program

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But: **apply** still useful for proof exploration

A typical Isar proof

proof
  assume $formula_0$
  have $formula_1$ **by** $simp$
  $\vdots$
  have $formula_n$ **by** $blast$
  show $formula_{n+1}$ **by** $\ldots$
qed

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proves $formula_0 \Longrightarrow formula_{n+1}$

Isar core syntax

$$\begin{array}{lcl} \text{proof} & = & \textbf{proof}\ [\text{method}]\ \ \text{step}^*\ \ \textbf{qed} \\ & \mid & \textbf{by}\ \text{method} \end{array}$$

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$$

`Isar_Demo.thy`

Isar by example

Further reading

- More detailed Isar introduction in Chapter 5 of "Concrete Semantics"
- Isabelle/Isar reference manual (`isar-ref.pdf`), in particular Chapter 6