University of Cambridge
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Category Theory
Exercise Sheet 2
by Andrew Pitts and Marcelo Fiore

Terminal objects

1. A *partial state transformer* is a structure

$$S \xrightarrow{\sigma} 1 + S$$

in the category of sets.

A homomorphism of partial state transformers $f : (S, \sigma) \to (T, \tau)$ is a function $f : S \to T$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & 1 + S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_1 + f} \\ T & \xrightarrow[\tau]{} & 1 + T \end{array}$$

commutes; that is, for all $s \in S$,

- if $\sigma(s) = \iota_1(\,) \in 1 + S$, then $\tau(f\,s) = \iota_1(\,) \in 1 + T$;
- if $\sigma(s) = \iota_2(s') \in 1 + S$, for $s' \in S$, then $\tau(f\,s) = \iota_2(f\,s') \in 1 + T$.

Consider the category of partial state transformers and their homomorphisms, with composition and identities as for functions.

Construct a terminal object in the category of partial state transformers and prove its universal property.

2. In a category with a terminal object 1, a morphism $p : 1 \to X$ is called a *point* or *global element* of the object X.

A category **C** with a terminal object 1 is said to be *well-pointed* if, for all objects $X, Y \in \mathbf{C}$, two morphisms $f, g : X \to Y$ in **C** are equal if their compositions with all points of X are equal:

$$(\forall p \in \mathbf{C}(1, X),\ f \circ p = g \circ p) \Rightarrow f = g \tag{1}$$

   (a) Show that **Set** is well-pointed.

   (b) Is the opposite category $\mathbf{Set}^{\mathrm{op}}$ well-pointed?
   [Hint: Observe that the left-hand side of the implication in (1) is vacuously true in the case that $\mathbf{C}(1, X)$ is empty.]

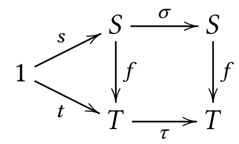
Initial objects

1. A *pointed state transformer* is a structure

$$1 \xrightarrow{s} S \xrightarrow{\sigma} S$$

in the category of sets.

A morphism of pointed state transformers $f : (S, s, \sigma) \to (T, t, \tau)$ is a function $f : S \to T$ such that the diagram



commutes.

Consider the category of pointed state transformers and their homomorphisms, with composition and identities as for functions.

Construct an initial object in the category of pointed state transformers and prove its universal property.

Products

1. Let $\mathbf{C}$ be a category with binary products.

   (a) For morphisms $f \in \mathbf{C}(X, Y)$, $g_1 \in \mathbf{C}(Y, Z_1)$ and $g_2 \in \mathbf{C}(Y, Z_2)$, show that

   $$\langle g_1, g_2 \rangle \circ f = \langle g_1 \circ f, g_2 \circ f \rangle \in \mathbf{C}(X, Z_1 \times Z_2)$$

   (b) For morphisms $f_1 \in \mathbf{C}(X_1, Y_1)$ and $f_2 \in \mathbf{C}(X_2, Y_2)$, define

   $$f_1 \times f_2 \triangleq \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle \in \mathbf{C}(X_1 \times X_2, Y_1 \times Y_2) \tag{2}$$

   For any $g_1 \in \mathbf{C}(Z, X_1)$ and $g_2 \in \mathbf{C}(Z, X_2)$, show that

   $$(f_1 \times f_2) \circ \langle g_1, g_2 \rangle = \langle f_1 \circ g_1, f_2 \circ g_2 \rangle \in \mathbf{C}(Z, Y_1 \times Y_2)$$

   (c) Show that the operation $f_1, f_2 \mapsto f_1 \times f_2$ defined in (2) satisfies

   $$(h_1 \times h_2) \circ (k_1 \times k_2) = (h_1 \circ k_1) \times (h_2 \circ k_2)$$
   $$\mathrm{id}_X \times \mathrm{id}_Y = \mathrm{id}_{X \times Y}$$

2. A *pairing* for a monoid $(M, \bullet, \iota)$ consists of elements $p_1, p_2 \in M$ and a binary operation $\langle \_, \_ \rangle : M \times M \to M$ satisfying, for all $x, y, z \in M$,

$$p_1 \cdot \langle x, y \rangle = x$$
$$p_2 \cdot \langle x, y \rangle = y$$
$$\langle p_1, p_2 \rangle = \iota$$
$$\langle x, y \rangle \cdot z = \langle x \cdot z, y \cdot z \rangle$$

Given such a pairing, show that the monoid, when regarded as a one-object category, has binary products.

3. Let $\mathbf{C}$ be a category with binary products $\_\times\_$ and a terminal object 1. Given objects $X, Y, Z \in \mathbf{C}$, construct natural isomorphisms

$$\begin{aligned}\alpha_{X,Y,Z} &: (X \times Y) \times Z \cong X \times (Y \times Z)\\ \lambda_X &: 1 \times X \cong X\\ \rho_X &: X \times 1 \cong X\\ \tau_{X,Y} &: X \times Y \cong Y \times X\end{aligned}$$

4. A category $\mathbf{C}$ is called *locally finite* if, for all $X, Y \in \operatorname{obj}\mathbf{C}$, the set of morphisms $\mathbf{C}(X, Y)$ is finite. $\mathbf{C}$ is said to be *finite* if it is both locally finite and $\operatorname{obj}\mathbf{C}$ is finite.

   (a) Prove that any finite category with binary products is a pre-order; that is, there is at most one morphism between any pair of objects.

   [Hint: If $f, g : X \to Y$ were distinct, use them to construct too large a number of morphisms from X to the product Y^n of Y with itself n (> 0) times, for some suitable number n.]

   (b) Is every locally finite category with binary products a pre-order? (Either prove it, or give a counterexample.)

Coproducts

1. A monoid $(M, \bullet, \iota)$ is said to be *abelian* (or commutative) if its multiplication is commutative: $\forall x, y \in M,\ x \bullet y = y \bullet x$.

   Let **AbMon** be the category whose objects are abelian monoids and whose morphisms, identity morphisms and composition are as in **Mon**.

   (a) Show that the product in **Mon** of two abelian monoids gives their product in **AbMon**.

   (b) Given $M_1, M_2 \in \operatorname{obj}\mathbf{AbMon}$ define morphisms

   $$\iota_1 \in \mathbf{AbMon}(M_1, M_1 \times M_2) \text{ and } \iota_2 \in \mathbf{AbMon}(M_2, M_1 \times M_2)$$

   that make $M_1 \times M_2$ into a *coproduct* in **AbMon**.

2. In this question I use the notation $A \xrightarrow{\mathrm{inl}_{A,B}} A + B \xleftarrow{\mathrm{inr}_{A,B}} Y$ for the coproduct of two objects A and B in a category, as it will be clearer to make explicit the objects A and B in the notation for the associated coproduct injections, $\mathrm{inl}_{A,B}$ and $\mathrm{inr}_{A,B}$.

   A category $\mathbf{C}$ is *distributive* if it has all binary products and binary coproducts, and for all $X, Y, Z \in \operatorname{obj}\mathbf{C}$, using the defining property of the coproduct

   $$X \times Y \xrightarrow{\mathrm{inl}_{X\times Y, X\times Z}} (X \times Y) + (X \times Z) \xleftarrow{\mathrm{inr}_{X\times Y, X\times Z}} X \times Z$$

   the unique morphism

   $$\delta_{X,Y,Z} : (X \times Y) + (X \times Z) \to X \times (Y + Z)$$

that makes the following diagram commute

$$\begin{array}{ccc}
X \times Y & & \\
{\scriptstyle \mathrm{inl}_{X\times Y, X\times Z}}\downarrow & \overset{\mathrm{id}_X \times \mathrm{inl}_{Y,Z}}{\searrow} & \\
(X \times Y) + (X \times Z) & \xrightarrow{\delta_{X,Y,Z}} & X \times (Y + Z) \\
{\scriptstyle \mathrm{inr}_{X\times Y, X\times Z}}\uparrow & \underset{\mathrm{id}_X \times \mathrm{inr}_{Y,Z}}{\nearrow} & \\
X \times Z & &
\end{array}$$

is an isomorphism.

(a) Using the usual product and coproduct constructs in the category **Set**, show that it is a distributive category.

(b) Give, with justification, an example of a category with binary products and coproducts that is not distributive.

(c) If **C** is a distributive category and 0 is an initial object in **C**, prove that for all $X \in \mathrm{obj}\,\mathbf{C}$, the unique morphism $0 \longrightarrow X \times 0$ is an isomorphism.

Algebras

1. A *monoid object* in a cartesian category **C** is a structure

$$M \times M \xrightarrow{m} M \xleftarrow{u} 1$$

in **C** such that the following diagrams commute:

$$\begin{array}{ccccc}
(M \times M) \times M & \xrightarrow[\cong]{\alpha_{M,M,M}} & M \times (M \times M) & \xrightarrow{\mathrm{id}_M \times m} & M \times M \\
{\scriptstyle m \times \mathrm{id}_M}\downarrow & & & & \downarrow{\scriptstyle m} \\
M \times M & & \xrightarrow[m]{\qquad\qquad} & & M
\end{array} \qquad \text{(associativity)}$$

$$\begin{array}{ccccc}
1 \times M & \xrightarrow{u \times \mathrm{id}_M} & M \times M & \xleftarrow{\mathrm{id}_m \times u} & M \times 1 \\
& {\scriptstyle \lambda_M}\underset{\cong}{\searrow} & \downarrow{\scriptstyle m} & \underset{\cong}{\swarrow}{\scriptstyle \rho_M} & \\
& & M & &
\end{array} \qquad \text{(left and right unit)}$$

(a) Show that a monoid object in **Set** is equivalently a monoid.

(b) Define a notion of morphism between monoid objects in **C** such that for $\mathbf{C} = \mathbf{Set}$ it is equivalently the notion of monoid homomorphism.

(c) If $\underline{M} = (M, \bullet, \imath)$ is an abelian monoid, show that the functions $m : M \times M \longrightarrow M$ and $u : 1 \longrightarrow M$ defined by

$$\begin{aligned} m(x, y) &= x \bullet y & \qquad\qquad (x, y \in M) \\ u(\,) &= \imath \end{aligned}$$

determine morphisms $m \in \mathbf{Mon}(\underline{M} \times \underline{M}, \underline{M})$ and $u \in \mathbf{Mon}(1, \underline{M})$. Show further that (M, m, u) is a monoid object in **Mon**.

Show that every monoid object in **Mon** arises as above. [Hint: If necessary, search the internet for "Eckmann-Hilton argument".]