Commutative diagrams

In a category $\mathbf{C}$:

a diagram is

> a directed graph whose vertices are $\mathbf{C}$-objects and whose edges are $\mathbf{C}$-morphisms

and the diagram is commutative (or commutes) if

> any two finite paths in the graph between any two vertices determine equal morphisms in the category under composition

Examples:

- The diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f"'] & Y \arrow[d, "\mathtt{id}_Y"] \arrow[dr, "g"] & \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

  commutes by the unity laws.
- The diagram

$$\begin{tikzcd} X \arrow[r, "g\,f"] \arrow[d, "f"'] & Y \arrow[d, "h"] \\ Y \arrow[ur, "g"] \arrow[r, "h\,g"'] & Z \end{tikzcd}$$

  commutes by the associativity law.

One-object categories

Problem: Give an equivalent elementary description of categories with a singleton set of objects.

Each monoid determines a category

Given a monoid $\underline{M} = (M, \bullet, \imath)$, we get a category

$$\mathbf{C}_{\underline{M}}$$

by taking

- objects: $\mathtt{obj}\,\mathbf{C}_{\underline{M}} = \{*\}$ (a singleton set)
- morphisms: $\mathbf{C}_{\underline{M}}(*, *) = M$
- identity morphism: $\mathtt{id}_* = \imath \in M = \mathbf{C}_{\underline{M}}(*, *)$
- composition $g \circ f \in \mathbf{C}_{\underline{M}}$ of $f \in \mathbf{C}_{\underline{M}}(*, *)$ and $g \in \mathbf{C}_{\underline{M}}(*, *)$ is $g \bullet f \in M = \mathbf{C}_{\underline{M}}(*, *)$

Isomorphism

Let $\mathbf{C}$ be a category. A $\mathbf{C}$-morphism $f : X \to Y$ is an isomorphism if there is some $g : Y \to X$ for which

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\mathtt{id}_X"'] & Y \arrow[d, "g"] \arrow[dr, "\mathtt{id}_Y"] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

is a commutative diagram.

Isomorphism

Let $\mathbf{C}$ be a category. A $\mathbf{C}$-morphism $f : X \to Y$ is an isomorphism if there is some $g : Y \to X$ with $g \circ f = \mathtt{id}_X$ and $f \circ g = \mathtt{id}_Y$.

- Such a g is uniquely determined by f (why?) and we write $\boxed{f^{-1}}$ for it.
- Given $X, Y \in \mathbf{C}$, if such an f exists, we say the objects X and Y are isomorphic in $\mathbf{C}$ and write $\boxed{X \cong Y}$

NB: There may be many different morphisms witnessing the fact that two objects are isomorphic.

Proposition. A function $f \in \mathbf{Set}(X, Y)$ is an isomorphism in the category $\mathbf{Set}$ iff f is a bijection, equivalently:

- injective: $\forall x, x' \in X,\ f\,x = f\,x' \Rightarrow x = x'$

and

- surjective: $\forall y \in Y, \exists x \in X,\ f\,x = y$

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Proposition. A monoid morphism $f \in \mathbf{Mon}((M_1, \bullet_1, \imath_1), (M_2, \bullet_2, \imath_2))$ is an isomorphism in the category $\mathbf{Mon}$ iff $f \in \mathbf{Set}(M_1, M_2)$ is a bijection.

Categories with trivial hom-sets

Problem: Give an equivalent elementary description of categories with trivial hom-sets in the sense of being either empty or a singleton set.

Preorders and posets

A **preorder** $\underline{P} = (P, \sqsubseteq)$ consists of a set P to equipped with a binary relation on it $\_ \sqsubseteq \_ \subseteq P \times P$ that is

reflexive: $\forall x \in P,\ x \sqsubseteq x$

transitive: $\forall x, y, z \in P,\ x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z$

A **poset** or (**partial order**) is a preorder that is also

anti-symmetric: $\forall x, y \in P,\ x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$

Examples:

- $(\mathbb{N}, \leq)$, $(\mathbb{N}, \geq)$
- $(\mathscr{P}(X), \subseteq)$, $(\mathscr{P}(X), \supseteq)$
- $(\mathbb{Z}, \mid)$ where $n | m \overset{\Delta}{\Leftrightarrow} n$ divides m

Proposition.

1. If $\underline{P} = (P, \sqsubseteq)$ is a preorder, then so is $\underline{P}^{\mathrm{op}} \triangleq (P, \sqsupseteq)$ where $x \sqsupseteq y \overset{\Delta}{\Leftrightarrow} y \sqsubseteq x$.
2. $(\underline{P}^{\mathrm{op}})^{\mathrm{op}} = \underline{P}$

Each preorder determines a category

Given a preorder $\underline{P} = (P, \sqsubseteq)$, we get a category

$$\mathbf{C}_{\underline{P}}$$

by taking

- objects: $\texttt{obj}\,\mathbf{C}_{\underline{P}} = P$
- morphisms: $\mathbf{C}_{\underline{P}}(x, y) \triangleq \begin{cases} \{\,(x, y)\,\} & \text{, if } x \sqsubseteq y \\ \emptyset & \text{, if } x \not\sqsubseteq y \end{cases}$
- identity morphisms and composition are uniquely determined (why?)

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E.g. when $\underline{P}$ has just one element 0

$$\mathbf{C}_{\underline{P}} = \boxed{\begin{tikzcd} 0 \arrow[loop left, "{(0,0)=\mathtt{id}_0}"] \end{tikzcd} \\ \text{one object, one morphism}}$$

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- identity morphisms and composition are uniquely determined (why?)

E.g. when $\underline{P}$ has just two elements $0 \sqsubseteq 1$

$$\mathbf{C}_{\underline{P}} = \boxed{\begin{tikzcd} 0 \arrow[loop left, "{(0,0)=\mathtt{id}_0}"] \arrow[r, "{(0,1)}"] & 1 \arrow[loop right, "{\mathtt{id}_1=(1,1)}"] \end{tikzcd}}$$

two objects, one non-identity morphism

Category of preorders: **Preord**

- objects: preorders
- morphisms:

$$\mathbf{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$$
$$\triangleq \{f \in \mathbf{Set}(P_1, P_2) \mid f \text{ is monotone}\}$$

monotonicity: $\forall x, x' \in P_1,\ x \sqsubseteq_1 x' \Rightarrow f\,x \sqsubseteq_2 f\,x'$

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- identities and composition: as for **Set**

Q: why is this well-defined?
A: because the set of monotone functions contains identity functions and is closed under composition.

Subcategory of posets: **Poset**

Define **Poset** to be the category whose objects are posets, and is otherwise defined like the category **Preord** of preorders.

NB: Pre and partial orders are relevant to the denotational semantics of programming languages (among other things).

Proposition. A morphism $f \in \mathbf{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$ is an isomorphism in the category **Poset** iff the function $f \in \mathbf{Set}(P_1, P_2)$ is

surjective: $\forall y \in P_2,\ \exists x \in P_1,\ f(x) = y$

and

reflective: $\forall x, x' \in P_1,\ f\,x \sqsubseteq_2 f\,x' \Rightarrow x \sqsubseteq_1 x'$

(Why does this characterisation not work for **Preord**?)

Problem: Recalling that categories generalise both monoids and preorders, find a notion of morphism between categories that generalises both monoid homomorphisms and monotone functions.

Category-theoretic properties

Any two isomorphic objects in a category should have the same category-theoretic properties – statements that are provable in a formal logic for category theory, whatever that is.

Instead of trying to formalize such a logic, we will just look at examples of category-theoretic properties.

Here is our first one…

Terminal object

An object T of a category $\mathbf{C}$ is terminal if for all $X \in \mathbf{C}$, there is a unique $\mathbf{C}$-morphism from X to T, which we write as $\boxed{\langle\rangle_X : X \to T}$.

So we have $\begin{cases} \forall X \in \mathbf{C},\ \langle\rangle_X \in \mathbf{C}(X,T) \\ \forall X \in \mathbf{C}, \forall f \in \mathbf{C}(X,T),\ f = \langle\rangle_X \end{cases}$

(In particular, $\mathtt{id}_T = \langle\rangle_T$.)

Convention: Sometimes we write $!_X$ or X for $\langle\rangle_X$ —there is no commonly accepted notation— and also just write $\langle\rangle$ or $!$.

Examples of terminal objects

- In **Set** a set is terminal iff it is a singleton.
- Any one-element set has a unique preorder and this makes it terminal in **Preord** and **Poset**.
- Any one-element set has a unique monoid (group) structure and this makes it terminal in **Mon** (**Grp**).

Examples of terminal objects

- In **Set** a set is terminal iff it is a singleton.
- Any one-element set has a unique preorder and this makes it terminal in **Preord** and **Poset**.
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- A preorder $\underline{P} = (P, \sqsubseteq)$, regarded as a category $\mathbf{C}_{\underline{P}}$, has a terminal object iff it has a greatest element $\top$, that is: $\forall x \in P,\ x \sqsubseteq \top$.

Terminal object

$\boxed{\text{terminal objects are unique up to unique isomorphism}}$

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Proposition. In a category,

(a) If T and T' are both terminal, then $T \cong T'$ (and there is only one isomorphism between T and T').

(b) If T is terminal and $T \cong T'$, then T' is terminal.

Notation: If a category $\mathbf{C}$ has a terminal object we will write that object as $\boxed{1_{\mathbf{C}}}$ or $\boxed{1}$.

Global elements

Given a category $\mathbf{C}$ with a terminal object $\mathtt{1}$.

A global element of an object $X \in \mathtt{obj}\,\mathbf{C}$ is by definition a morphism $\mathtt{1} \to X$ in $\mathbf{C}$.

E.g. $\mathbf{Set}(1_{\mathbf{Set}}, X) \cong X$; in $\mathbf{Mon}(1_{\mathbf{Mon}}, \underline{M}) \cong 1_{\mathbf{Set}}$.

Global elements

Given a category $\mathbf{C}$ with a terminal object $\mathtt{1}$.

A global element of an object $X \in \mathtt{obj}\,\mathbf{C}$ is by definition a morphism $\mathtt{1} \to X$ in $\mathbf{C}$.

E.g. $\mathbf{Set}(1_{\mathbf{Set}}, X) \cong X$; in $\mathbf{Mon}(1_{\mathbf{Mon}}, \underline{M}) \cong 1_{\mathbf{Set}}$.

Say that $\mathbf{C}$ is well-pointed if for all $f, g : X \to Y$ in $\mathbf{C}$ we have:

$$\left(\forall 1 \xrightarrow{x} X \text{ in C},\ f \circ x = g \circ x\right) \Rightarrow f = g$$

E.g. $\mathbf{Set}$ is well-pointed (by function extensionality); $\mathbf{Mon}$ is not.

Generalising elements

- **Proposition**. For all $f, g : M \longrightarrow M'$ in **Mon**, if

$$\forall\, e : \mathbb{N} \longrightarrow M \text{ in } \mathbf{Mon},\ f \circ e = g \circ e : \mathbb{N} \longrightarrow M'$$

  then

$$f = g$$

Directed graphs

Let **DiGph** be the category with

- objects: $(\, E, N \in \mathtt{obj}\,\mathbf{Set}\,,\ \ s, t : E \to N \text{ in } \mathbf{Set}\,)$;
- morphisms:

$$h = (h_e, h_n) : (\, E \overset{s}{\underset{t}{\rightrightarrows}} N\,) \longrightarrow (\, E' \overset{s'}{\underset{t'}{\rightrightarrows}} N'\,)$$

  given by functions $h_e : E \to E'$ and $h_n : N \to N'$ such that

$$\forall\, a \in E.\ s'(h_e\, a) = h_n(s\, a)$$

  and

$$\forall\, a \in E.\ t'(h_e\, a) = h_n(t\, a)$$

- identities and composition: given pointwise as in **Set**.

Directed graphs in a category

- For a category $\mathbf{C}$, let $\mathbf{DiGph}(\mathbf{C})$ be the category with
  - objects: diagrams $E \underset{t}{\overset{s}{\rightrightarrows}} N$ in $\mathbf{C}$;
  - morphisms:

$$h = (h_e, h_n) : (\ E \underset{t}{\overset{s}{\rightrightarrows}} N\ ) \longrightarrow (\ E' \underset{t'}{\overset{s'}{\rightrightarrows}} N'\ )$$

given by $h_e : E \to E'$ and $h_n : N \to N'$ in $\mathbf{C}$ such that

$$\begin{tikzcd} E \arrow[r, "s"] \arrow[d, "h_e"'] & N \arrow[d, "h_n"] \\ E' \arrow[r, "s'"'] & N' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \arrow[r, "t"] \arrow[d, "h_e"'] & N \arrow[d, "h_n"] \\ E' \arrow[r, "t'"'] & N' \end{tikzcd}$$

commute;
  - identities and composition: given pointwise as in $\mathbf{C}$.

NB. $\mathbf{DiGph}(Set) = \mathbf{DiGph}$.

Proposition. $\mathbf{DiGph}(\mathbf{Set})$ is not well-pointed. (Why?)

$\boxed{?}$ Is there a single $C \in \mathbf{DiGph}(\mathbf{Set})$ such that, for all $f, g : G \longrightarrow G'$ in $\mathbf{DiGph}(\mathbf{Set})$,

$\forall\, c : C \longrightarrow G$ in $\mathbf{DiGph}(\mathbf{Set})$, $f \circ c = g \circ c : C \longrightarrow G'$

implies

$$f = g \quad ?$$

Proposition. There exist $A, B \in \mathbf{DiGph(Set)}$ such that for all $f, g : G \longrightarrow G'$ in $\mathbf{DiGph(Set)}$,
if

$\forall\, a : A \longrightarrow G$ in $\mathbf{DiGph(Set)}$, $f \circ a = g \circ a : A \longrightarrow G'$

and

$\forall\, b : B \longrightarrow G$ in $\mathbf{DiGph(Set)}$, $f \circ b = g \circ b : B \longrightarrow G'$

then

$$f = g$$

Generalised elements

Idea:

Replace global elements $\mathtt{1} \xrightarrow{x} X$ of X

by morphisms $C \xrightarrow{x} X$ for $C \in \mathtt{obj}\,\mathbf{C}$

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Some people say that x is a generalised element of X at stage C and use the notation $x \in_C X$. For instance, $\langle\rangle_C \in_C \mathtt{1}$ is the unique generalised element of $\mathtt{1}$ at stage C.

One may also think that x inhabits X in context C and use the notation $C \vdash x : X$; for instance, $C \vdash \langle\rangle_C : \mathtt{1}$.

NB: One has to take into account "change of stage or context": for $\sigma : D \to C$,

$$x \in_C X \;\Rightarrow\; x\,\sigma \in_D X$$

$$C \vdash x : X \;\Rightarrow\; D \vdash x\,\sigma : X$$

(cf. Kripke's "possible world" semantics of intuitionistic and modal logics)

Opposite of a category

Given a category $\mathbf{C}$, its opposite category $\boxed{\mathbf{C}^{\mathrm{op}}}$ is defined by interchanging the operations of $\mathtt{dom}$ and $\mathtt{cod}$ in $\mathbf{C}$:

- $\mathtt{obj}\,\mathbf{C}^{\mathrm{op}} \triangleq \mathtt{obj}\,\mathbf{C}$
- $\mathbf{C}^{\mathrm{op}}(X, Y) \triangleq \mathbf{C}(Y, X)$, for all objects X and Y
- identity morphism on $X \in \mathtt{obj}\,\mathbf{C}^{\mathrm{op}}$ is $\mathtt{id}_X \in \mathbf{C}(X, X) = \mathbf{C}^{\mathrm{op}}(X, X)$
- composition in $\mathbf{C}^{\mathrm{op}}$ of $f \in \mathbf{C}^{\mathrm{op}}(X, Y)$ and $g \in \mathbf{C}^{\mathrm{op}}(Y, Z)$ is given by the composition $f \circ g \in \mathbf{C}(Z, X) = \mathbf{C}^{\mathrm{op}}(X, Z)$ in $\mathbf{C}$ (associativity and unity properties hold for this operation because they do in $\mathbf{C}$)