

Introduction to Probability

Lecture 7: Independence, Covariance and Correlation

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Independence of Random Variables

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Two random variables X and Y are **independent** if for all values a, b :

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Using the **joint probability distribution**, the above is equivalent to for all a, b ,

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All these definitions extend in the natural way to **more than two** variables!



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The definition of independence of X and Y implies the following **factorisation** formula: for any “suitable” sets A and B ,

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Example

Let X and Y be two independent variables. Let $I = (a, b]$ be any interval and define $U := \mathbf{1}_{X \in I}$ and $V := \mathbf{1}_{Y \in I}$. Prove U and V are independent.

Answer



Buffon's Needle Problem (1/2)



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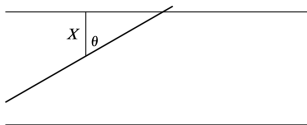
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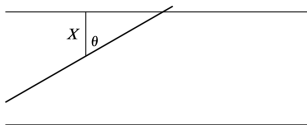
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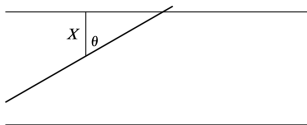
Let X be the distance of the **middle point** of the needle to the closest parallel line. Needle intersects a line if hypotenuse of the triangle is less than $L/2$, i.e.,

$$\frac{X}{\cos(\theta)} < \frac{L}{2} \quad \Leftrightarrow \quad X < \frac{L}{2} \cos(\theta).$$

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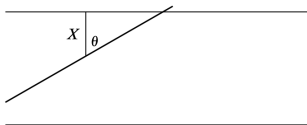
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We assume that $X \in [0, D/2]$ and $\theta \in [0, \pi/2]$ are **independent** and **uniform**.

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Can be thought of as: 1. Sample the middle point of needle, 2. Sample the angle.



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This gives us a method to estimate π !



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- Two variables X, Y with $\mathbf{Cov}[X, Y] = 0$ are **uncorrelated**.



Illustration of 3 Cases for $\text{Cov}[X, Y]$

500 outcomes of randomly generated pairs of RVs (X, Y) with different joint distributions

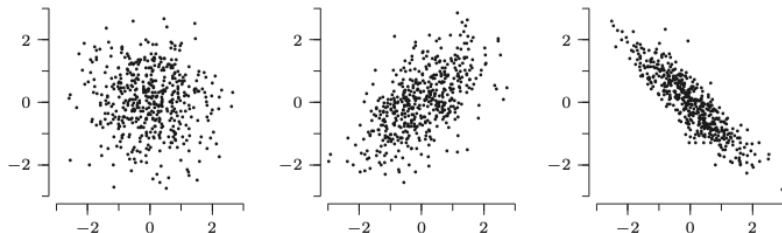


Fig. 10.1. Some scatterplots.
Source: Textbook by Dekking

1. What is the covariance (positive, negative, neutral)?
2. Where is the covariance the largest (in magnitude)?

Independence implies Uncorrelated

Example

Let X and Y be two **independent** random variables. Then X and Y are **uncorrelated**, i.e., $\mathbf{Cov}[X, Y] = 0$.

Answer

We give a proof for the discrete case:



Uncorrelated may not imply Independence

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$$1 = \mathbf{P}[X \cdot Y = 0] > \mathbf{P}[X = 0] \cdot \mathbf{P}[Y = 0] = 2/9.$$



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Generalisation of the case where X and Y are even **independent**!

- For any random variables $X_1, X_2, \dots, X_n$:

$$\mathbf{V}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbf{V}[X_i] + 2 \cdot \sum_{i=1}^n \sum_{j=i+1}^n \mathbf{Cov}[X_i, X_j].$$

Computing Variances of Sums of Uncorrelated Variables

Example

Recall the example where $X \in \{-1, 0, +1\}$ uniformly and $Y := \mathbf{1}_{X=0}$. Compute $\mathbf{V}[X + Y]$.

Answer



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Let X and Y be two random variables. The **correlation coefficient** $\rho(X, Y)$ is defined as:

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If $\mathbf{V}[X] = 0$ or $\mathbf{V}[Y] = 0$, then it is defined as 0.



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Properties:

1. The correlation coefficient is **scaling-invariant**, i.e.,
 $\rho(X, Y) = \rho(\alpha \cdot X, \beta \cdot Y)$ for any $\alpha, \beta > 0$.



Correlation Coefficient: Normalising the Covariance

The definition of covariance is **not scaling invariant**:

- If X increases by a factor of α , then $\mathbf{Cov}[X, Y]$ increases by a factor of α .
- ⇒ Even if X and Y both increase by α , then $\mathbf{Cov}[X, Y]$ will change.
(Exercise: It changes by?)

Correlation Coefficient

Let X and Y be two random variables. The **correlation coefficient** $\rho(X, Y)$ is defined as:

$$\rho(X, Y) = \frac{\mathbf{Cov}[X, Y]}{\sqrt{\mathbf{V}[X] \cdot \mathbf{V}[Y]}}.$$

If $\mathbf{V}[X] = 0$ or $\mathbf{V}[Y] = 0$, then it is defined as 0.

Properties:

1. The correlation coefficient is **scaling-invariant**, i.e.,
 $\rho(X, Y) = \rho(\alpha \cdot X, \beta \cdot Y)$ for any $\alpha, \beta > 0$.
2. For any two random variables X, Y , $\rho(X, Y) \in [-1, 1]$.



Range of the Correlation Coefficient

Example

Verify that the correlation coefficients' range satisfies $\rho(X, Y) \in [-1, 1]$.

Answer



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