Introduction to Probability

Lecture 4: More discrete distributions – Poisson, Geometric, Negative Binomial, Hypergeometric

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Outline

Poisson discrete random variable

Geometric discrete random variable

Negative binomial discrete random variable

Hypergeometric discrete random variable

Preliminaries:

The natural exponent e

e is a mathematical constant AKA the Euler number. e is very important for exponential functions. Here are some important identities:
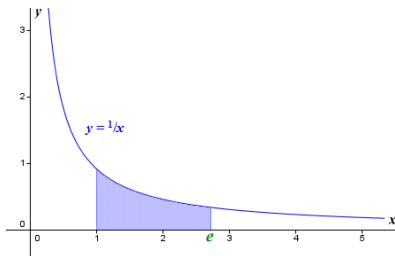
$$e \approx 2.71828$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$e^{-\lambda} = \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n$$

$$e^r = \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n$$

$$e^r = \sum_{n=0}^{\infty} \frac{r^n}{n!}$$

$y = 1/x$

Binomial RV example: large *n*, small *p*

We are trying to predict footfall in a store. We know, based on previous data, that on average 8 people enter the store per hour. What is the probability of *k* people entering the store in the next 1 hour?

1. Break an hour into **minutes**.

   What if 2 people enter in the same minute?

   - At each **minute**, independent Bernoulli trial with 1 for a person entering the store and 0 for nobody entering the store.
   - X is a Binomial RV: # people entering in an hour, so $\mathbf{E}[X] = np = \lambda = 8$.
   - $X \sim Bin(n = 60, p = \frac{\lambda}{n})$, so $\mathbf{P}[X=k] = \binom{n}{k}p^k(1-p)^{n-k} = \binom{60}{k}\left(\frac{8}{60}\right)^k\left(1-\frac{8}{60}\right)^{n-k}$

2. Break an hour into **milliseconds**.

   What if 2 people enter in the same millisecond?

   - At each **millisecond**, independent Bernoulli trial: 1 for enter, 0 for not enter.
   - X is a Binomial RV: # people entering in an hour, so $\mathbf{E}[X] = np = \lambda = 8$.
   - $X \sim Bin(n = 3600000, p = \frac{\lambda}{n})$, so $\mathbf{P}[X=k] = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$

3. Break an hour into **infinitely small units**.
   - At each **unit**, independent Bernoulli trial: 1 for enter, 0 for not enter.
   - X is a Binomial RV: # people entering in an hour, so $\mathbf{E}[X] = np = \lambda = 8$.
   - $X \sim Bin(n, p = \frac{\lambda}{n})$, thus $\mathbf{P}[X=k] = \lim_{n\to\infty} \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$

Computing Binomial in the limit

$$\mathbf{P}\,[\,X = k\,] = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \underset{expand}{=} \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \frac{\lambda^k}{n^k} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

$$\underset{rearrange}{=} \lim_{n\to\infty} \frac{n!}{n^k(n-k)!} \frac{\lambda^k}{k!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} \underset{def\ of\ e}{=} \lim_{n\to\infty} \frac{n!}{n^k(n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

$$\underset{expand}{=} \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

as $n \to \infty$ $\qquad \dfrac{n(n-1)\cdots(n-k+1)}{n^k} \approx \dfrac{n^k}{n^k} = 1$

$$\left(1 - \frac{\lambda}{n}\right)^k \approx 1^k = 1$$

$$\left(1 - \frac{\lambda}{n}\right)^n \approx e^{-\lambda} \text{ because } e^{-\lambda} = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n \text{ thus we have}$$

Therefore, in our store footfall example: the probability of k people entering the store in the next 1 hour is:

$$\mathbf{P}\,[X=k] = \lim_{n\to\infty} \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!}e^{-\lambda}$$

$$\mathbf{P}\,[\,X = k\,] = (1)\frac{\lambda^k}{k!} \frac{e^{-\lambda}}{1} = \frac{\lambda^k}{k!}e^{-\lambda}$$

Poisson

Poisson discrete random variable

A Poisson RV *X* approximates Binomial where *n* is large, *p* is small, and $\lambda = np$ is "moderate". Thus we no longer need to know *n* and *p*, we only need to provide **rate** λ. *X* is the number of successes over the duration of the experiment.

$$X \sim \textbf{Pois}(\lambda)$$

Range: $\{0, 1, 2, \ldots\}$

PMF: $\textbf{P}[X = k] = \dfrac{\lambda^k}{k!} e^{-\lambda}$

Expectation: $\textbf{E}[X] = \lambda$

Variance: $\textbf{V}[X] = \lambda$

Examples: # earthquakes in a given year, # goals scored during a 90 minute football game, # misprints per page in a book, # emails per day.

Key idea: Divide time into a **large number** of small increments. Assume that during each increment, there is some **small probability** of the event happening (independent of other increments).
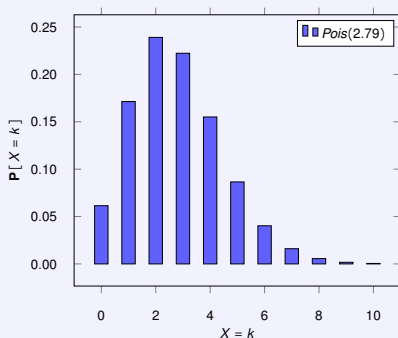
Earthquake example

Example

Suppose there are an average of 2.79 major earthquakes in the world each year. What is the probability of getting 3 major earthquakes next year?
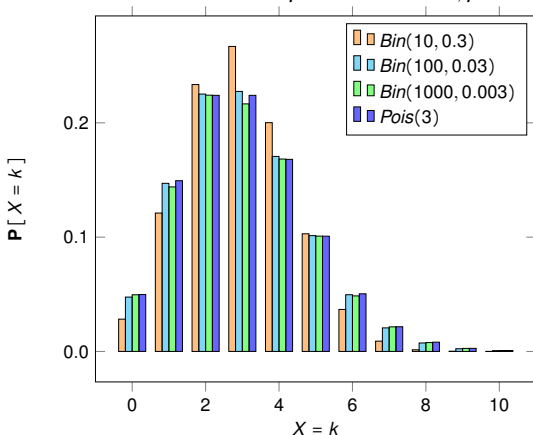
Answer

Define RVs: $\lambda = 2.79, k = 3, X \sim Pois(2.79)$

Poisson paradigm

- Poisson approximates Binomial when *n* is large, *p* is small, and $\lambda = np$ is "moderate".
- Different interpretations of "moderate". Commonly accepted ranges are:
    - *n* > 20 and *p* < 0.05
    - *n* > 100 and *p* < 0.1
- Poisson is Binomial in the limit: $\lambda = np$ where $n \to \infty, p \to 0$.

$$\text{PMF: } = k \in \{0, 1, 2, \ldots \infty\}; \mathbf{P}\left[\, X = k \,\right] = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\mathbf{E}\left[\, X \,\right] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} =$$

$$= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \text{ (let } i = k - 1)$$

$$= \lambda e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Poisson variance

$$\mathbf{E}\left[X^2\right] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \text{ (let } i = k - 1)$$

$$= \lambda \sum_{i=0}^{\infty} (i+1) \frac{\lambda^i}{i!} e^{-\lambda} = \lambda \left(\underbrace{\sum_{i=0}^{\infty} i \frac{\lambda^i}{i!} e^{-\lambda}}_{\text{same as before}} + \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda}}_{\text{sum of PMFs=1}} \right) =$$

$$= \lambda(\lambda + 1) \qquad \text{thus}$$

$$\mathbf{V}\left[X\right] = \mathbf{E}\left[X^2\right] - \left(\mathbf{E}\left[X\right]\right)^2 = \lambda(\lambda + 1) - \lambda^2 = \lambda$$

$$\mathbf{E}\left[X^k\right] = \lambda \mathbf{E}\left[(X+1)^{k-1}\right]$$

Bernoulli, Poisson, and random processes

- A Poisson process is a model for a series of discrete events where the average time between events is known, but the exact timing of events is random.
    - The arrival of an event is independent of the event before (waiting time between events is memoryless).
    - The average rate (events per time period) is constant.
    - Two events cannot occur at the same time: each sub-interval of a Poisson process is a Bernoulli trial that is either a success or a failure.

- Example: your website goes down on average twice per 60 days; calling a help centre; movements of stock price...
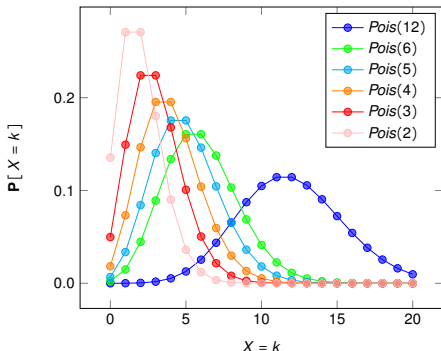
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Geometric

Geometric discrete random variable

X is a geometric RV if X is a number of independent Bernoulli trials until the first success, and p is the probability of success on each Bernoulli trial.

$$X \sim \textbf{Geo}(\textbf{p})$$

Range: $\{1, 2, \ldots\}$

PMF: $\mathbf{P}[\, X = n \,] = (1 - p)^{n-1} p$

Expectation: $\mathbf{E}[\, X \,] = \dfrac{1}{p}$

Variance: $\mathbf{V}[\, X \,] = \dfrac{1 - p}{p^2}$

Examples: tossing a coin ($\mathbf{P}[\, head \,] = p$) until first heads appears, generating bits with $\mathbf{P}[\, bit = 1 \,] = p$ until first 1 is generated.

Geometric RV properties

PMF (E_i is the event that the i-th trial succeeds):

$$\mathbf{P}\left[\, X = n \,\right] = \mathbf{P}\left[\, E_1^c E_2^c \ldots E_{n-1}^c E_n \,\right] =$$
$$= \mathbf{P}\left[\, E_1^c \,\right]\mathbf{P}\left[\, E_2^c \,\right] \ldots \mathbf{P}\left[\, E_{n-1}^c \,\right]\mathbf{P}\left[\, E_n \,\right] =$$
$$= (1-p)^{n-1}p$$

CDF ($\mathbf{P}\left[\, X > n \,\right]$ is the probability that at least the first n trials fail):

$$\mathbf{P}\left[\, X \leq n \,\right] = 1 - \mathbf{P}\left[\, X > n \,\right] =$$
$$= 1 - \mathbf{P}\left[\, E_1^c E_2^c \ldots E_n^c \,\right] =$$
$$= 1 - \mathbf{P}\left[\, E_1^c \,\right]\mathbf{P}\left[\, E_2^c \,\right] \ldots \mathbf{P}\left[\, E_n^c \,\right] =$$
$$= 1 - (1-p)^n$$

Die example

You roll a fair 6-sided die until it comes up with # 6. What is the probability that it will take 3 rolls?

Answer

Let X be a RV for # of rolls. Probability for any # on die is $\frac{1}{6}$.

Define RVs: $X \sim Geo(\frac{1}{6})$, want $\mathbf{P}[X = 3]$.

Solve:

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Negative binomial

Negative binomial discrete random variable

X is a negative binomial RV if X is the number of independent Bernoulli trials until r successes and p is the probability of success on each trial.

$$X \sim \textbf{NegBin}(\textbf{r}, \textbf{p})$$

Range: $\{r, r + 1, \ldots\}$

PMF: $\mathbf{P}[X = n] = \binom{n - 1}{r - 1}(1 - p)^{n-r}p^r$

Expectation: $\mathbf{E}[X] = \dfrac{r}{p}$

Variance: $\mathbf{V}[X] = \dfrac{r(1 - p)}{p^2}$

Examples: tossing a coin until r-th heads appears, generating bits until the first r 1's are generated.

Note: $Geo(p) = NegBin(1, p)$.

NegBin example

Example (not real life!)

A PhD student is expected to publish 2 papers to graduate. A conference accepts each paper randomly and independently with probability $p = 0.25$. On average, how many papers will the student need to submit to a conference in order to graduate?

——————————————— Answer ———————————————

Adding NegBin example

Example

Let $X \sim NegBin(m, p)$ and $Y \sim NegBin(n, p)$ be two independent RVs. Define a new RV as $Z = X + Y$. Find PMF of Z.

Answer

- Need to show that $Z \sim NegBin(m + n, p)$.

- Consider the sequence of independent events tossing a coin with $\mathbf{P}[\,heads\,] = p$.

- Let X be a RV for # of coin tosses until m heads are observed. Thus $X \sim NegBin(m, p)$.

- Now, continue to toss a coin after m heads are observed, until n more heads are observed. Thus, for this part of the sequence, $Y \sim NegBin(n, p)$.

- Looking at it from the beginning we tossed independently the coin until we observed $m + n$ heads, thus $Z = X + Y$ and thus $Z \sim NegBin(m + n, p)$.

- Note: if $X_1, X_2, \ldots X_m$ are m independent $Geo(p)$ RVs, then the RV $X = X_1 + X_2 + \cdots + X_m$ has $NegBin(m, p)$ distribution.

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Hypergeometric

Hypergeometric discrete random variable

X is a hypergeometric RV that samples n objects, without replacement, with i successes (random draw for which the object drawn has a specified feature), from a finite population of size N that contains exactly m objects with that feature.

$$X \sim \textbf{Hyp}(\textbf{N}, \textbf{n}, \textbf{m})$$

Range: $\{0, 1, \ldots, n\}$

PMF: $\mathbf{P}[X = i] = \dfrac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$

Expectation: $\mathbf{E}[X] = n\dfrac{m}{N}$

Variance: $\mathbf{V}[X] = n\dfrac{m}{N}\left(1 - \dfrac{m}{N}\right)\left(1 - \dfrac{n-1}{N-1}\right)$

Example: an urn has N balls of which m are white and $N - m$ are black; we take a random sample **without replacement** of size n and measure X: # of white balls in the sample.

Survey sampling

A street has 40 houses of which 5 houses are inhabited by families with an income below the poverty line. In a survey, 7 houses are sampled at random from this street. What is the probability that: (a) none of the 5 families with income below poverty line are sampled? (b) 4 of them are sampled? (c) no more than 2 are sampled? (d) at least 3 are sampled?

—————————————— Answer ——————————————

Let X: # of families sampled which are below the poverty line.

$X \sim Hyp(N = 40, n = 7, m = 5)$.

Summary of discrete RV

	$Ber(p)$	$Bin(n,p)$	$Pois(\lambda)$	$Geo(p)$	$NegBin(r,p)$	$Hyp(N,n,m)$
PMF	$\mathbf{P}[X=1]=p$	$\mathbf{P}[X=k]=\binom{n}{k}p^k(1-p)^{n-k}$	$\mathbf{P}[X=k]=\frac{\lambda^k}{k!}e^{-\lambda}$	$\mathbf{P}[X=n]=(1-p)^{n-1}p$	$\mathbf{P}[X=n]=\binom{n-1}{r-1}(1-p)^{n-r}p^r$	$\mathbf{P}[X=i]=\frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$
$\mathbf{E}[X]$	p	np	λ	$\frac{1}{p}$	$\frac{r}{p}$	$n\frac{m}{N}$
$\mathbf{V}[X]$	$p(1-p)$	$np(1-p)$	λ	$\frac{1-p}{p^2}$	$\frac{r(1-p)}{p^2}$	$n\frac{m}{N}\left(1-\frac{m}{N}\right)\left(1-\frac{n-1}{N-1}\right)$
Descr.	1 experiment with prob p of success	n independent trials with prob p of success	# successes over experiment duration, $\lambda = np$ rate of success	# independent trials until first success	# independent trials until r successes	# successes of drawing item with a feature (without replacement) in a sample of size n from a population of size N with m items with the feature