Introduction to Probability

Lecture 8: Basic Inequalities and Law of Large Numbers

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Easter 2025

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Compiled: May 18, 2025 at 21:54

Outline

Intro: Sum of Independent (Uniform) Random Variables

Example 1

Let X_1 and X_2 be two independent random variables, both uniformly distributed on $[0, 1]$. How does the probability density of $X_1 + X_2$ look like? What happens for $X_1 + X_2 + X_3$ etc.?

Answer

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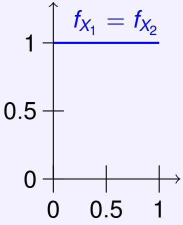

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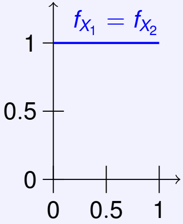

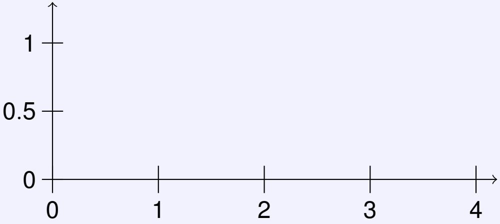

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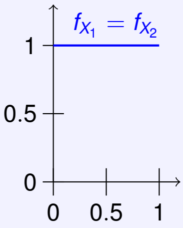

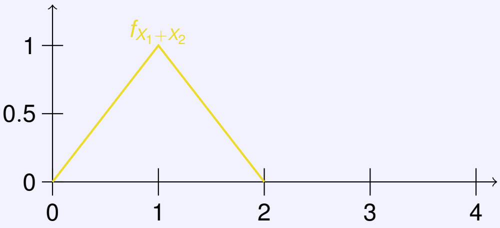

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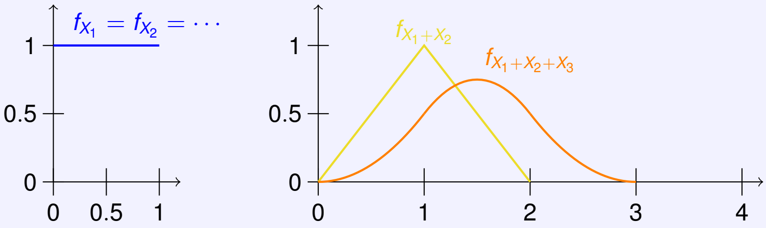

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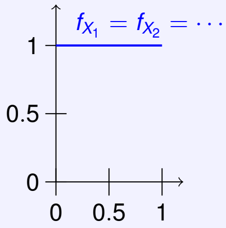
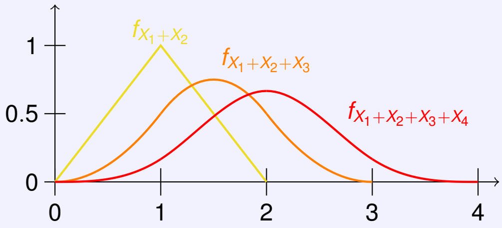
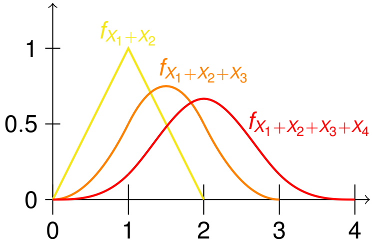


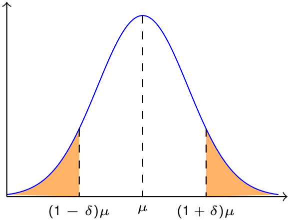


[a] This is also called "convolution". The detailed calculation for $f_{X_1+X_2}$ can be found at the end of these slides. The exact distribution is known for any number of random variables under the name Irwin-Hall distribution.

Motivation

We will study sums of independent and identically distributed variables. How does their distribution look like, and how well do they concentrate around the expectation?





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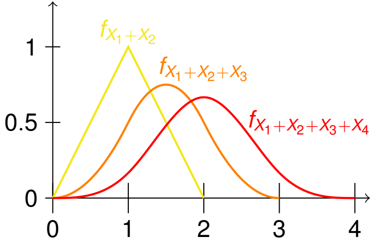

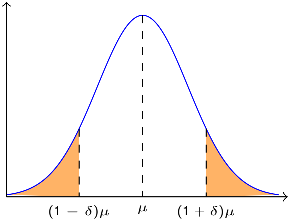

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2. Chebyshev's inequality
3. Law of Large Numbers
4. Central Limit Theorem

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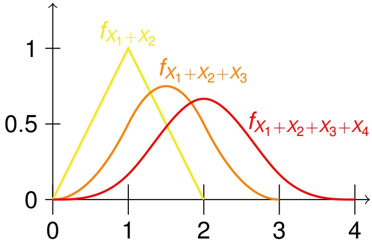

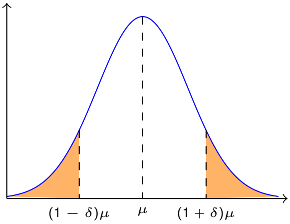

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2. Chebyshev's inequality
3. Law of Large Numbers
4. Central Limit Theorem

Re-use concepts from previous lectures:

1. Independence (Random Var.) (Lec. 1, 7)
2. Expectation and Variance (Lec. 2, 3)
3. Normal Distribution (Lec. 5)
4. Sums of Random Variables (Lec. 6)

Outline

Markov's Inequality

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$$\mathbf{P}[X \geq a] \leq \frac{\mathbf{E}[X]}{a}.$$

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- Advantage: Very basic inequality, we only need to know $\mathbf{E}[X]$
- Downside: For many distributions, the tail bound might be quite loose
- Proof is similar to the proof of Chebyshev's inequality (Exercise!)

Applying Markov's Inequality

Example 2

Consider throwing an unbiased, six-sided dice 120 times and let X denote the number of times we obtain a six.

1. Derive an upper bound on $\mathbf{P}[X \geq 30]$.
2. Can you also derive an upper bound on $\mathbf{P}[X \leq 10]$?

Answer

Chebyshev's Inequality

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For any random variable X with finite $\mathbf{E}[X]$ and $\mathbf{V}[X]$, for any $a > 0$,

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- Chebyshev's inequality is also known as Second Moment Method

Derivation of Chebychev's inequality

Proof

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Proof

Exercise: Can you find a proof that uses Markov's inequality?

Example: Chebychev is (usually) much stronger than Markov

Example 3

Throw an unbiased coin n times and let X be the total number of heads. In an experiment, with n large, we would usually expect a number of heads that is close to the expectation. Can we justify that?

Answer

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Introduction

Markov's Inequality and Chebyshev's Inequality

Weak Law of Large Numbers

Law of Large Numbers

The Weak Law of Large Numbers

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- "Power of Averaging": repeated samples allow us to estimate μ
- A similar statement holds even if the X_i's are not identically distributed
- There is also a strong law of large numbers:

$$\mathbf{P}\left[\lim_{n \to \infty} \overline{X}_n = \mu \right] = 1.$$

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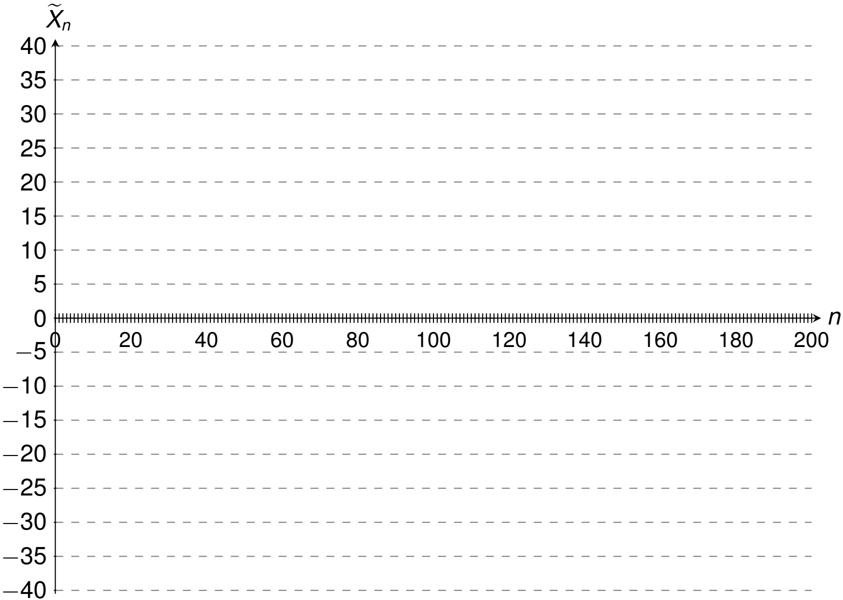
How does a "typical" realisation look like?

Illustration of Weak Law of Large Numbers (2/4)



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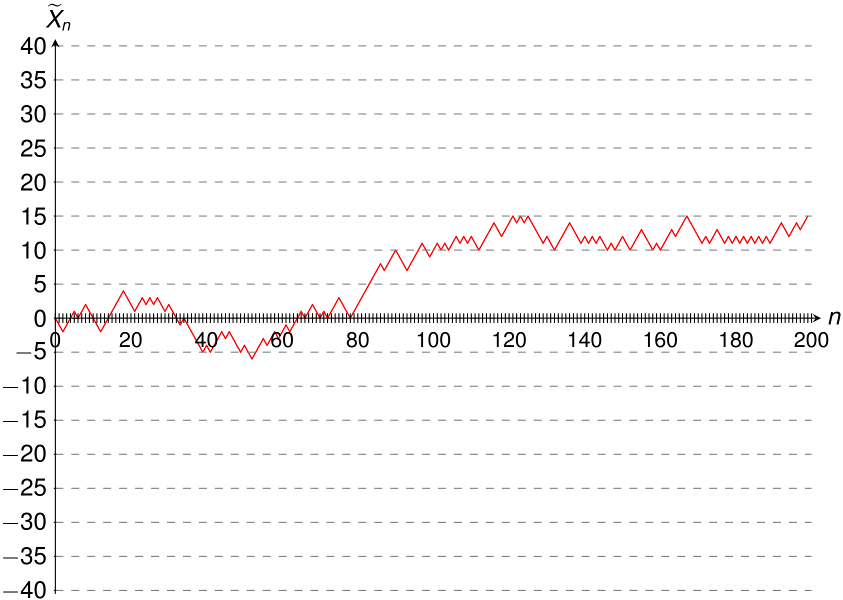

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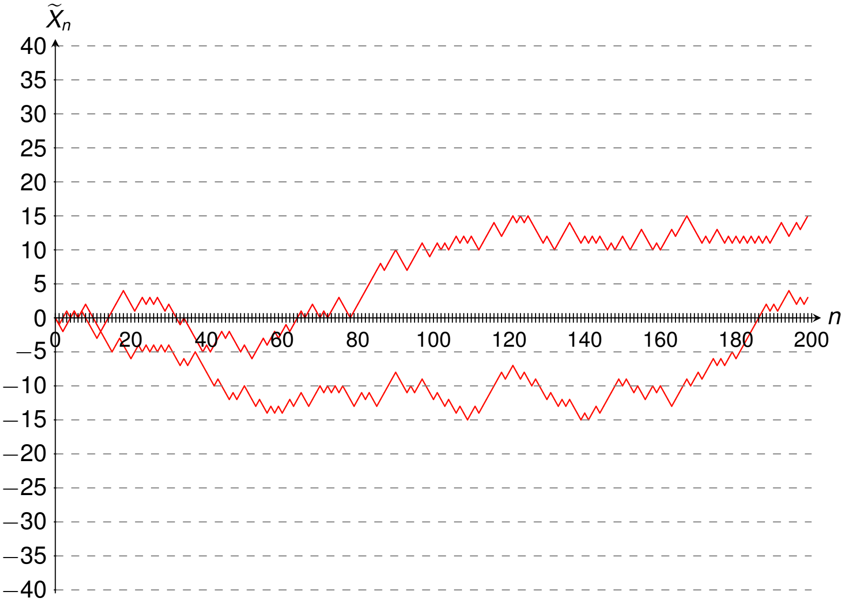

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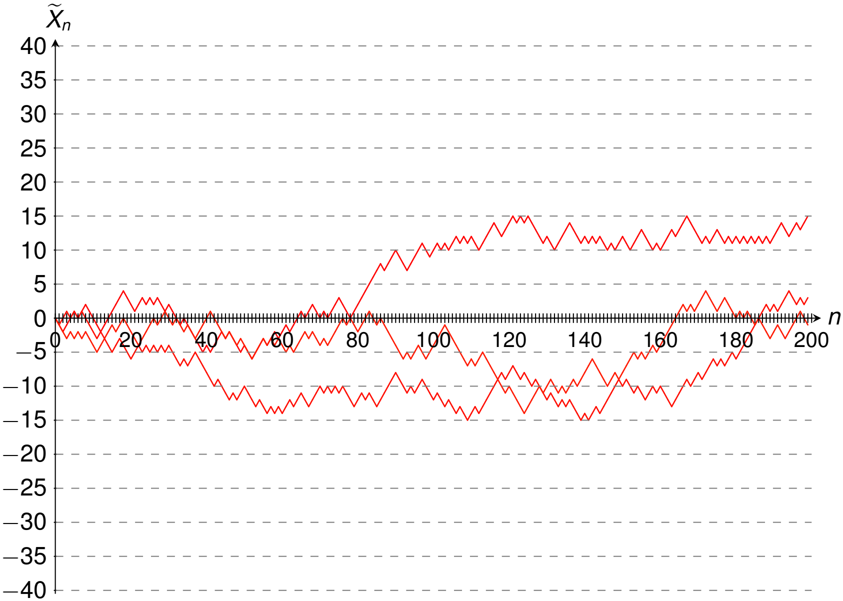

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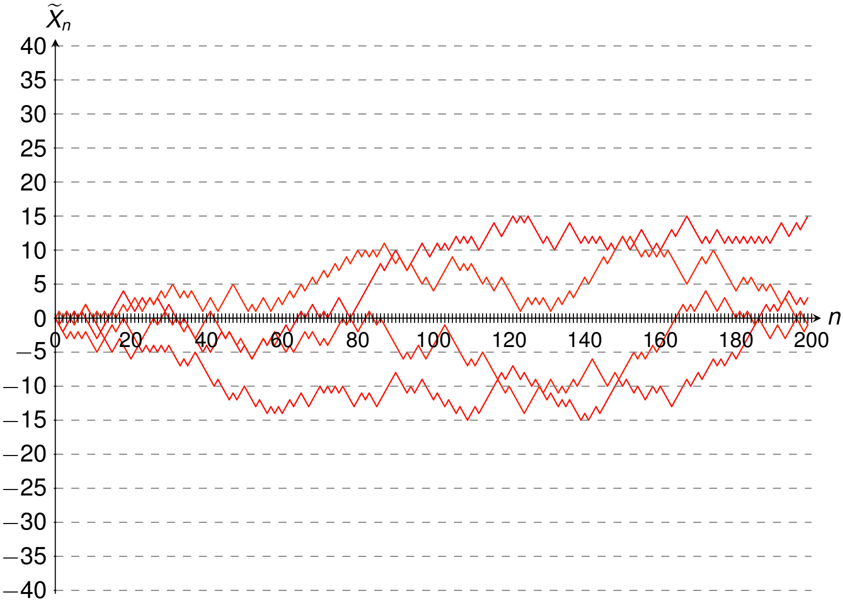

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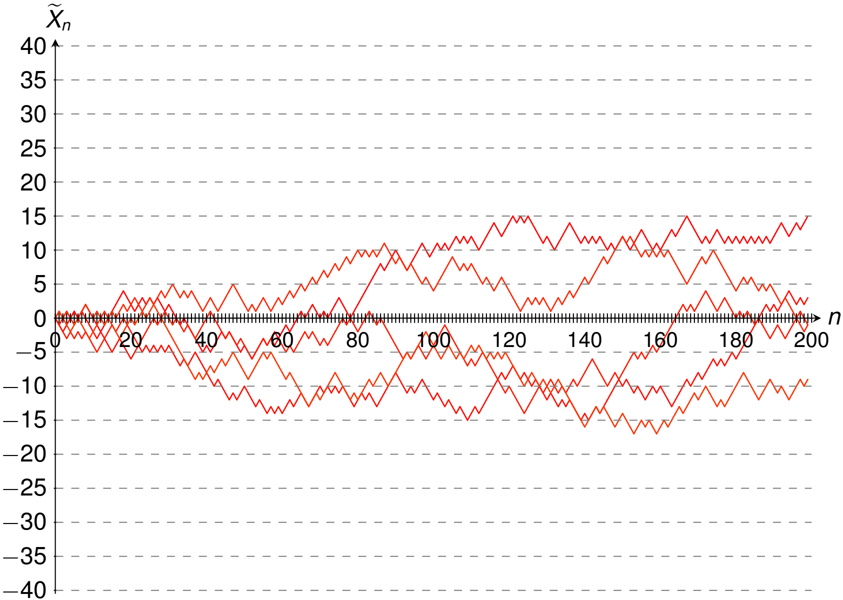

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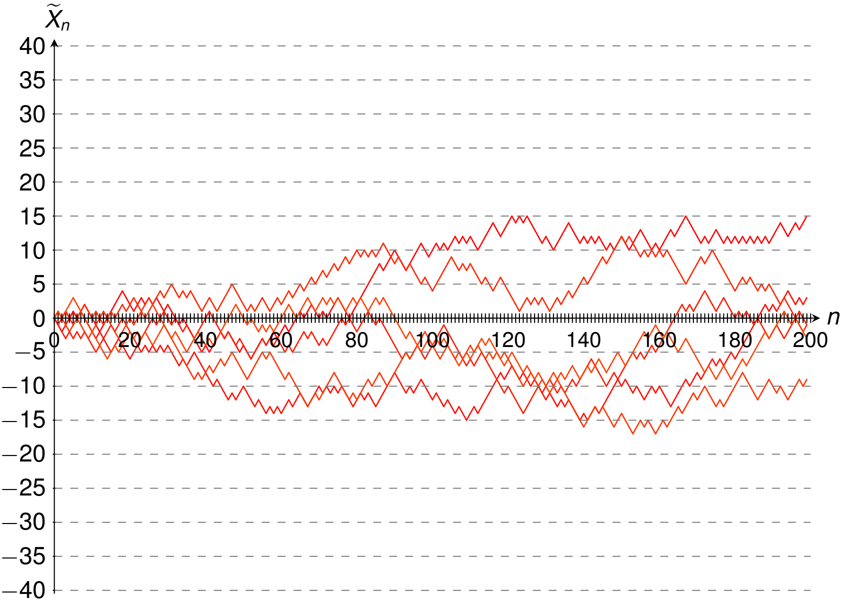

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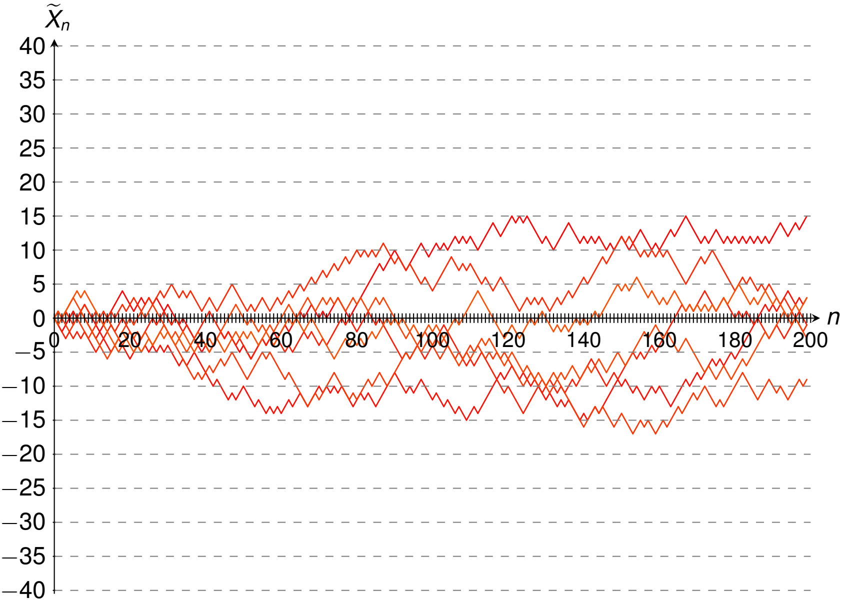

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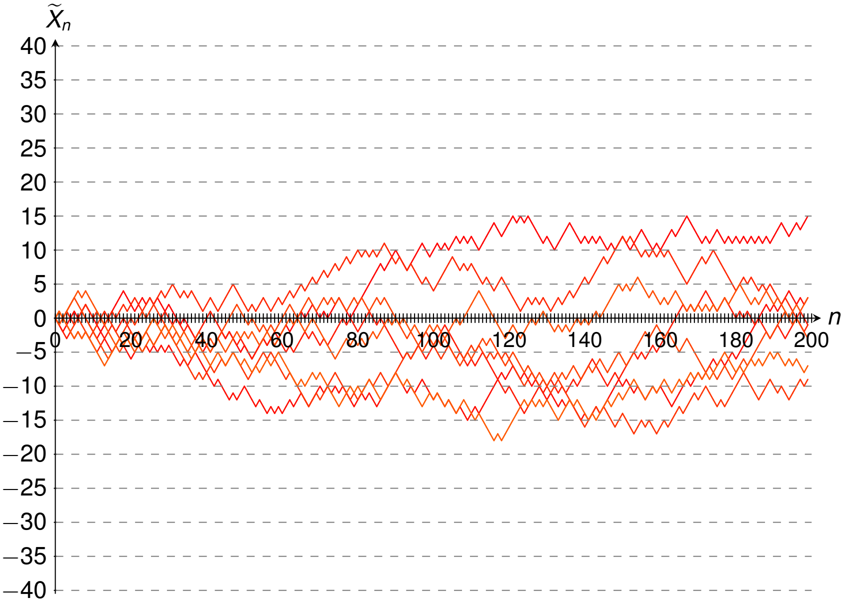

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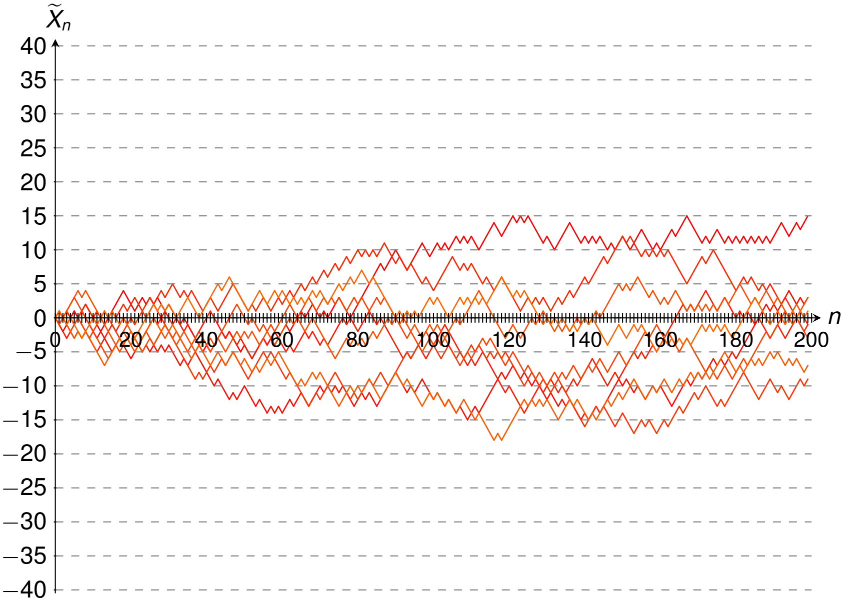

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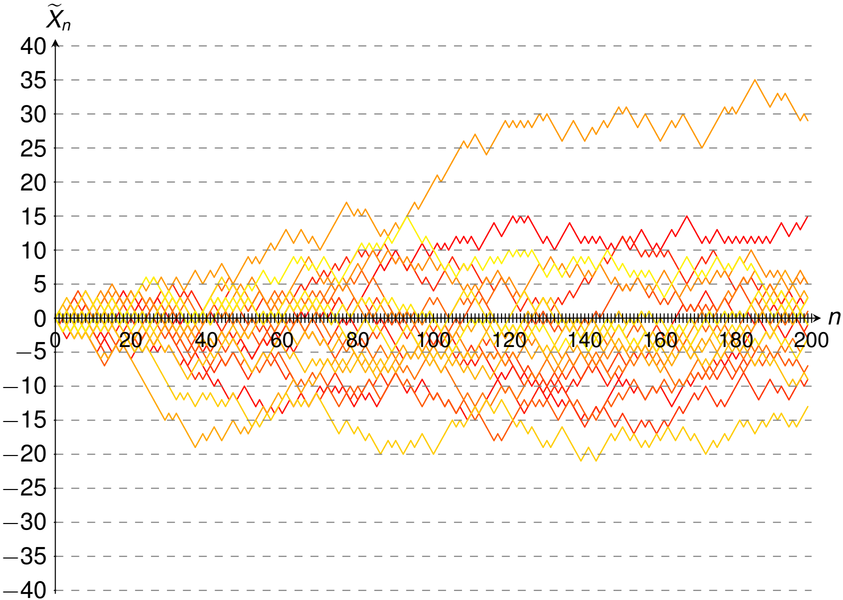

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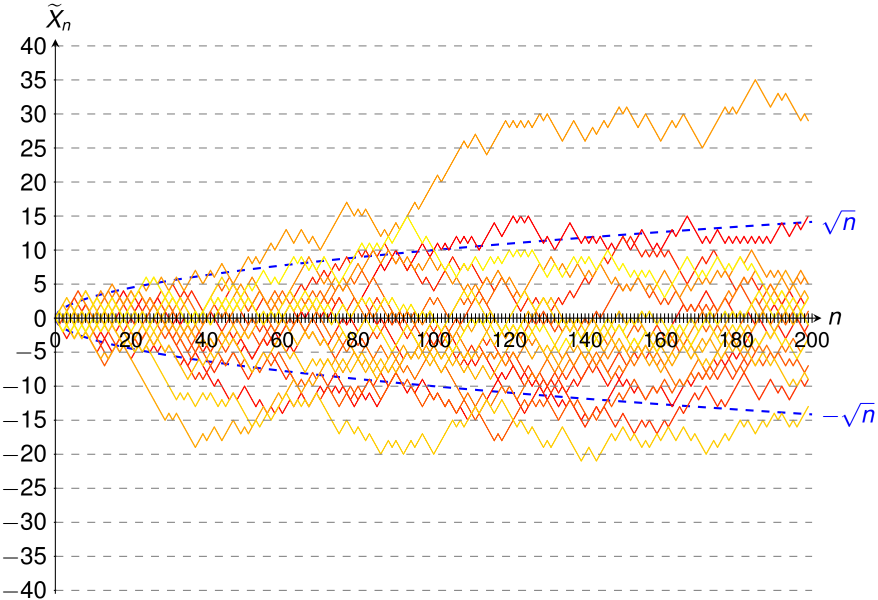
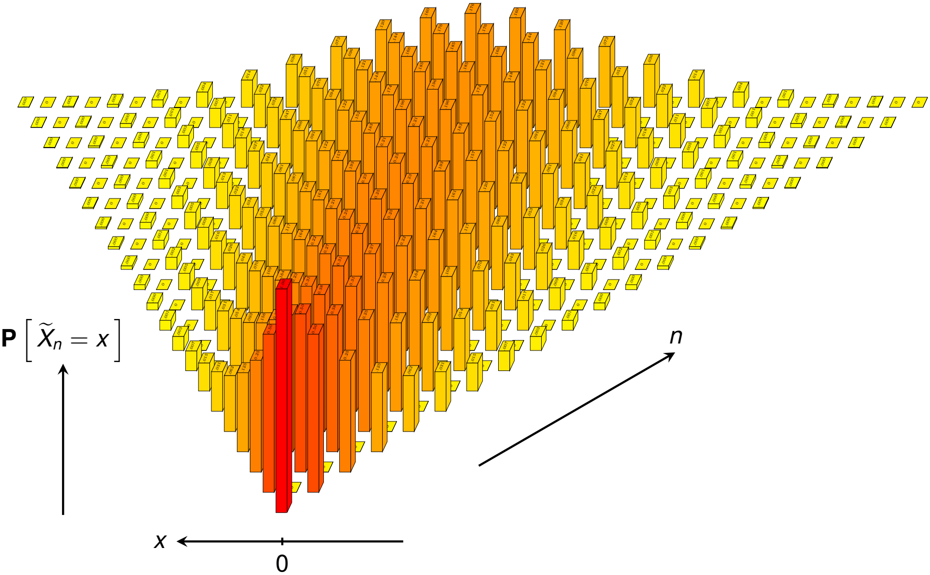
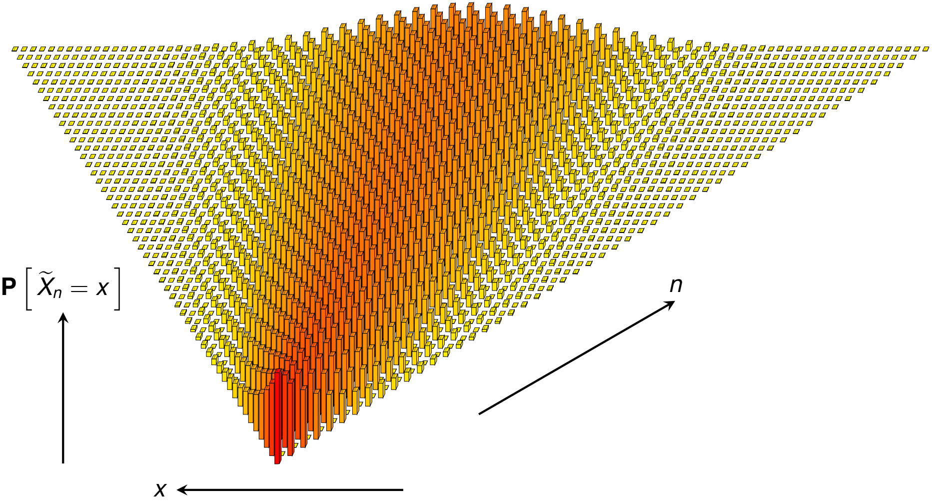


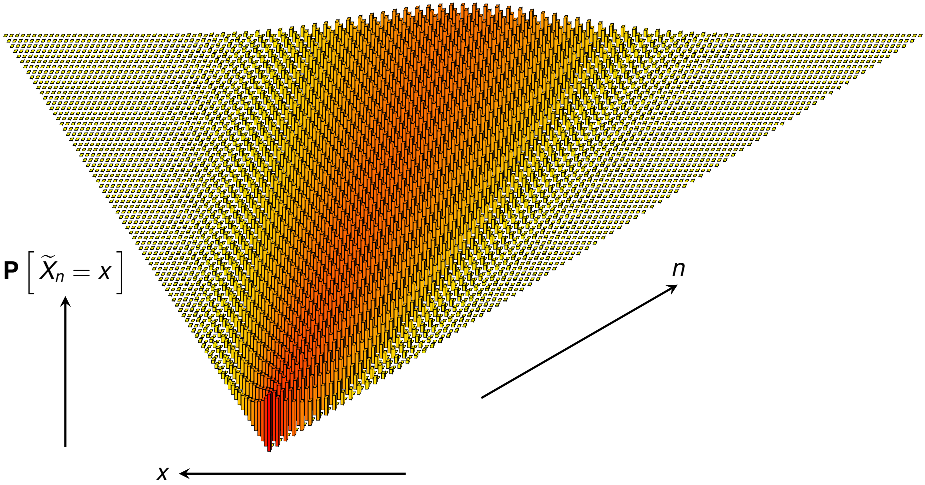
Plot of the Distributions for $n = 0, 1, \ldots, 20$



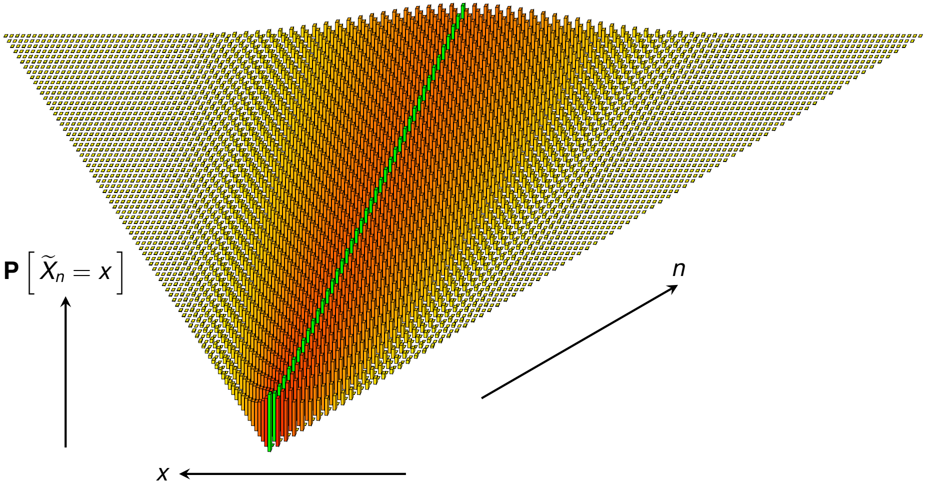
Plot of the Distributions for $n = 0, 1, \ldots, 50$



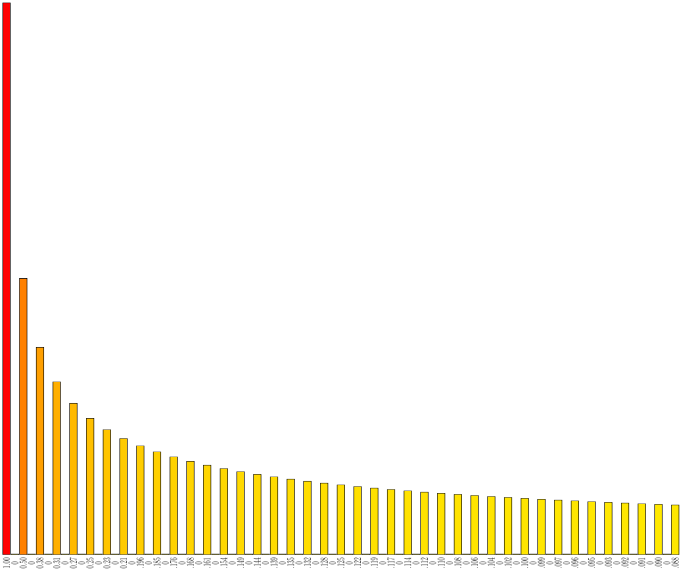
Plot of the Distributions for $n = 0, 1, \ldots, 80$



Plot of the Distributions for $n = 0, 1, \ldots, 80$



Interlude: Approximation of $\mathbf{P}[\,\widetilde{X}_n = 0\,]$



Exercise

Try to find an expression for $\mathbf{P}\left[\widetilde{X}_n = 0\right]$. Using Stirling's approximation for $n!$, conclude that $\mathbf{P}\left[\widetilde{X}_n = 0\right] = \Theta(1/\sqrt{n})$ for even integers n.

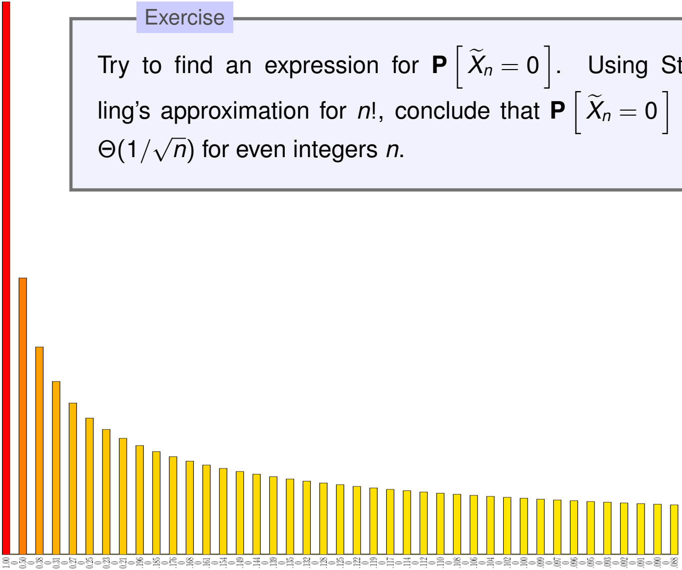


Illustration of Weak Law of Large Numbers (3/4)

- Let X_i be independent random variables taking values $\in \{-1, +1\}$ with probability $1/2$ each
- Consider $\widetilde{X}_n := \sum_{i=1}^{n} X_i$ for any for any $n = 0, 1, \ldots, 200$

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This does **not** converge!

Consider now the average (sample mean): $\overline{X}_n := 1/n \cdot \sum_{i=1}^{n} X_i$.

Illustration of Weak Law of Large Numbers (4/4)

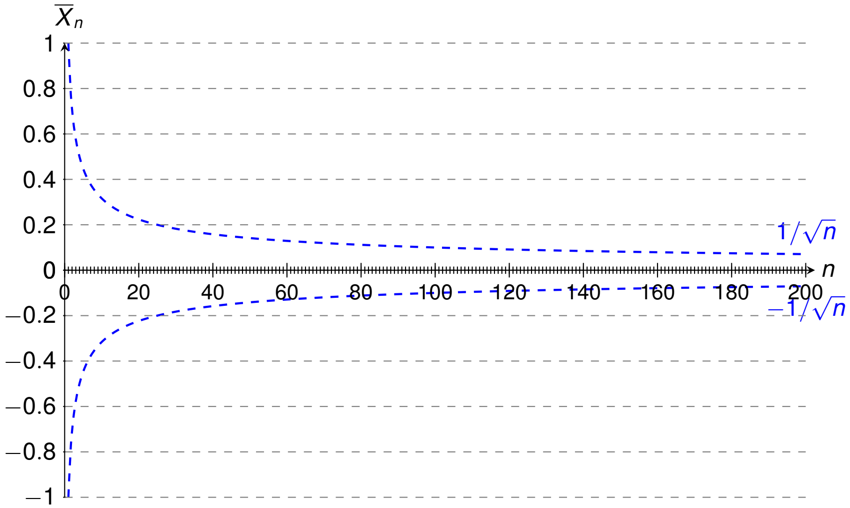


Illustration of Weak Law of Large Numbers (4/4)

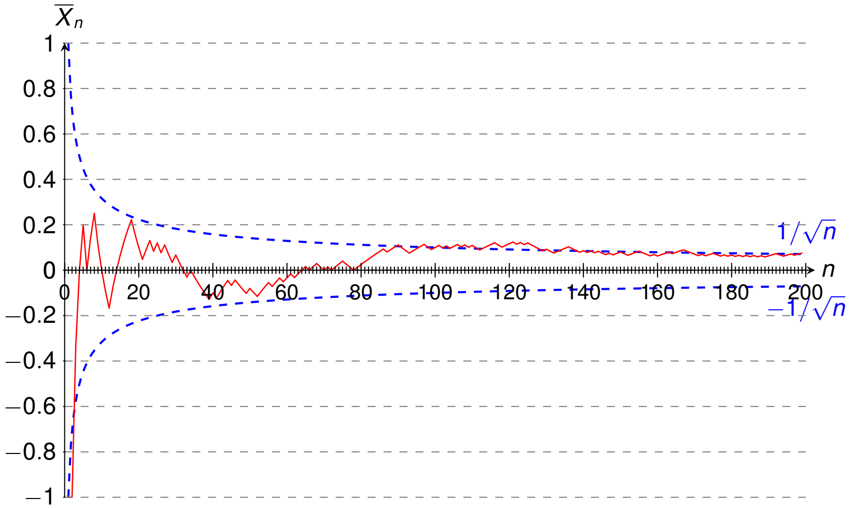


Illustration of Weak Law of Large Numbers (4/4)

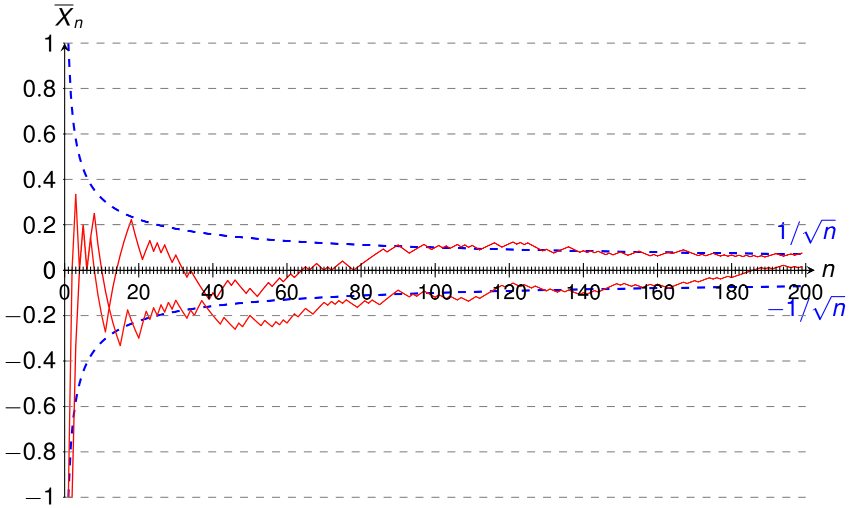
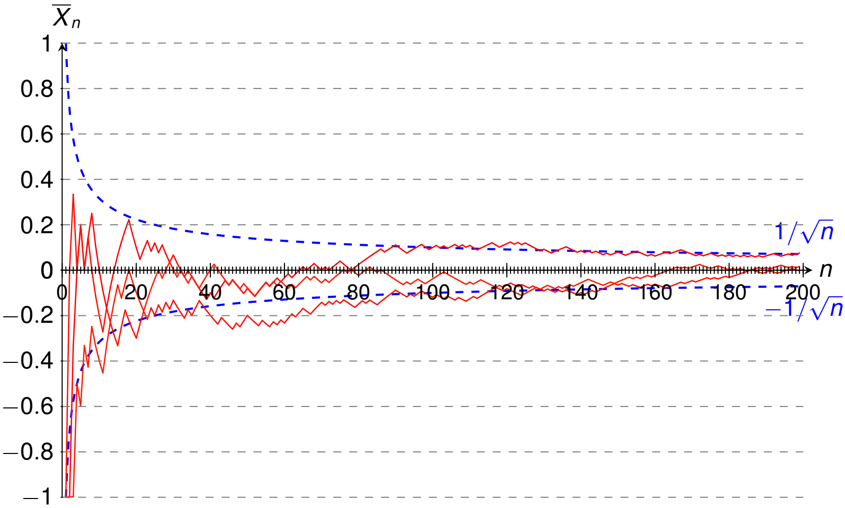


Illustration of Weak Law of Large Numbers (4/4)



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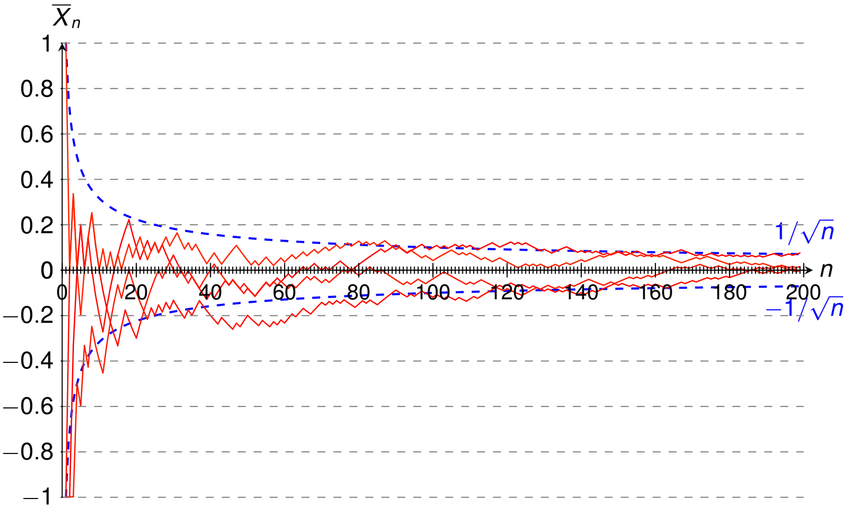
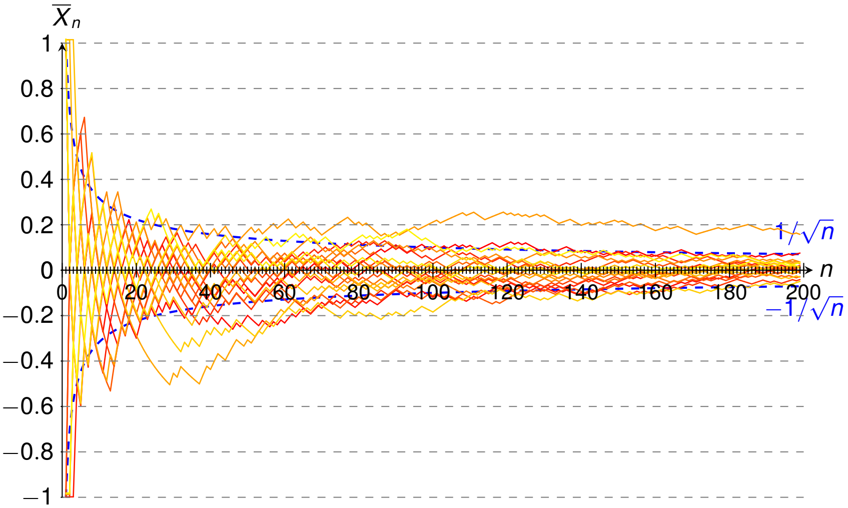


Illustration of Weak Law of Large Numbers (4/4)



Proof of the Weak Law of Large Numbers

The Weak Law of Large Numbers

Let $\overline{X}_n := 1/n \cdot \sum_{i=1}^{n} X_i$, where the X_i's are i.i.d. with finite expectation μ and finite variance σ^2. Then, for any $\epsilon > 0$,

$$\lim_{n \to \infty} \mathbf{P}\left[|\overline{X}_n - \mu| > \epsilon \right] = 0$$

Proof

Inferring Probabilities of an Event

Example 4

Suppose that, instead of the expectation μ, we want to estimate the probability of an event, e.g.,

$$p := \mathbf{P}\left[X \in (a, b]\right], \text{ where } a < b.$$

How can we use the Law of Large Numbers?

Answer

Appendix: Sum of Two Uniform R.V. (non-examinable)

Example

Let X and Y be two independent random variables, both uniformly distributed on $[0, 1]$. How does the probability density of $X + Y$ look like?

Answer

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Example

Let X and Y be two independent random variables, both uniformly distributed on $[0, 1]$. How does the probability density of $X + Y$ look like?

Answer

We have

$$f_{X+Y}(a) \overset{(\star)}{=} \int_{-\infty}^{+\infty} f_X(a-y) f_Y(y) dy,$$

where for $(\star)$, see Chapter 6.3 in Ross (Chapter 11.2 in Dekking et al.). Since $f_Y(y) = 1$ if $0 \leq y \leq 1$ and $f_Y(y) = 0$ otherwise, we have

$$f_{X+Y}(a) = \int_0^1 f_X(a-y) dy.$$

Further, for $0 \leq a \leq 1$ we have $f_X(a-y) = 1$ and $f_X(a-y) = 0$ otherwise, and thus

$$f_{X+Y}(a) = \int_0^a dy = a.$$

Similarly, for $1 < a < 2$, $f_{X+Y}(a) = \int_a^2 dy = 2 - a$. Therefore,

$$f_{X+Y}(a) = \begin{cases} a & \text{if } 0 \leq a \leq 1, \\ 2-a & \text{if } 1 \leq a \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$