Introduction to Probability

Lecture 12: Online Algorithms

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UNIVERSITY OF CAMBRIDGE

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Outline

Introduction: Dice Game

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This boils down to finding a threshold from which we `STOP` as soon as a 6 is thrown.

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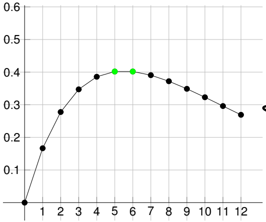

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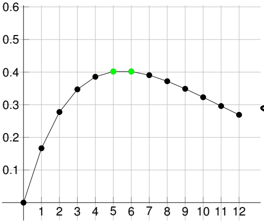

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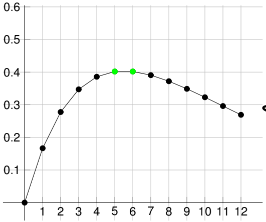

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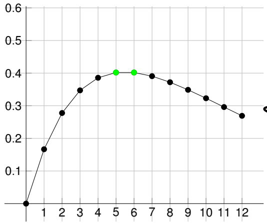

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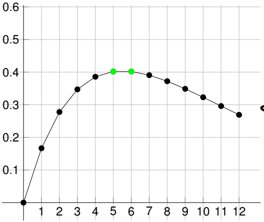

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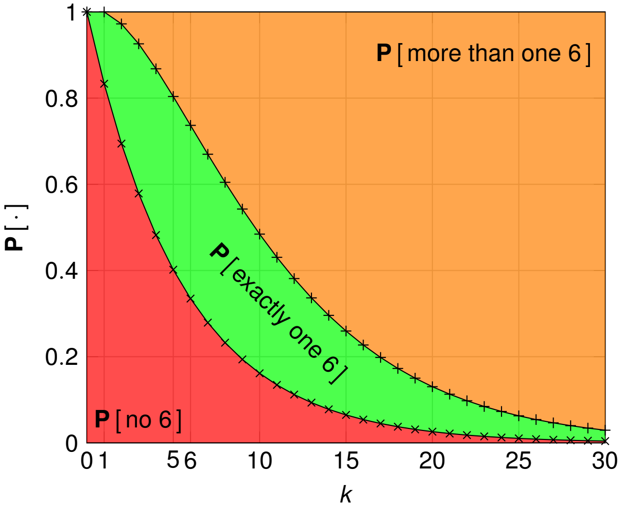
$$\left(\frac{5}{6}\right)^5 \approx 0.40.$$

Illustration of the Three Possibilities



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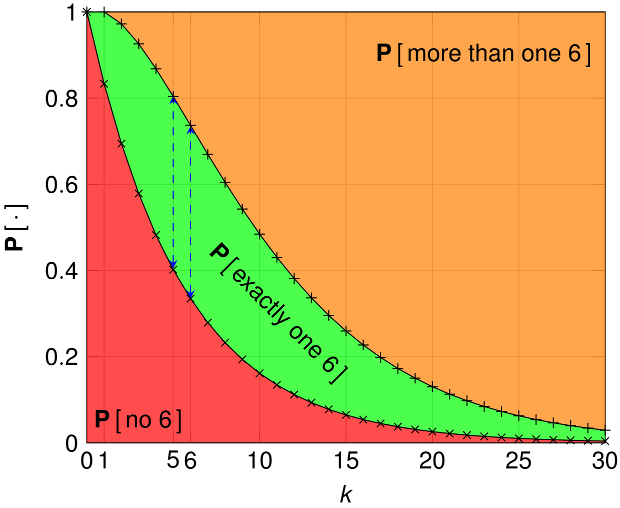

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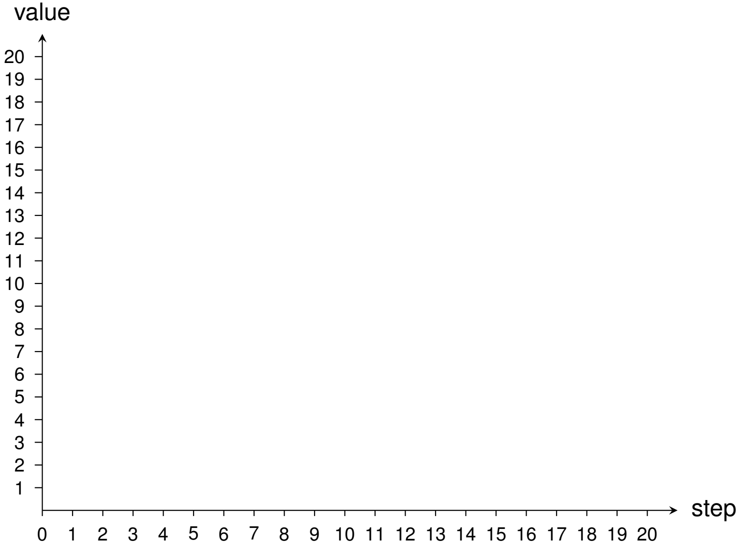
$\Rightarrow$ For our problem we may as well assume that the only information we have when interviewing candidate i is whether that candidate is best among $\{1, \ldots, i\}$ or not.

Illustration ($n = 20$)

unknown permutation:



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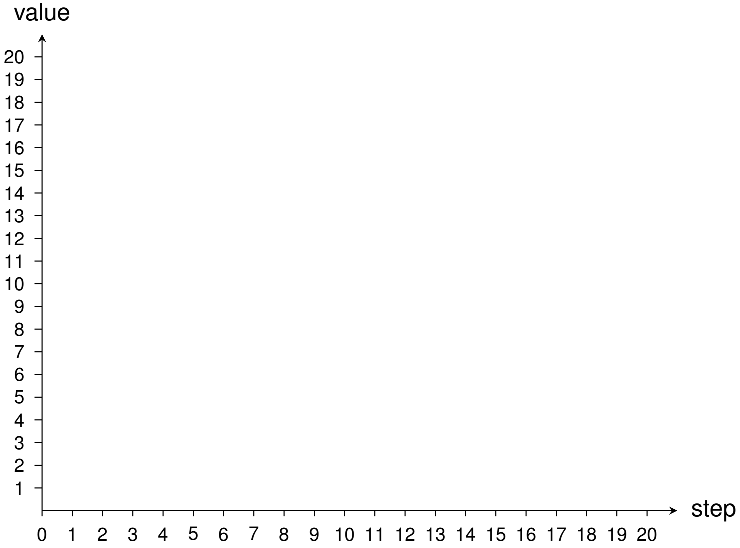
4,



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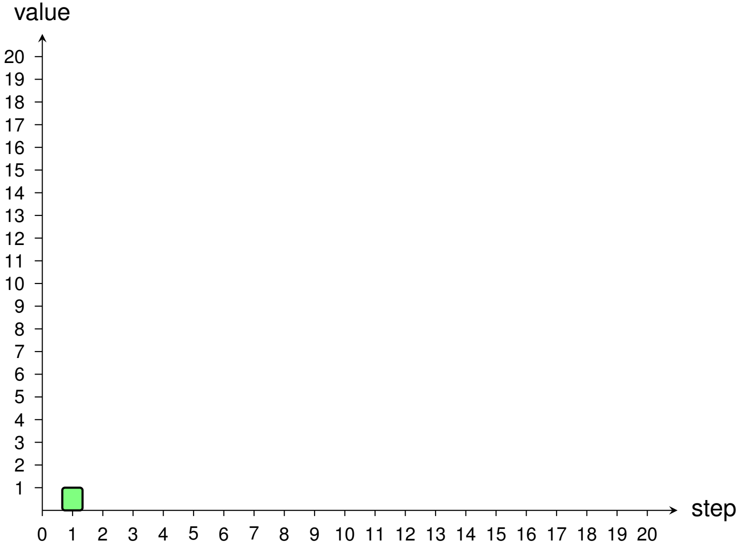

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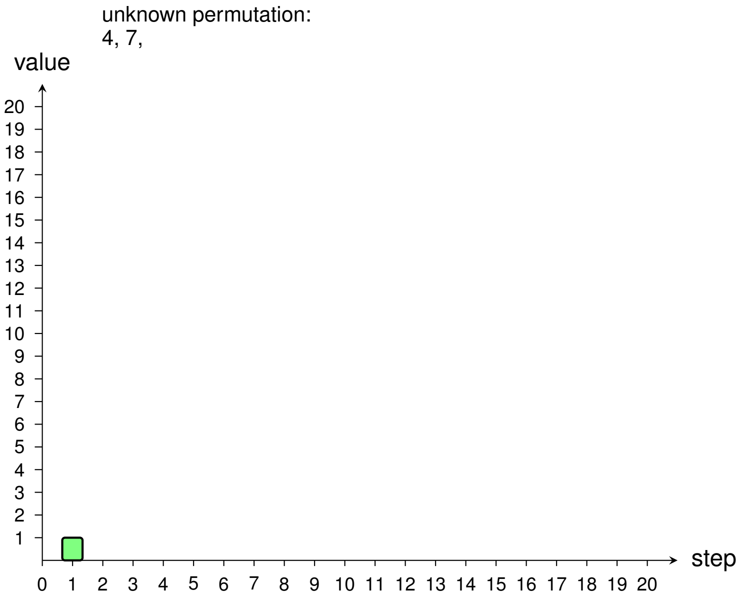

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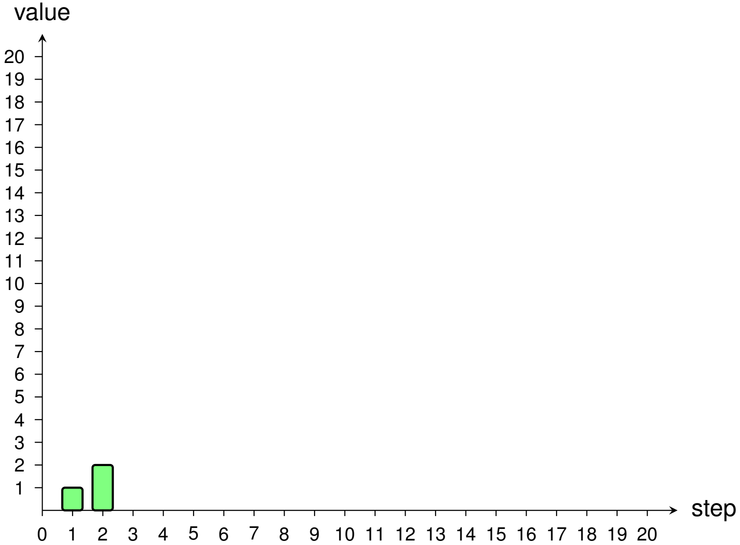
4, 7,



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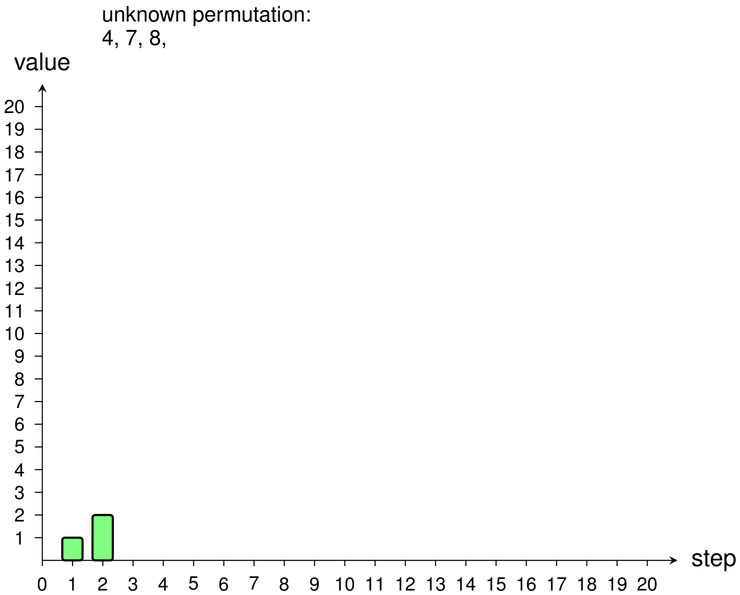

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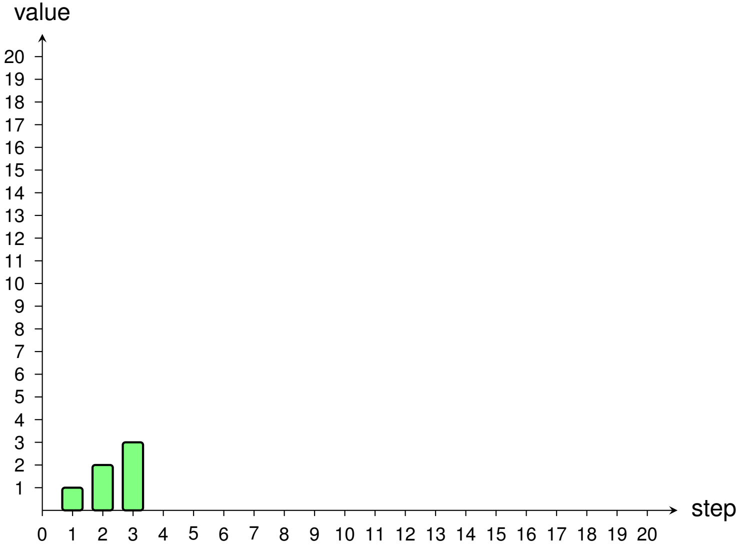
4, 7, 8,



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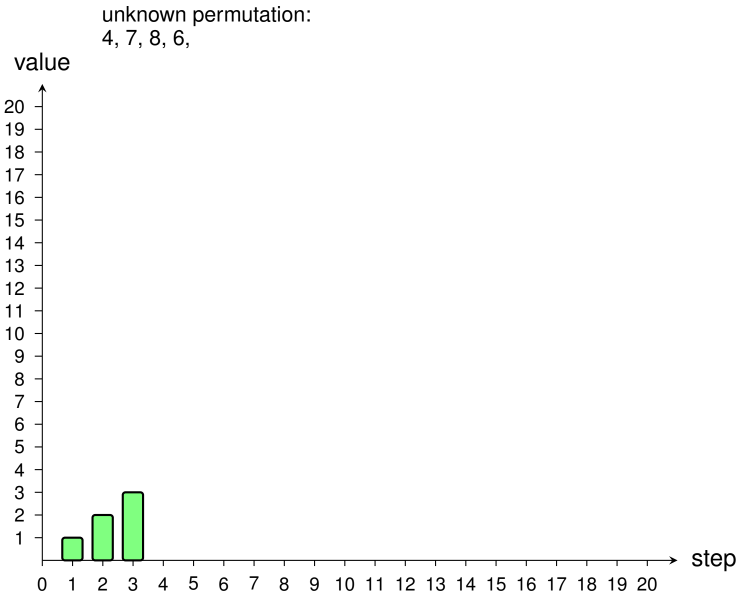

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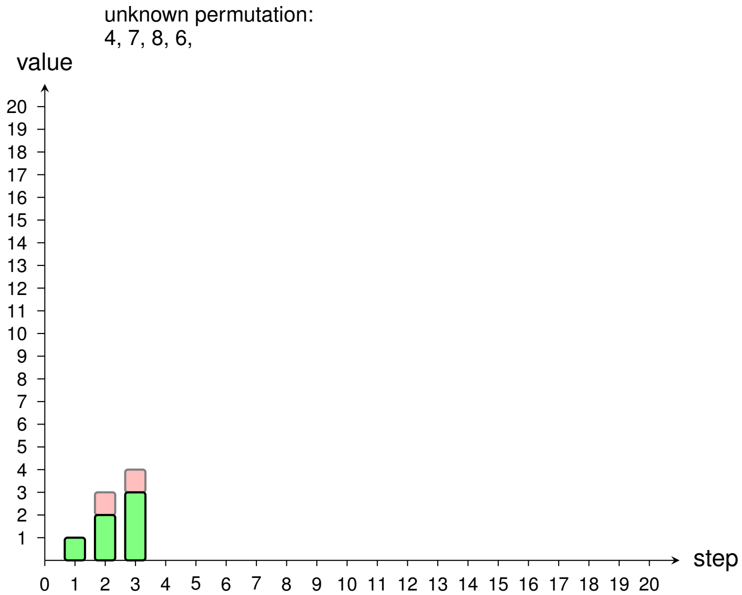

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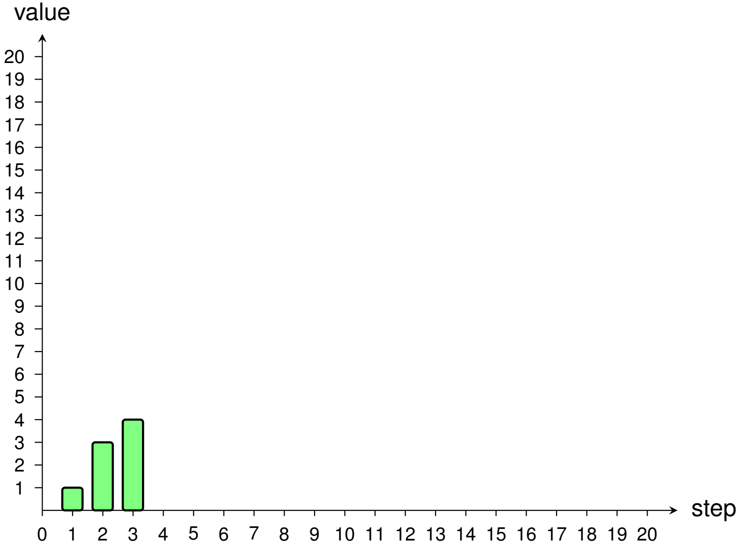
4, 7, 8, 6,



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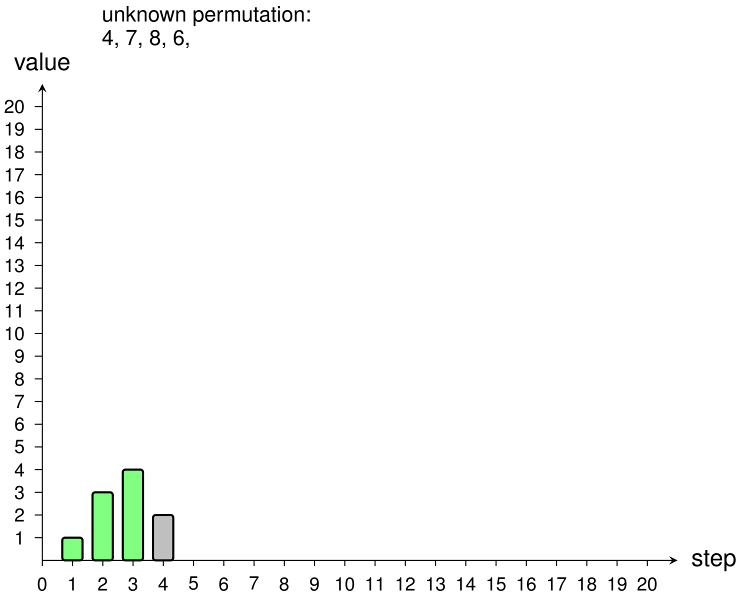
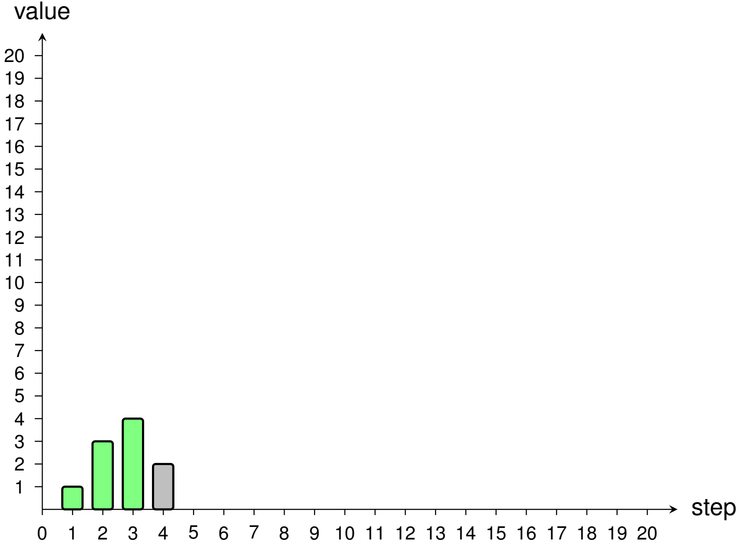


Illustration ($n = 20$)

unknown permutation:
4, 7, 8, 6, 18,



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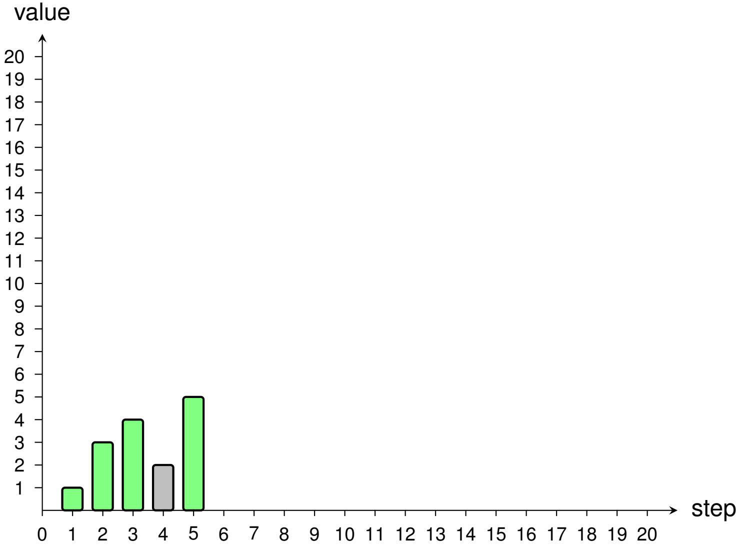

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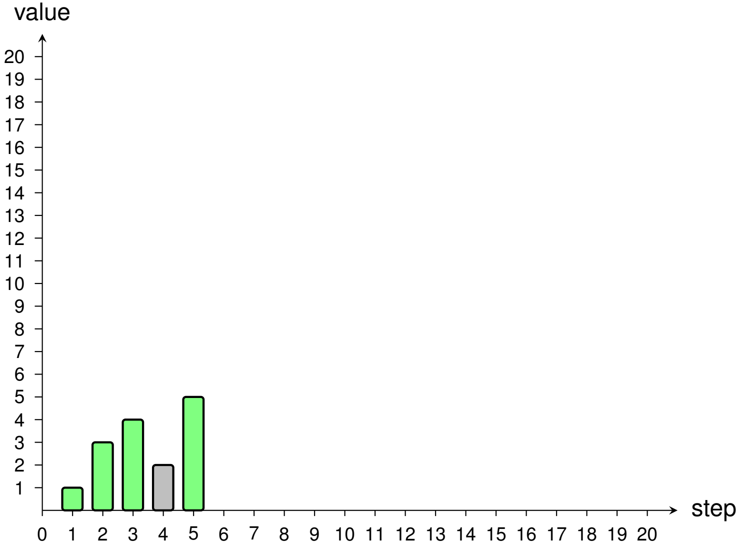

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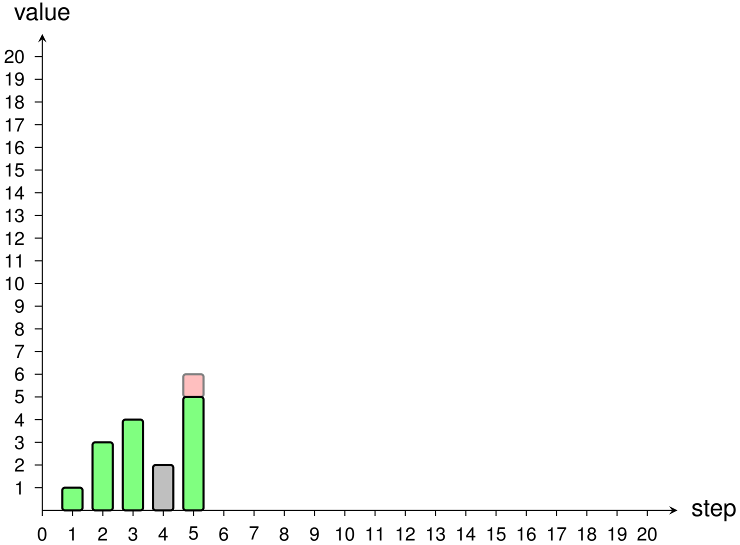

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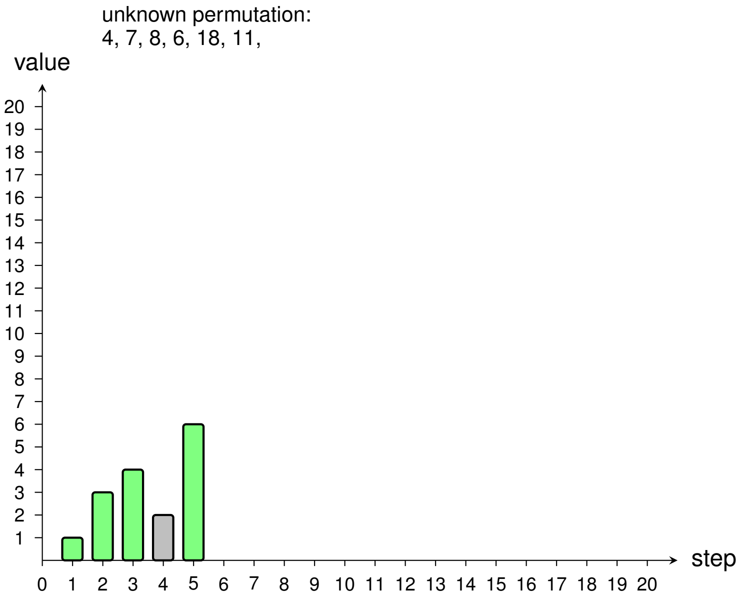

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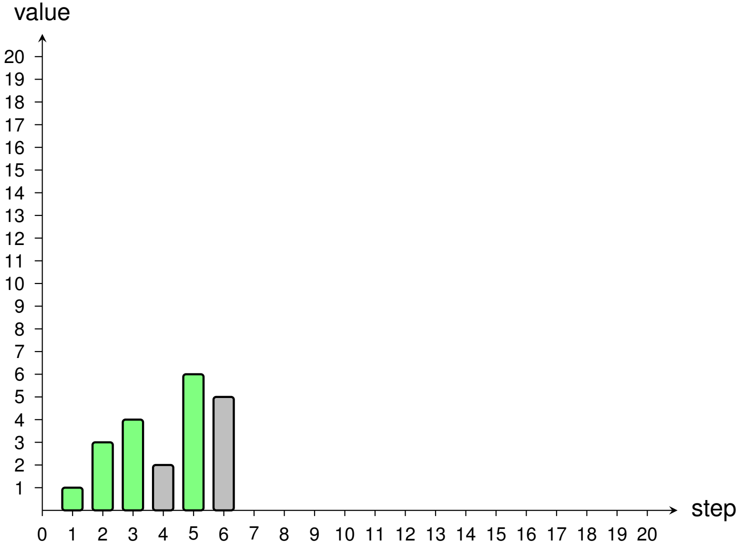

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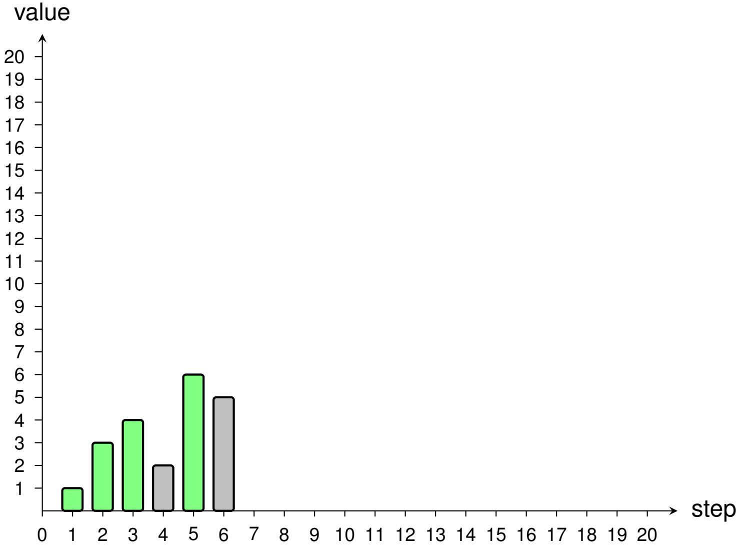

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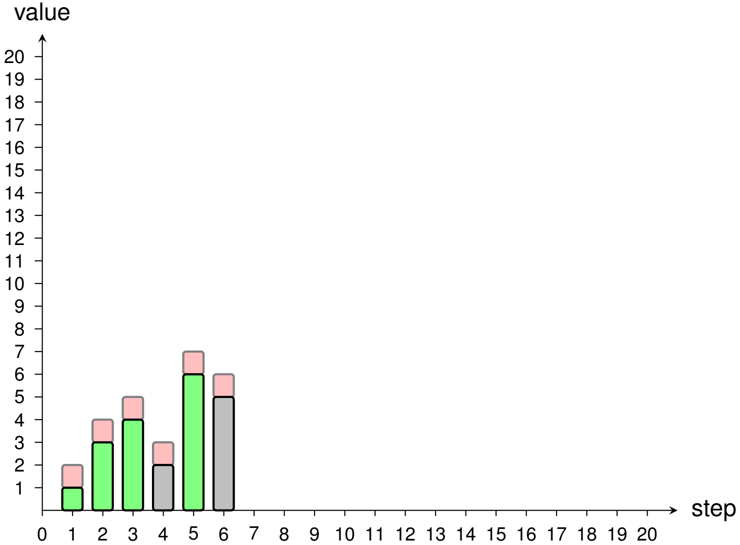

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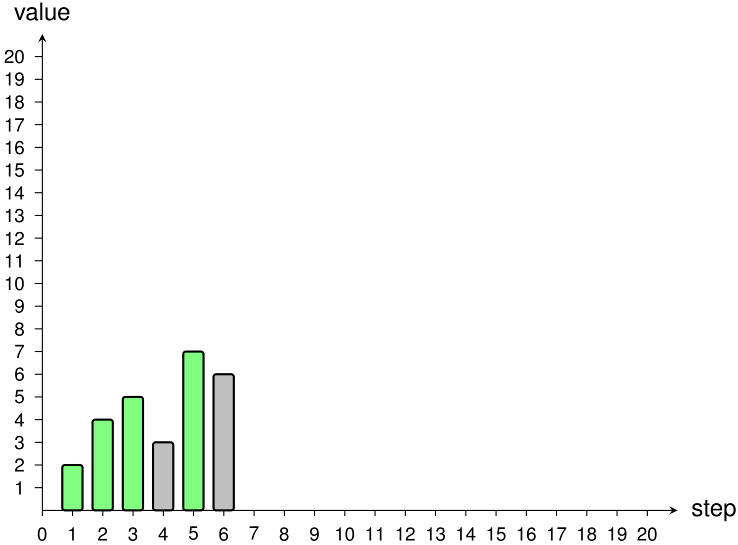

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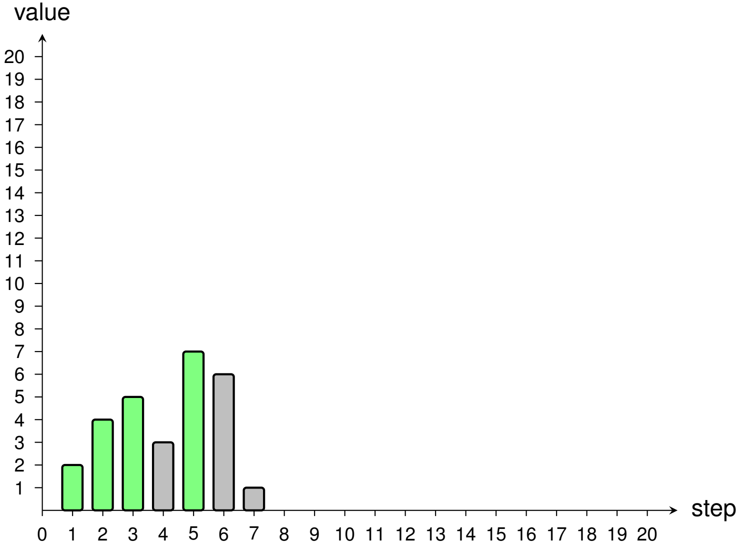

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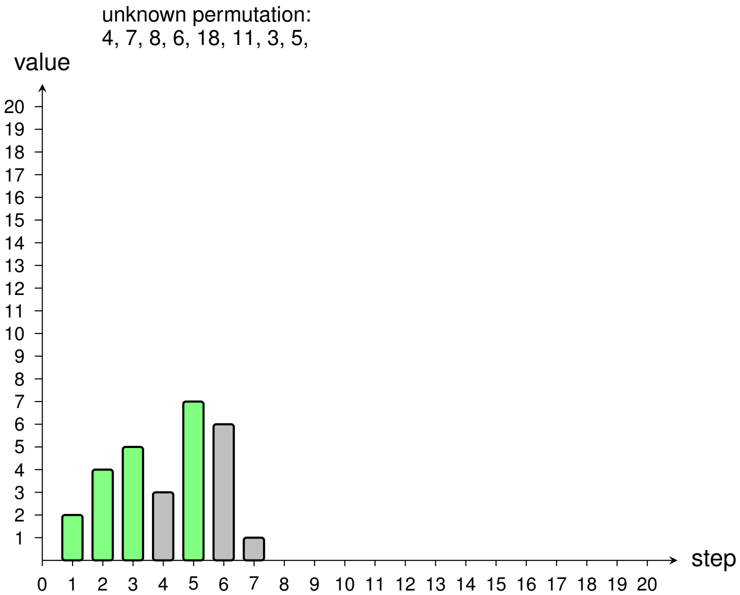

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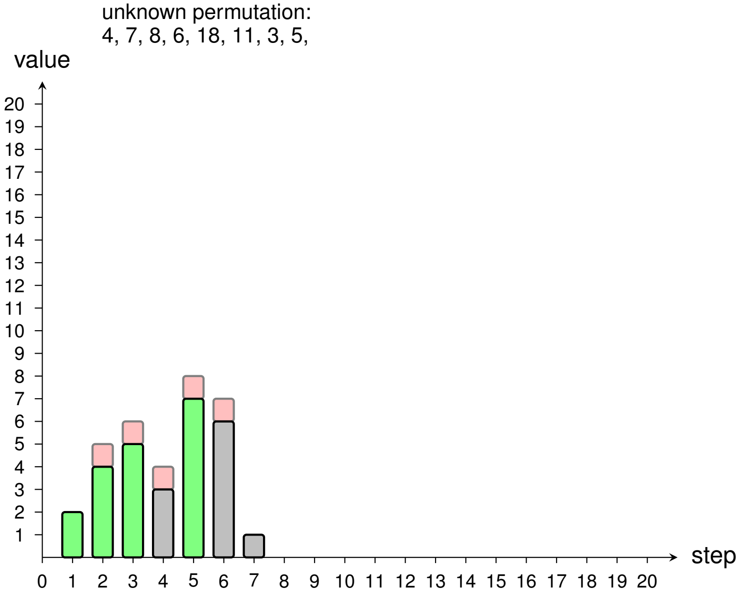

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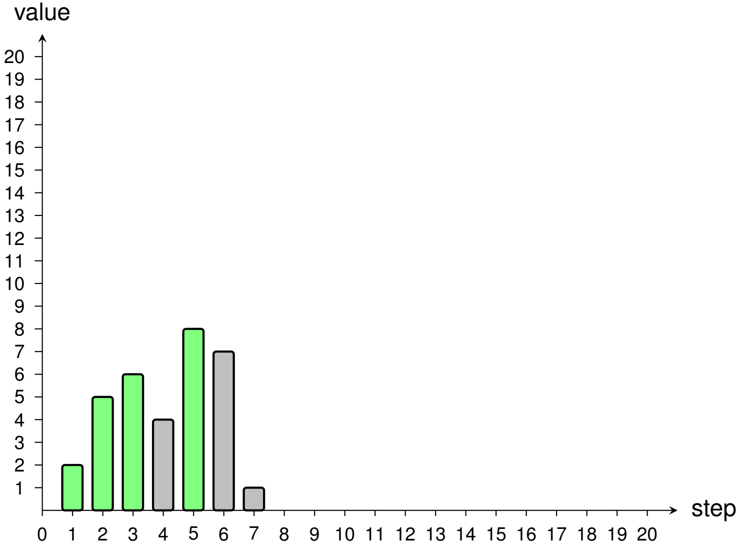

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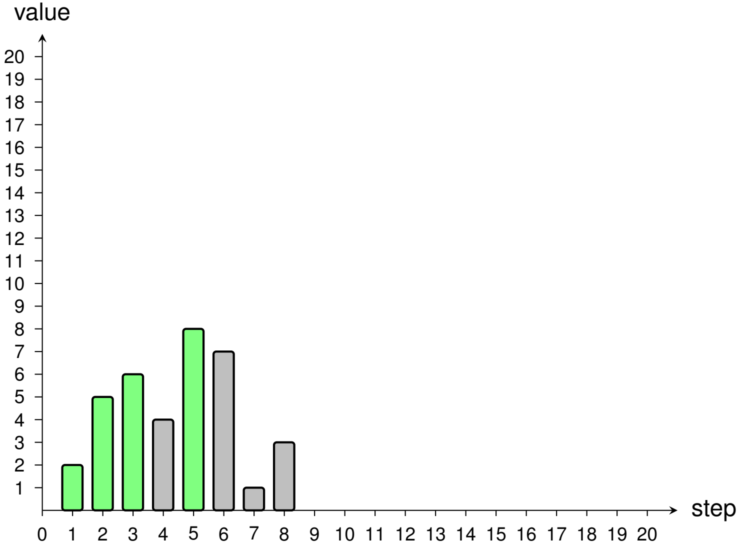

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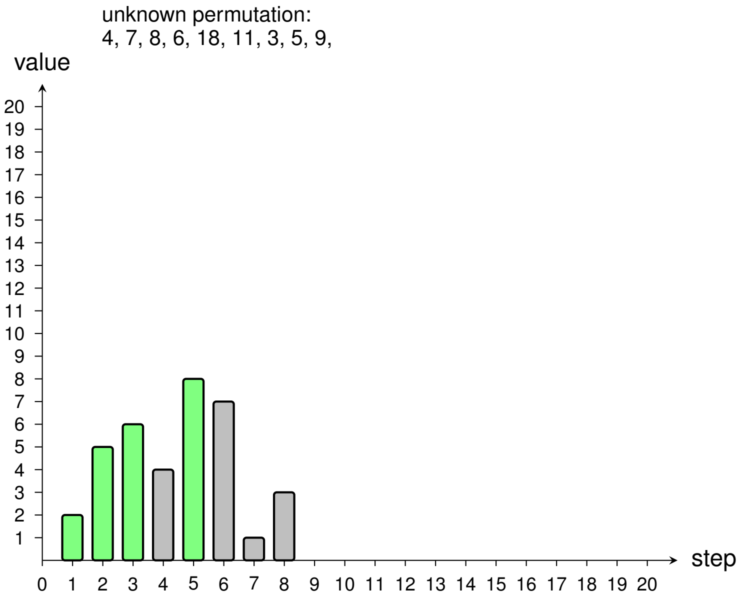

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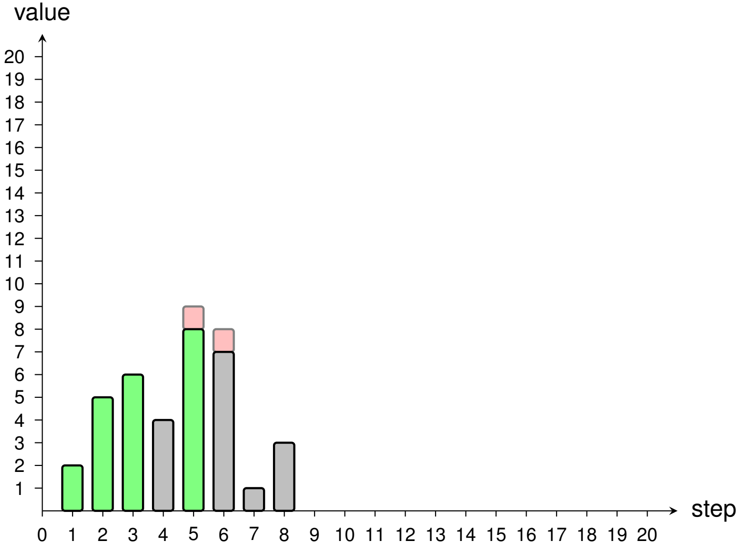

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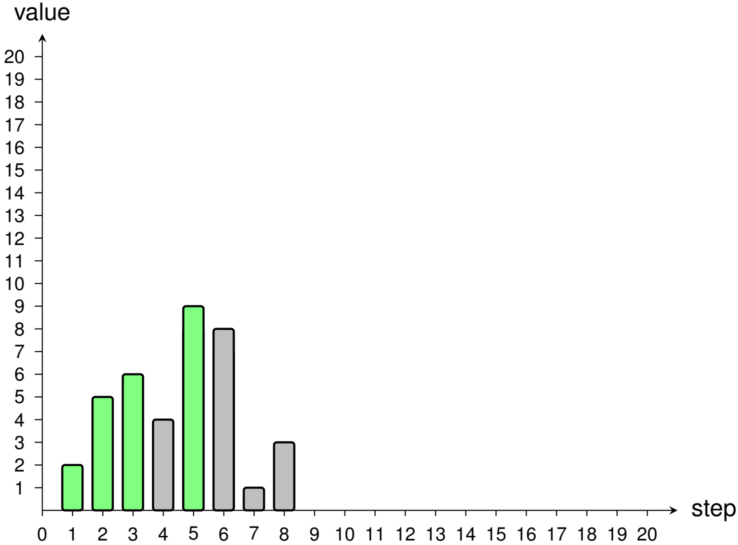

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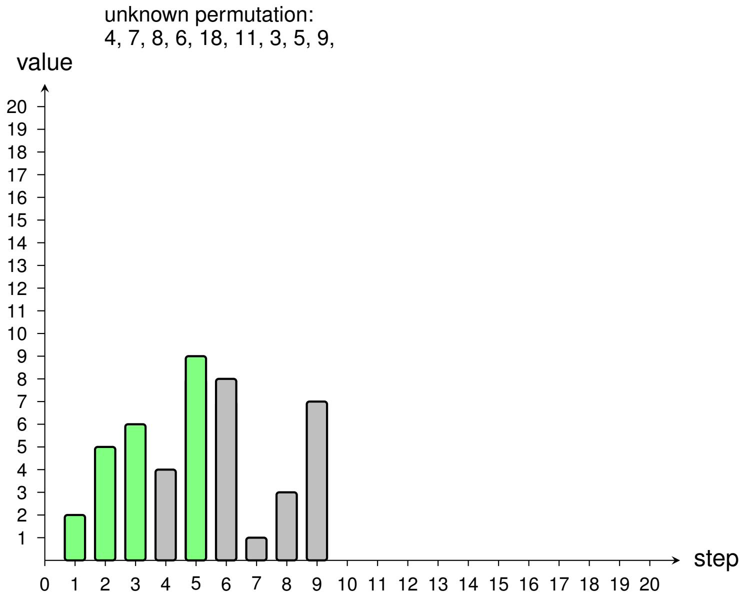

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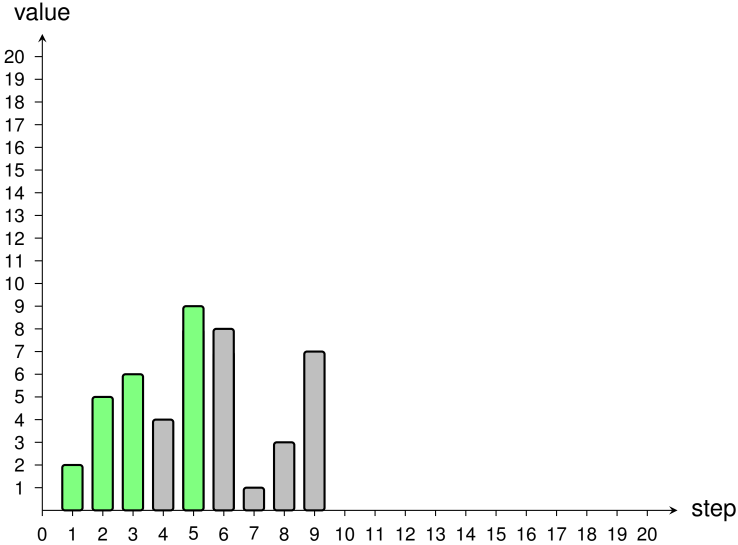

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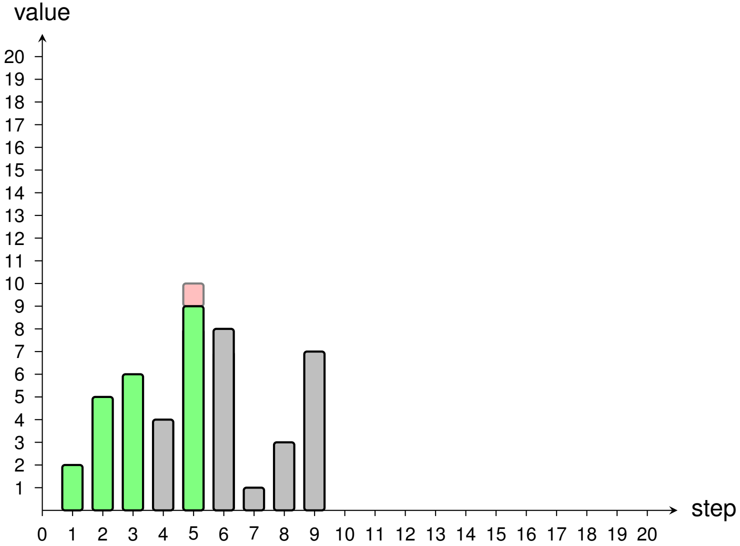

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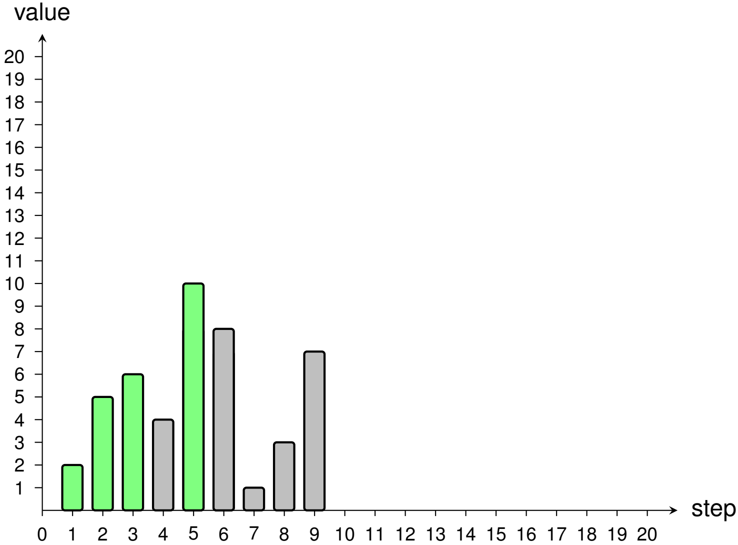

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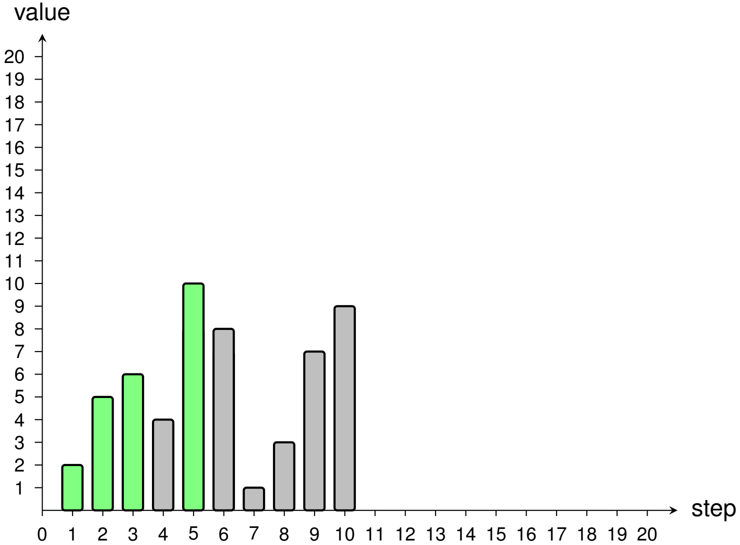

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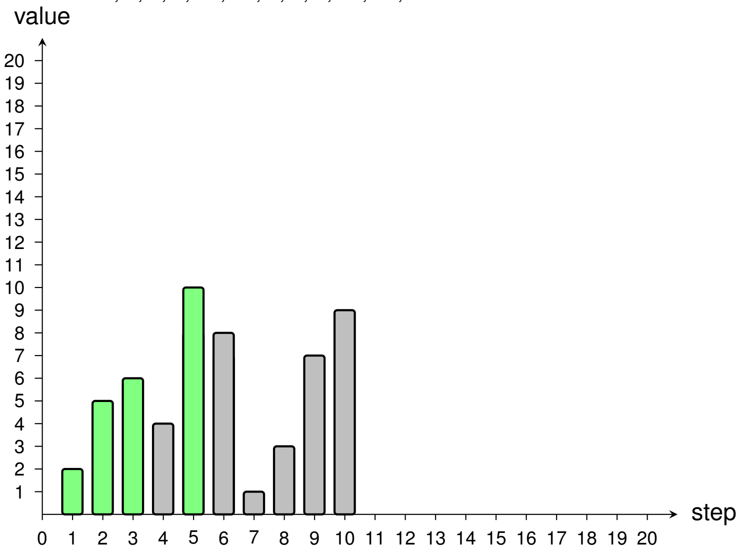

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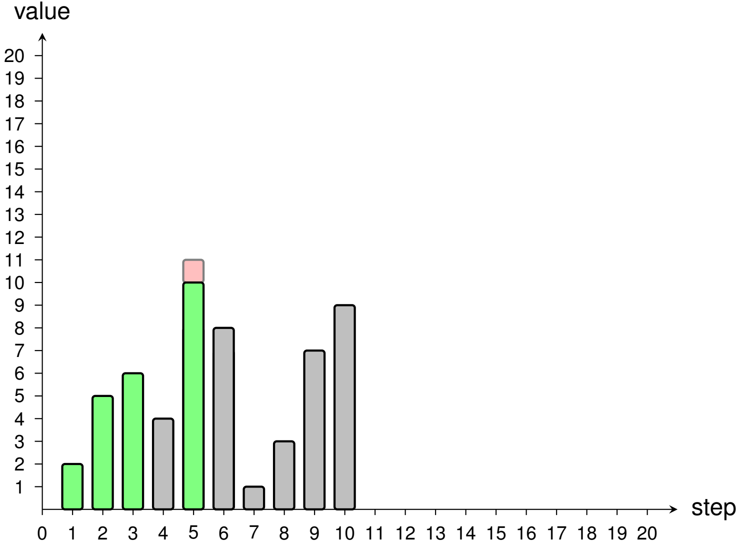
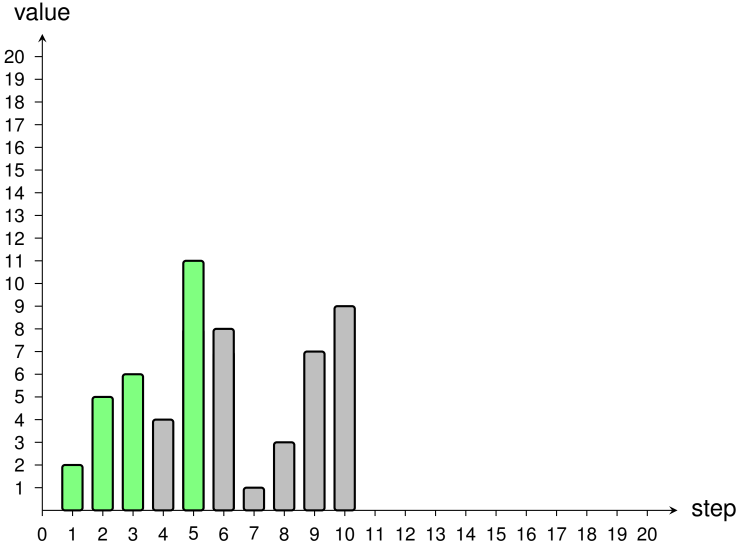


Illustration ($n = 20$)

unknown permutation:
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17,



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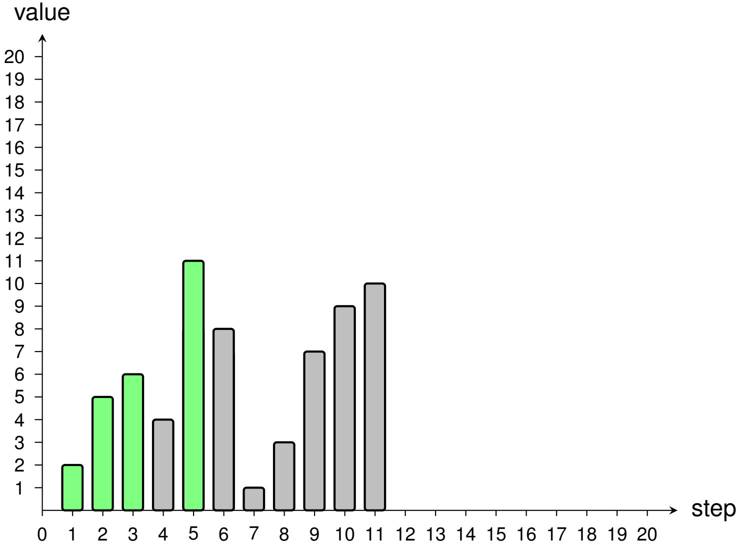

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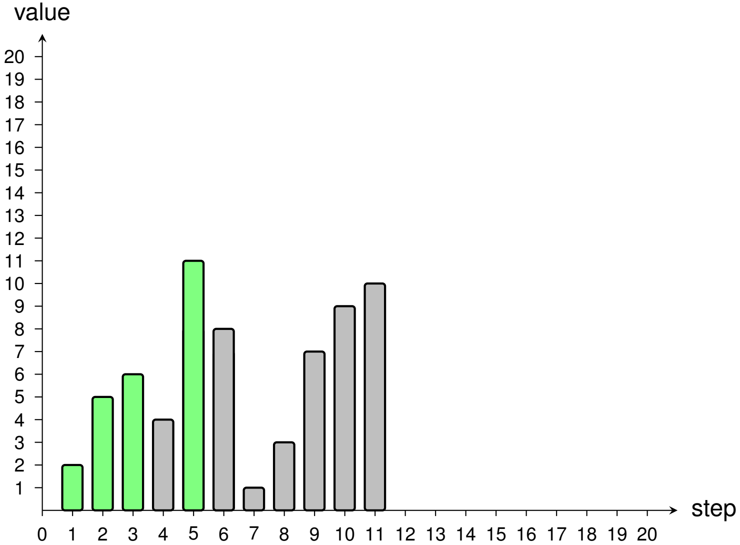
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17, 2,



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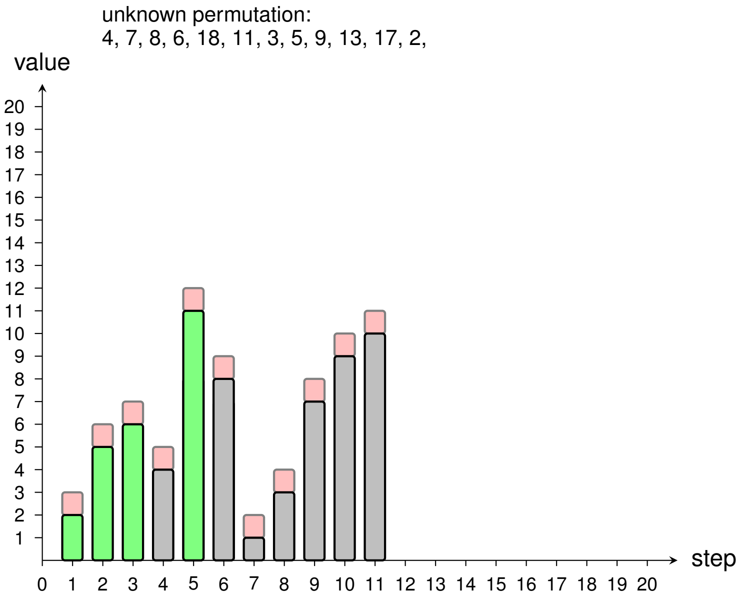

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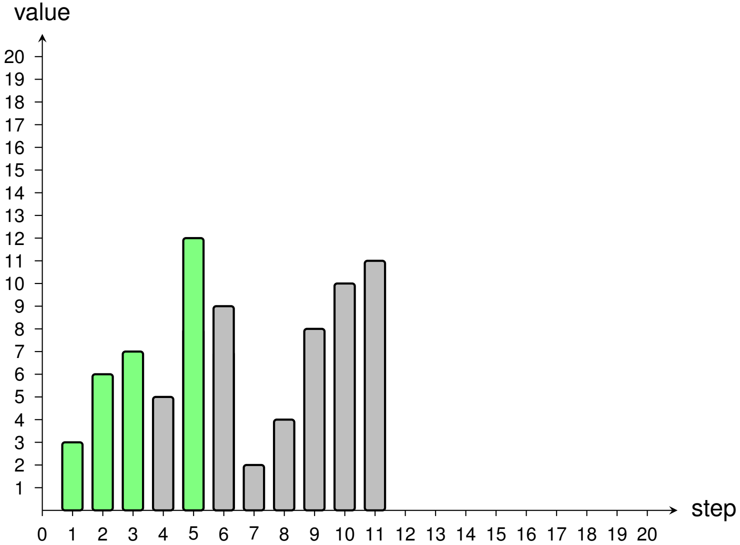

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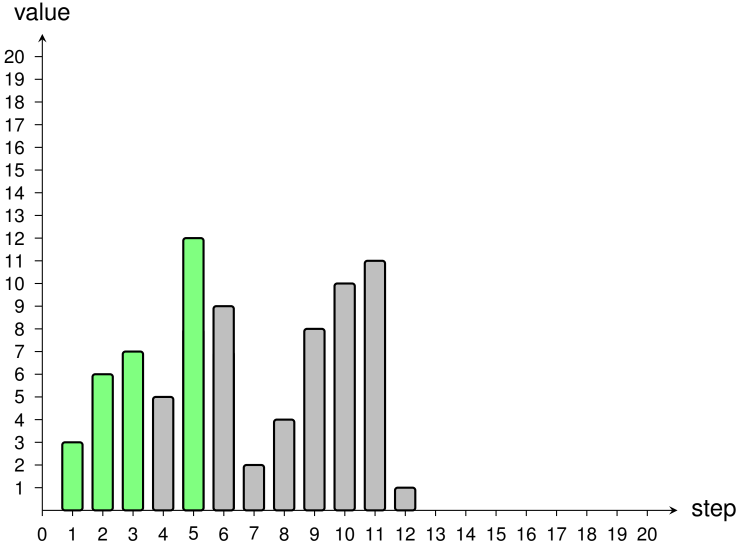

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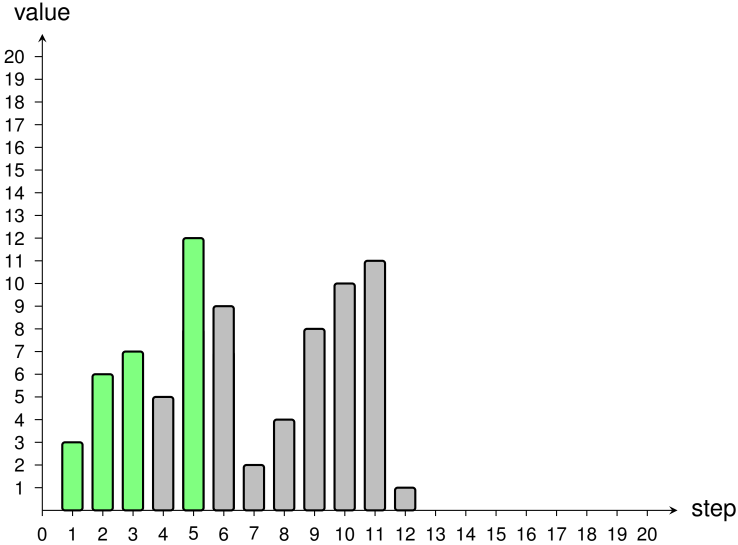

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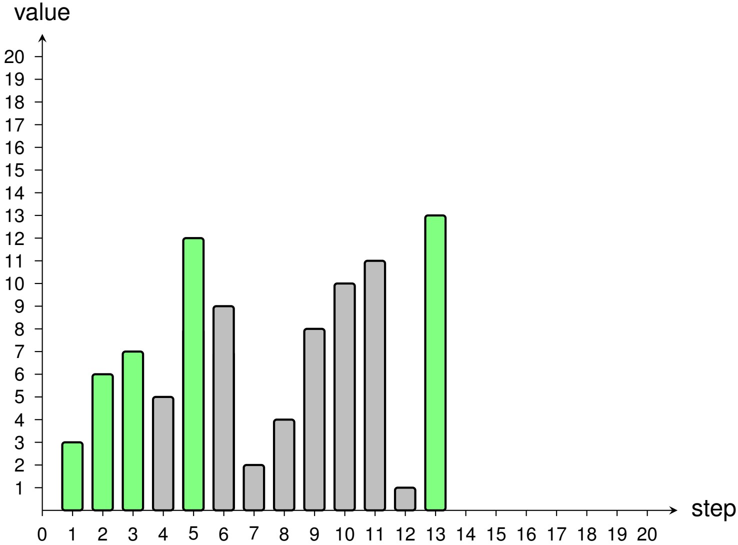
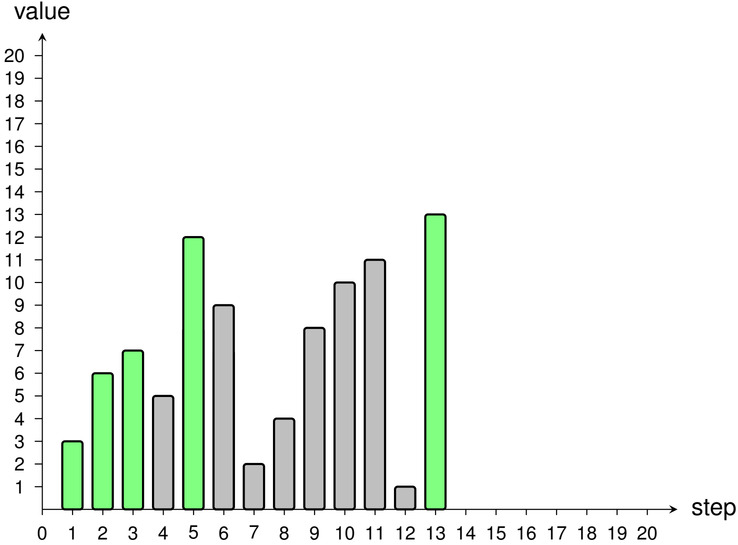


Illustration ($n = 20$)

unknown permutation:
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17, 2, 20, 14,



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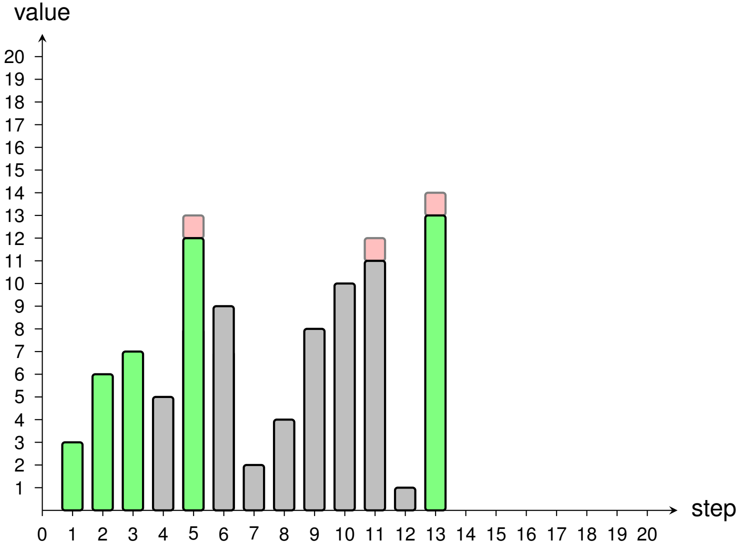

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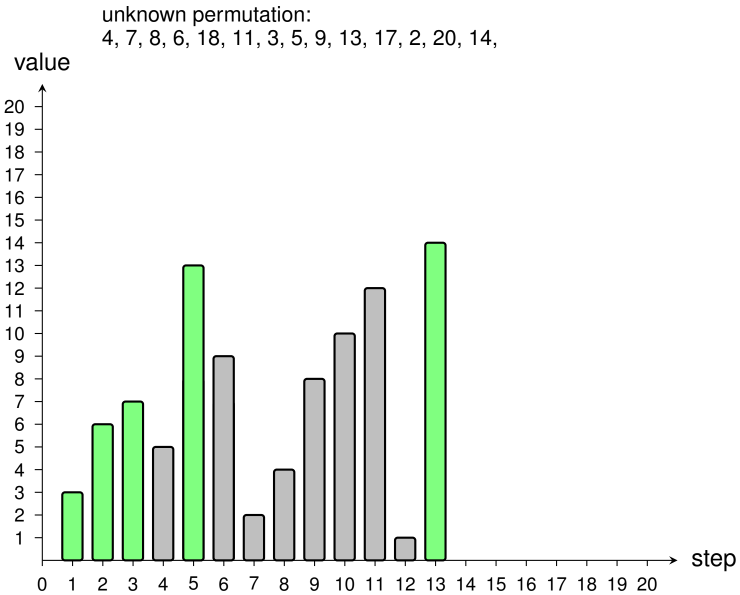

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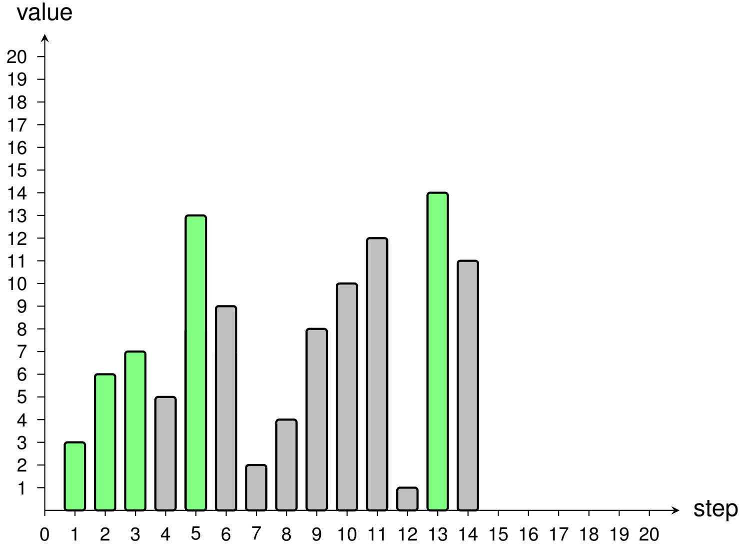

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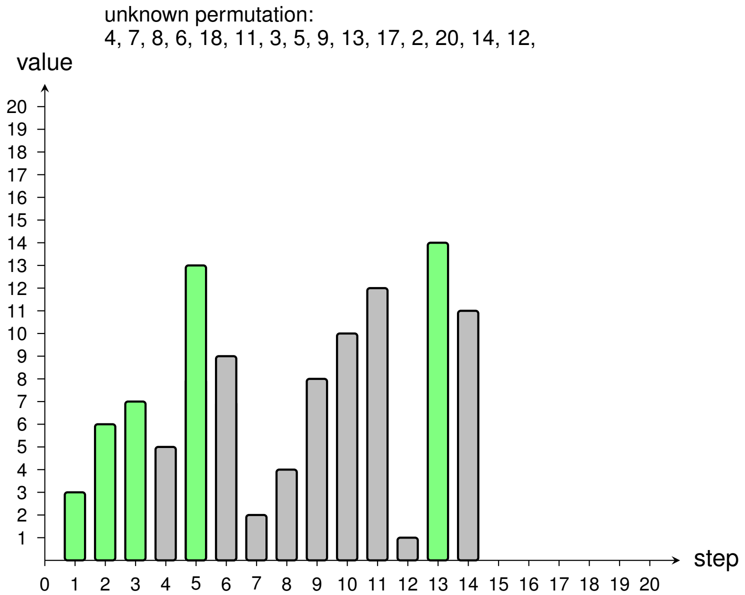

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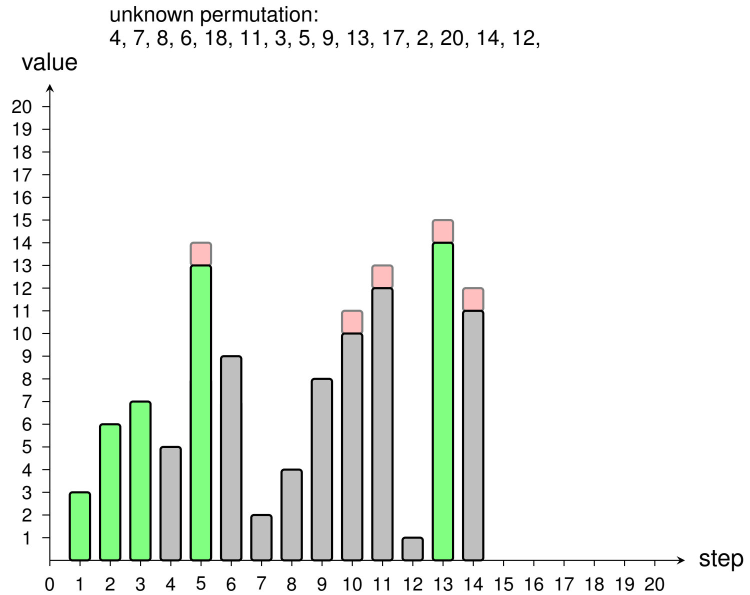

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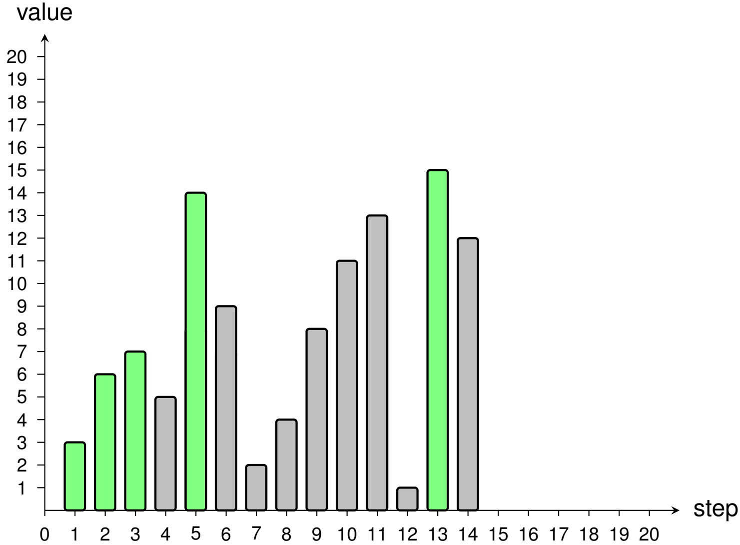

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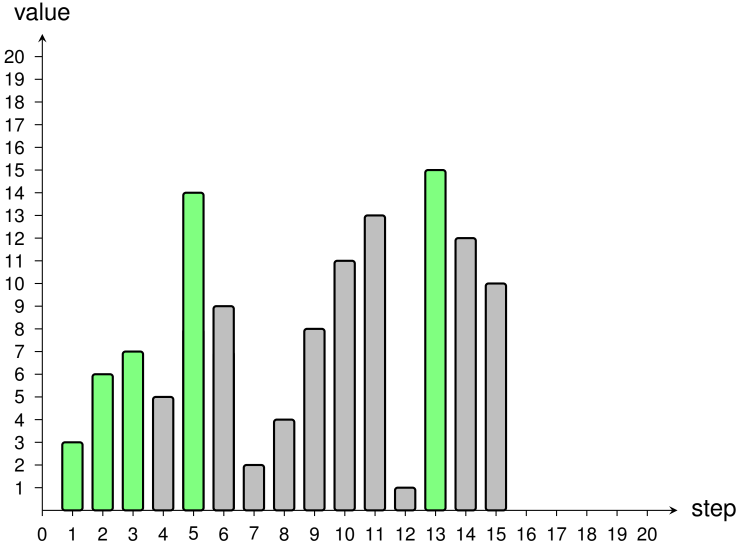

Illustration ($n = 20$)

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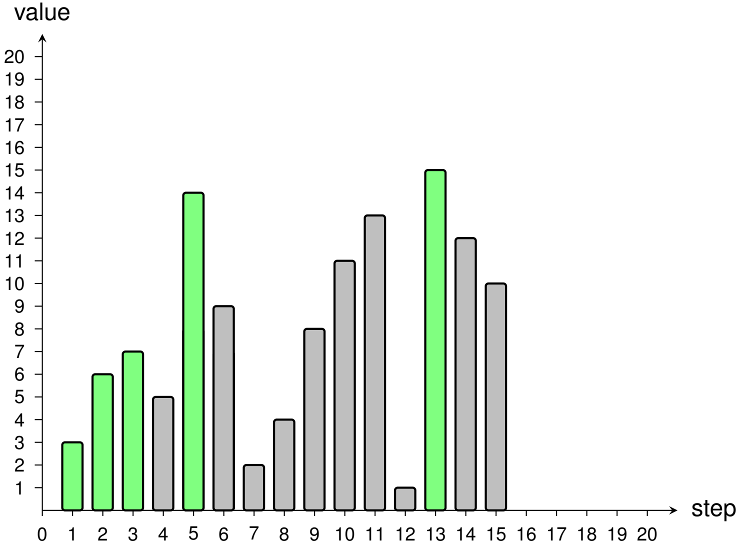
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17, 2, 20, 14, 12, 15,



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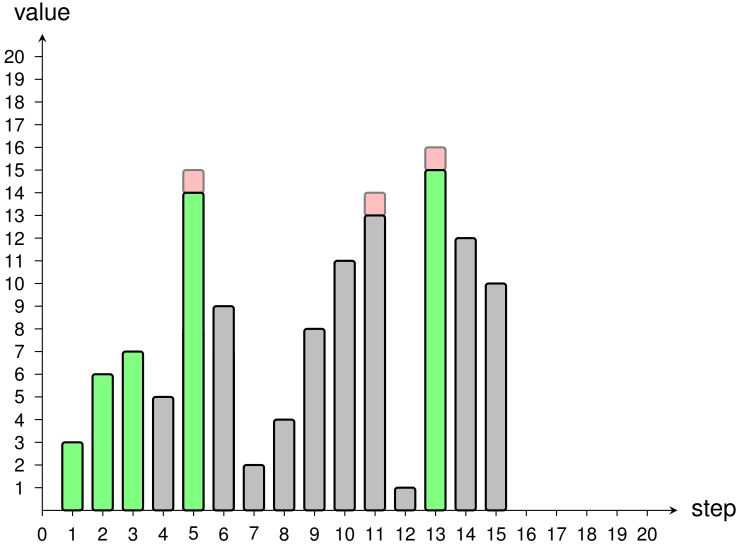

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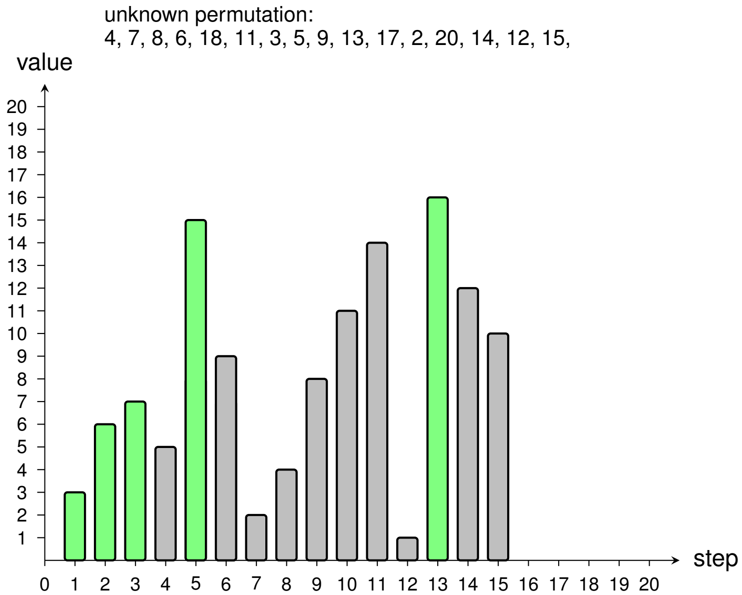

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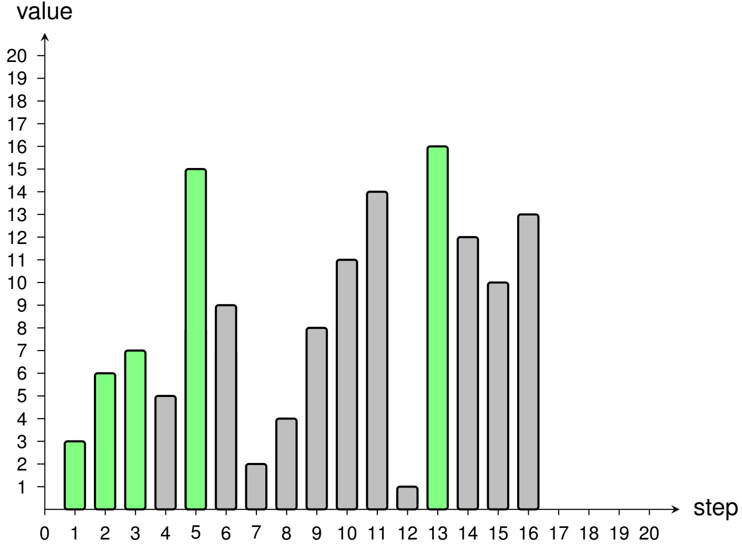

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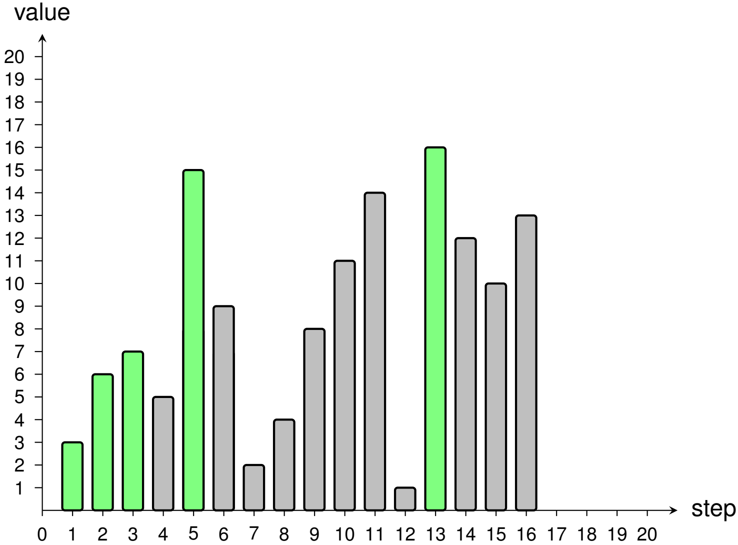
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17, 2, 20, 14, 12, 15, 10,



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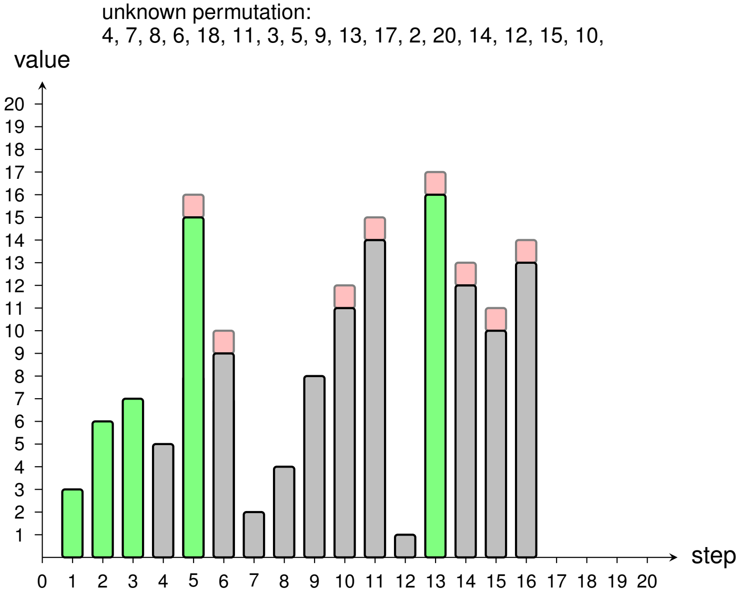

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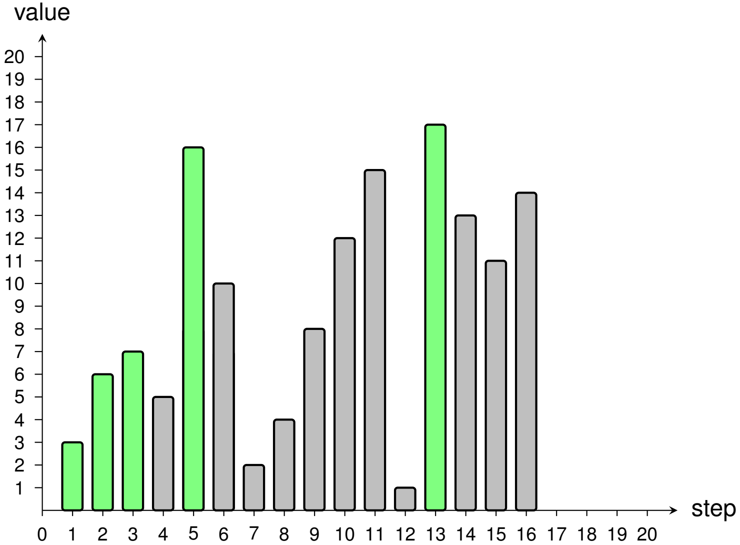

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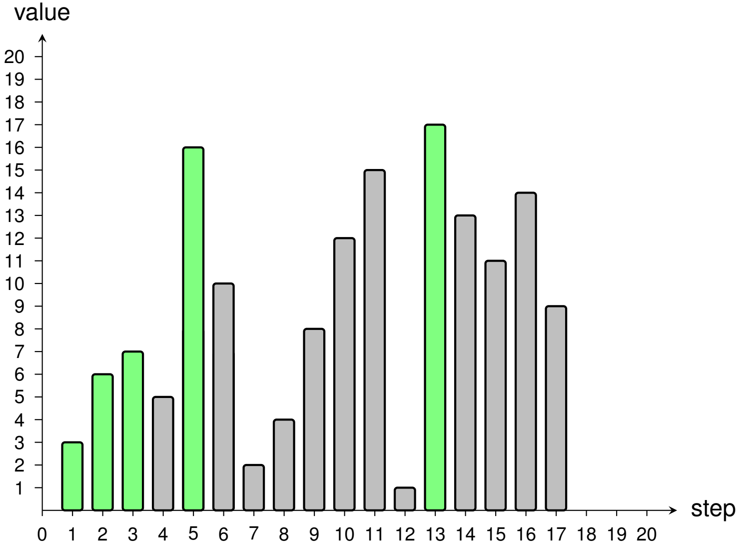

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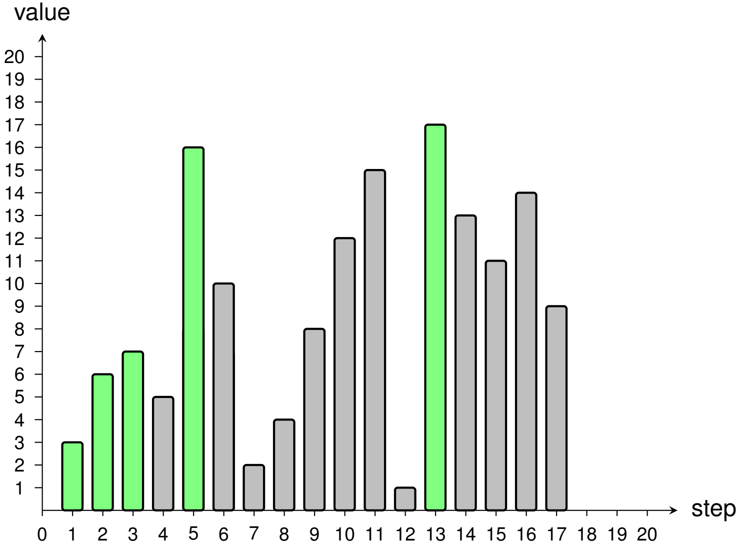

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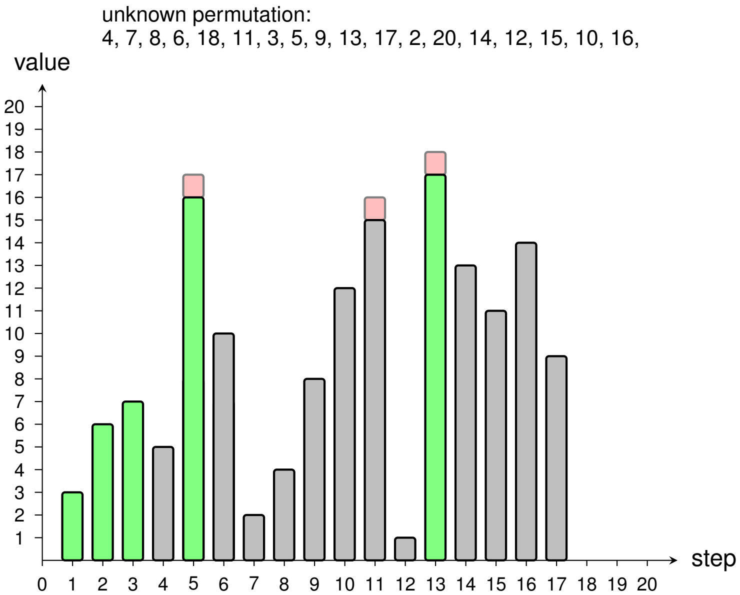

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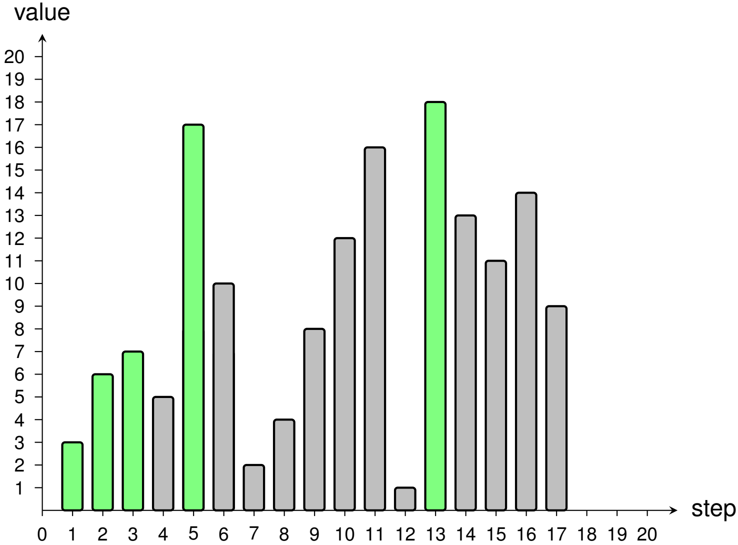

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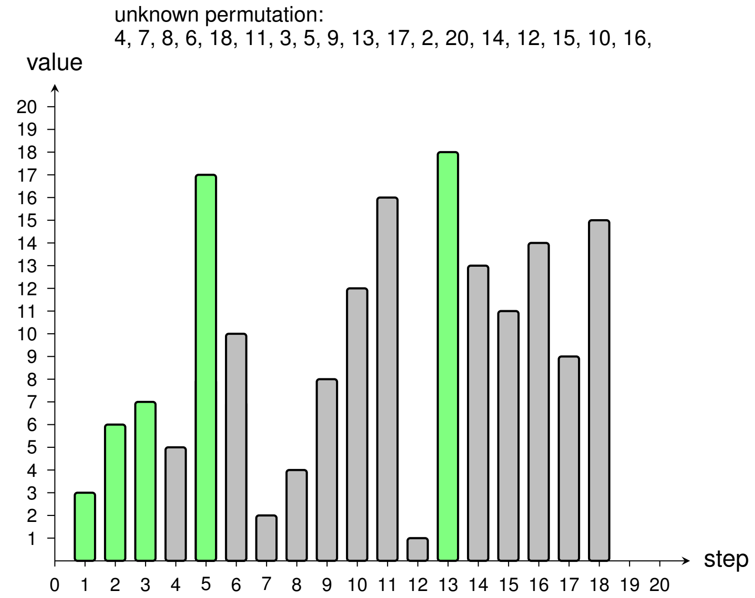

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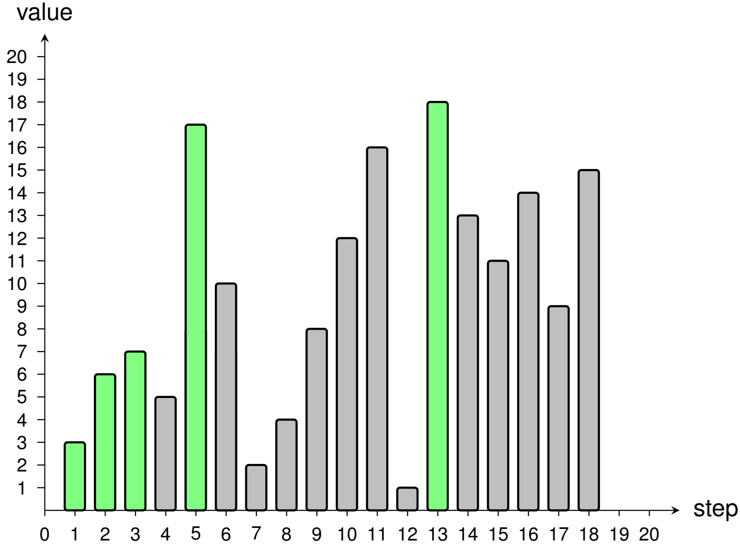

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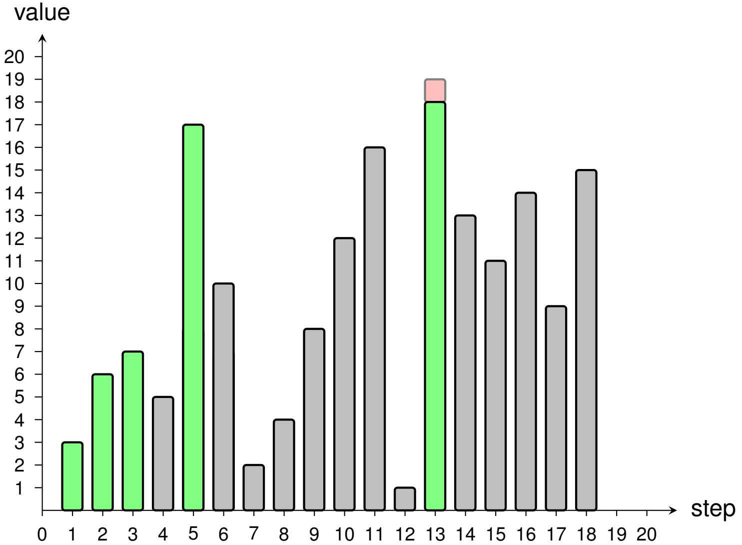

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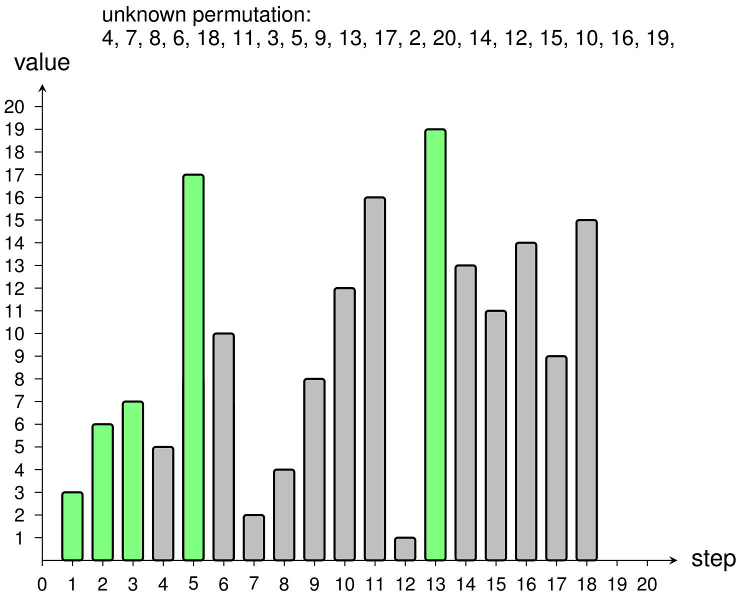

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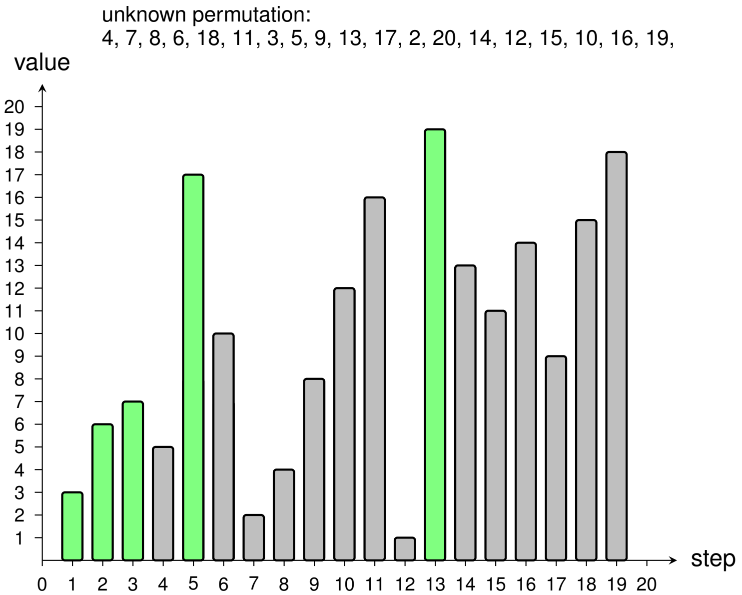

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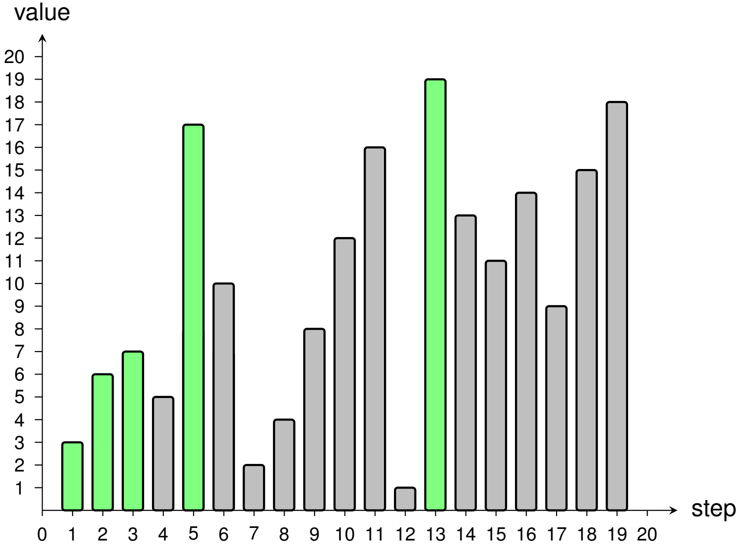

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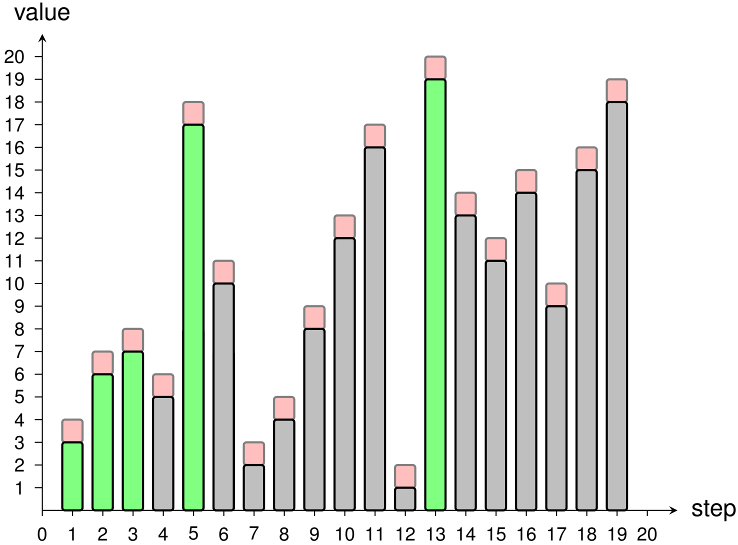

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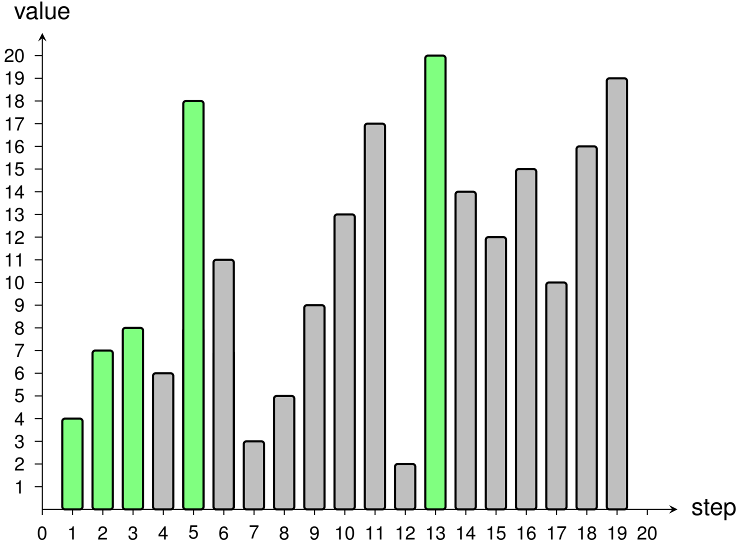

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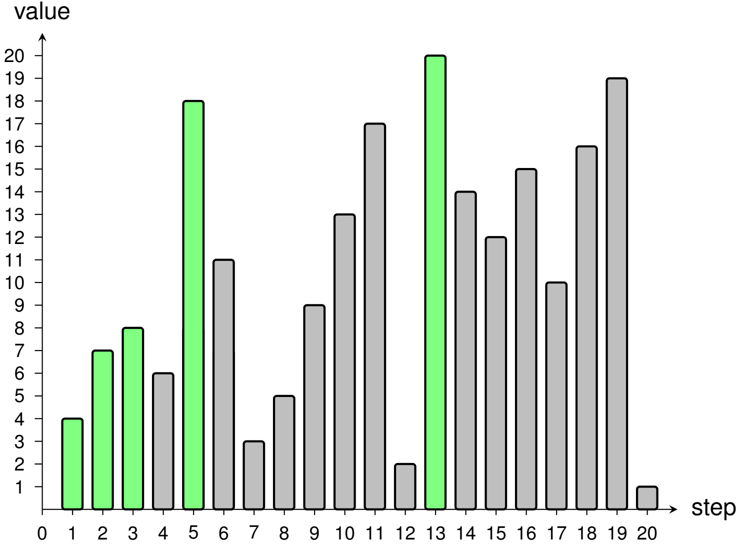

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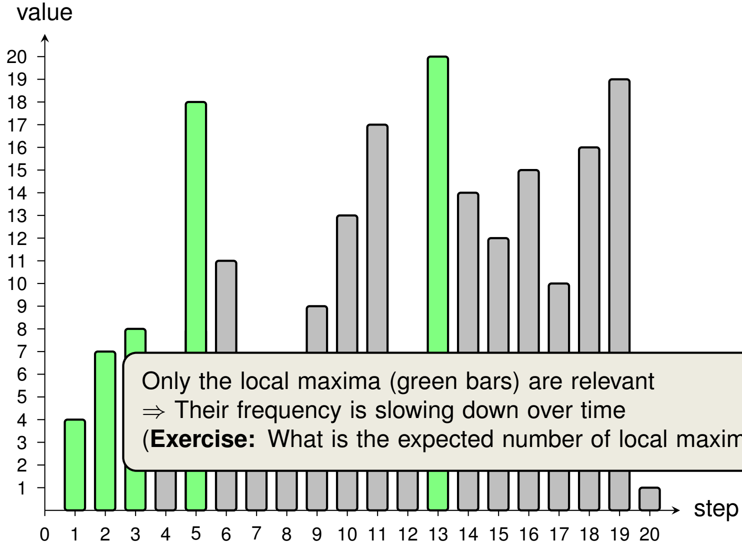

Only the local maxima (green bars) are relevant
$\Rightarrow$ Their frequency is slowing down over time
(**Exercise:** What is the expected number of local maxima?)

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How good is this approach?

Analysis of the Refined Approach

Example 1

Find a lower bound on the success probability of the refined approach (picking the first candidate better than the first $n/2$).

Answer

Finding the Optimal Strategy (1/2)

- Observation 1: At interview i, it only matters if current candidate is best so far (i.e., no benefit in counting how many “best-so-far” candidates we had).

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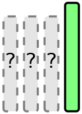

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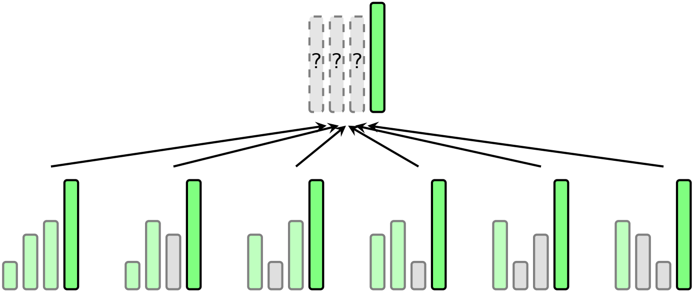

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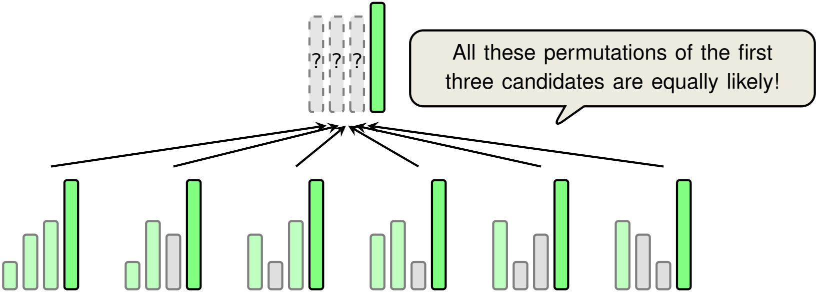

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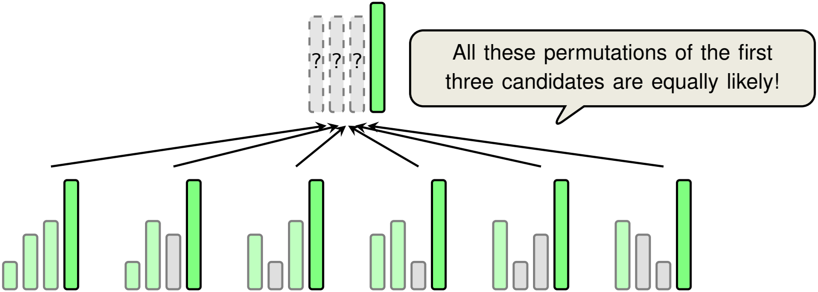

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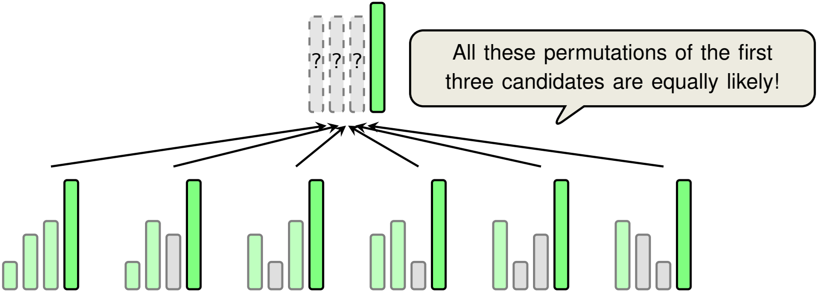

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Optimal Strategy

- Explore but reject the first $x - 1$ candidates
- Accept first candidate $i \geq x$ which is better than all candidates before

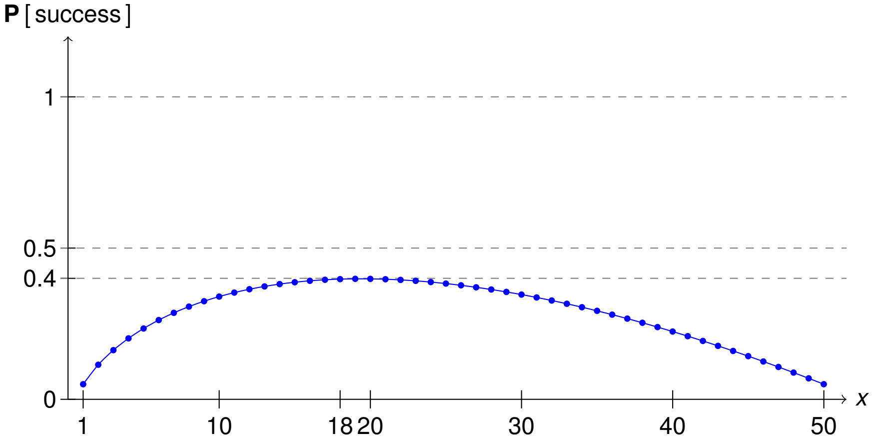
Example 2

Find x which maximises the probability of hiring the best candidate.

Answer

Probability for Success (Illustration)

Suppose $n = 50$:



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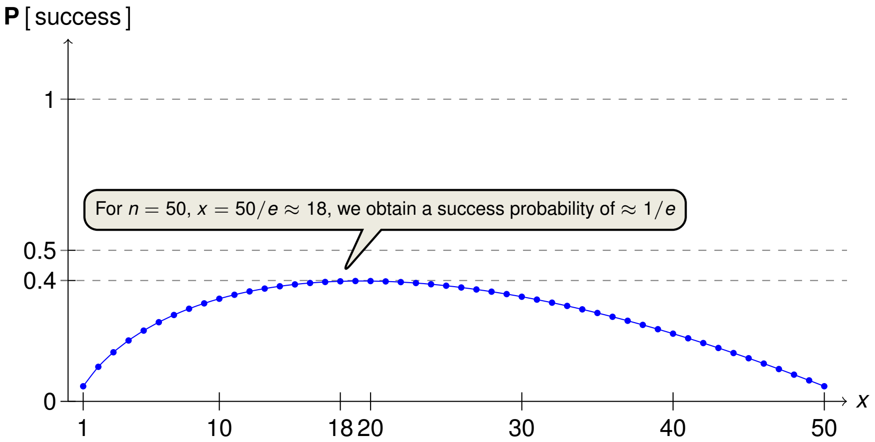

Another Variant of the Secretary Problem

"The Postdoc Variant of the Secretary Problem" (Vanderbei'80)

- same setup as in the secretary problem before
- difference: we want to pick the second-best ("the best [postdoc] is going to Harvard")
- Success probability of the optimal strategy is:

$$\frac{0.25n^2}{n(n-1)} \xrightarrow{n\to\infty} \frac{1}{4}$$

- Thus it is easier to pick the best than the second-best(!)

Outline

Details of the Odds Algorithm

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$$\mathbf{P}\left[\sum_{j=k}^{n} I_j = 1\right] = \sum_{j=k}^{n} p_j \cdot \prod_{k \leq j \leq n, j \neq i}^{n} (1 - p_i)$$

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$$\mathbf{P}\left[\sum_{j=k}^{n} I_j = 1\right] = \sum_{j=k}^{n} p_j \cdot \prod_{k \leq j \leq n, j \neq i}^{n} (1 - p_i) = \sum_{j=k}^{n} r_j \cdot \left(\prod_{i=k}^{n} (1 - p_i)\right)$$

Details of the Odds Algorithm

- Let $I_1, I_2, \dots, I_n$ be a sequence of independent indicators and let $p_j = \mathbf{E}[\, I_j \,]$
- Let $r_j := \frac{p_j}{1-p_j}$ (the odds) and $p_j \in (0,1)$ for all $j = 1, 2, \dots, n$

Example 3

What is the probability that after trial k, there is exactly one success?

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$$\mathbf{P}\left[\sum_{j=k}^{n} I_j = 1\right] = \sum_{j=k}^{n} p_j \cdot \prod_{\substack{i=k \\ k \leq j \leq n, j \neq i}}^{n} (1-p_i) = \sum_{j=k}^{n} r_j \cdot \left(\prod_{i=k}^{n} (1-p_i)\right)$$

- One can prove that $\mathbf{P}\left[\sum_{j=k}^{n} I_j = 1\right]$ is unimodal in k $\Rightarrow$ there is an ideal point from which on we should `STOP` at the first success!

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Odds Algorithm ("Sum the Odds to One and Stop", F. Thomas Bruss, 2000)

1. Let k^* be the largest k such that $\sum_{j=k}^{n} r_j \geq 1$
2. Ignore everything before the k^*-th trial, then STOP at the first success.

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- Let $I_1, I_2, \ldots, I_n$ be a sequence of independent indicators and let $p_j = \mathbf{E}[\, I_j \,]$
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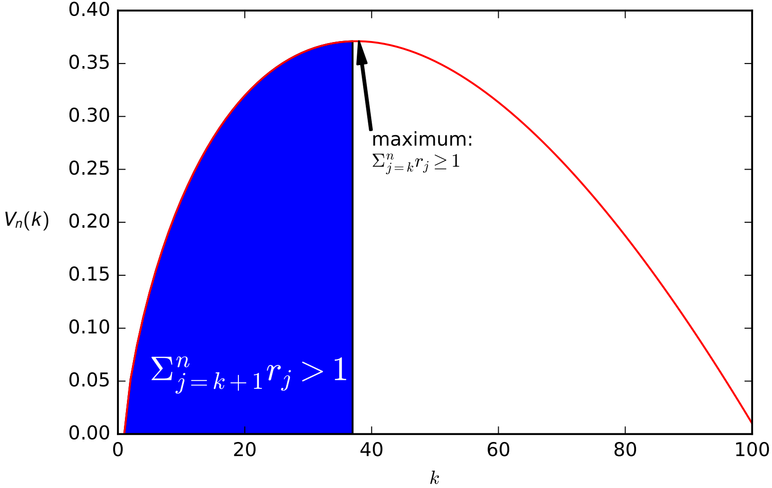
- The success probability is $\sum_{j=k^*}^{n} r_j \cdot \left(\prod_{i=k^*}^{n} (1-p_i)\right)$.
- This algorithm always executes the optimal strategy!

Illustration of the probability of having the last success ($n = 100$)



Source: Group Fibonado

Example 4

Use the Odds Algorithm to analyse the Secretary Problem.

Answer

Outline

List of Lectures

- **Part I: Introduction to Probability**
  - Lecture 1: Conditional probabilities and Bayes' theorem

- **Part II: Random Variables**
  - Lecture 2: Random variables, probability mass function, expectation
  - Lecture 3: Expectation properties, variance, discrete distributions
  - Lecture 4: More discrete distributions: Poisson, Geometric, Negative Binomial
  - Lecture 5: Continuous random variables
  - Lecture 6: Marginals and Joint Distributions
  - Lecture 7: Independence, Covariance and Correlation

- **Part III: Moments and Limit Theorems**
  - Lecture 8: Basic Inequalities and Law of Large Numbers
  - Lecture 9: Central Limit Theorem

- **Part IV: Applications and Statistics**
  - Lecture 10: Estimators (Part I)
  - Lecture 11: Estimators (Part II)
  - Lecture 12: Online Algorithms

List of Distributions

Very Important:

- Bernoulli, Binomial, Poisson
- (Continuous) Uniform, Normal, Exponential

(Somewhat Less) Important:

- Geometric, Negative Binomial, Hypergeometric, Discrete Uniform

Not used or not defined in this course (and thus not examinable):

- Cauchy, Gamma, bivariate Normal
- Beta

Thank you and Best Wishes for the Exam!