Introduction to Probability

Lecture 12: Online Algorithms

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Outline

Stopping Problem 1: Dice Game

Stopping Problem 2: The Secretary Problem

A Generalisation: The Odds Algorithm (non-examinable)

The End...

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This boils down to finding a threshold from which we `STOP` as soon as a 6 is thrown.

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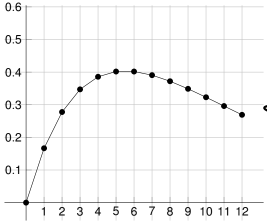

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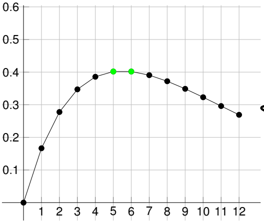

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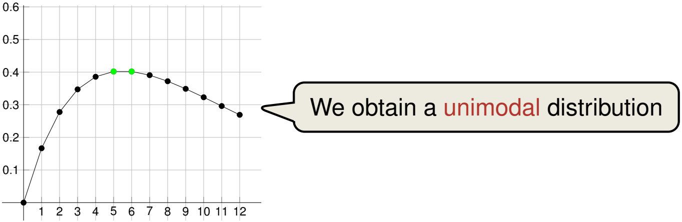

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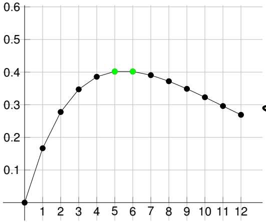

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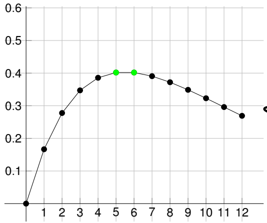

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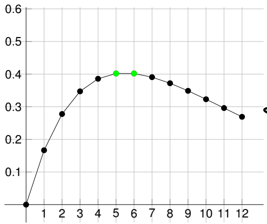

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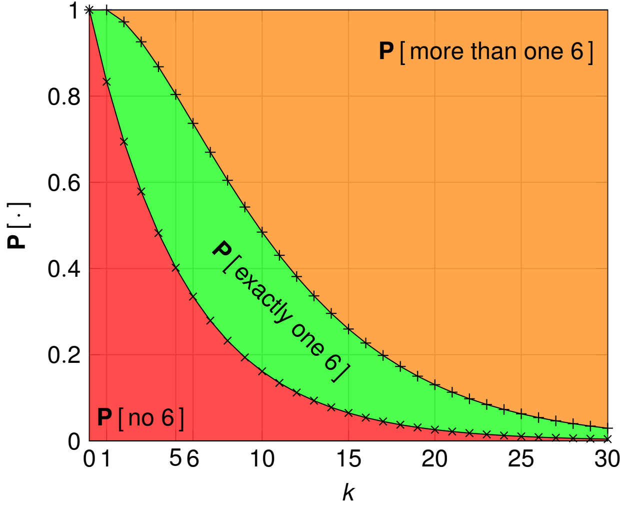
$$\left(\frac{5}{6}\right)^5 \approx 0.40.$$

Illustration of the Three Possibilities



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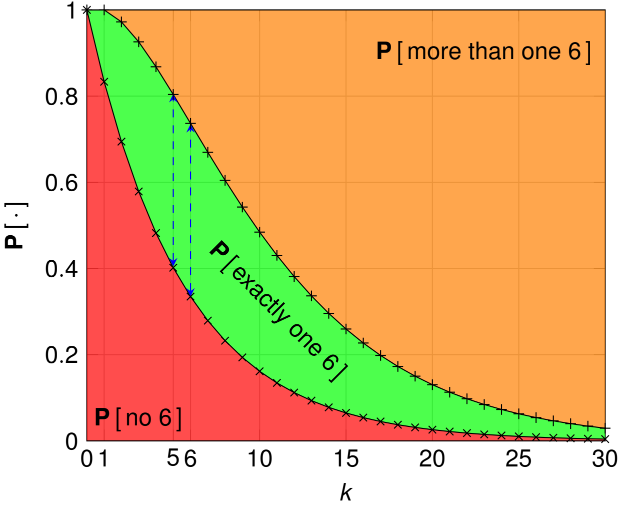

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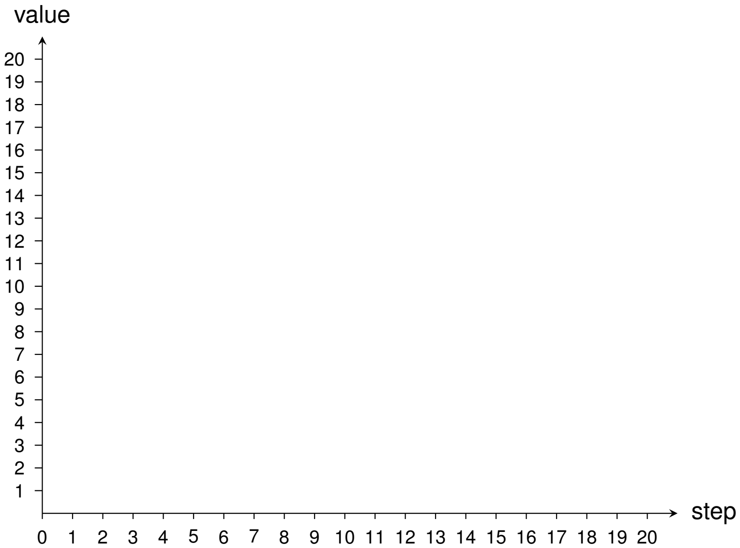
$\Rightarrow$ For our problem we may as well assume that the only information we have when interviewing candidate i is whether that candidate is best among $\{1, \ldots, i\}$ or not.

Illustration ($n = 20$)

unknown permutation:



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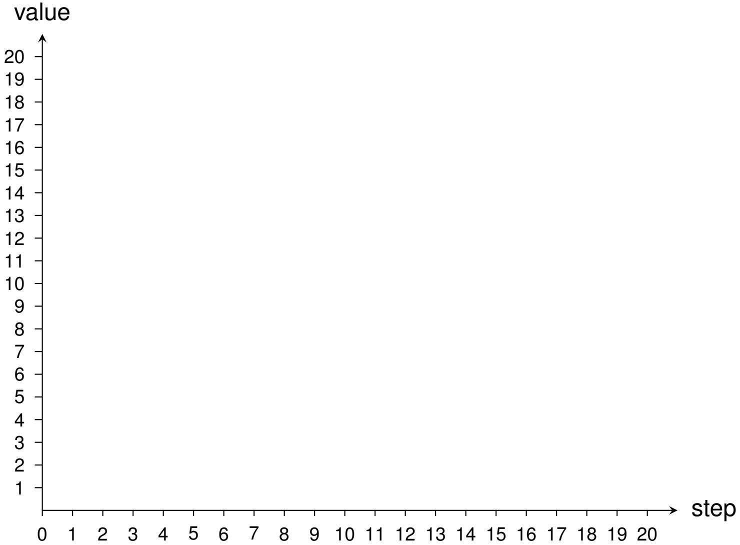

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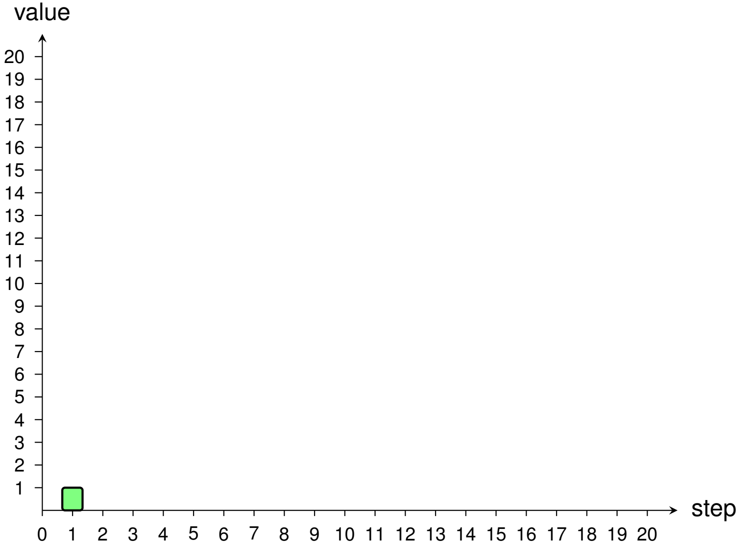

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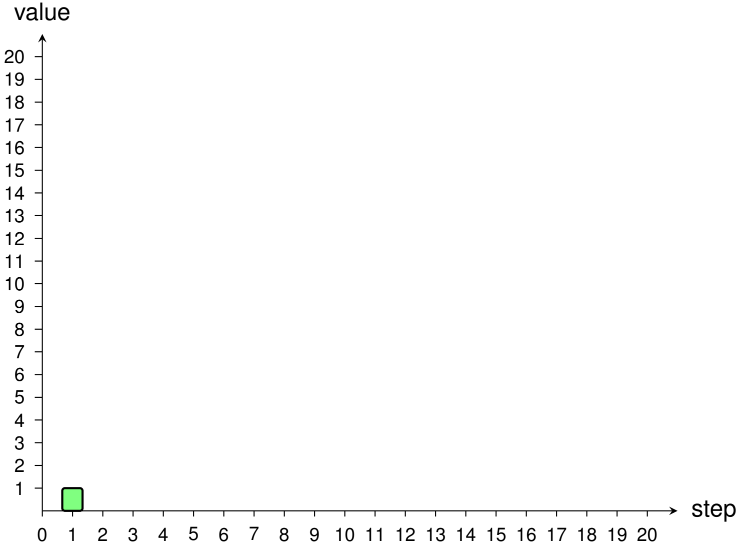
4, 7,



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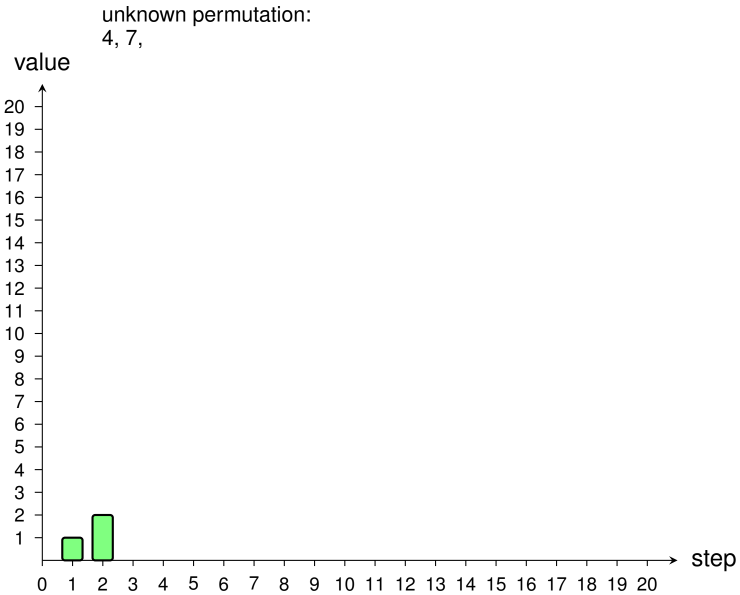

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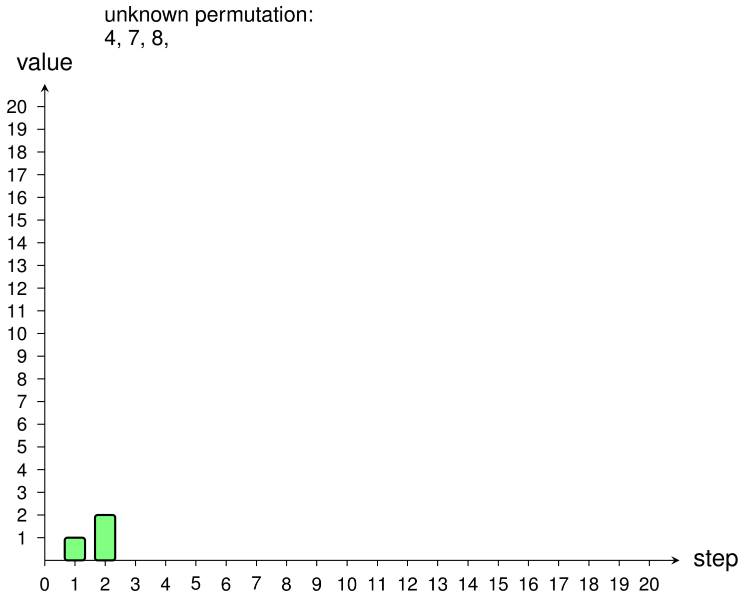

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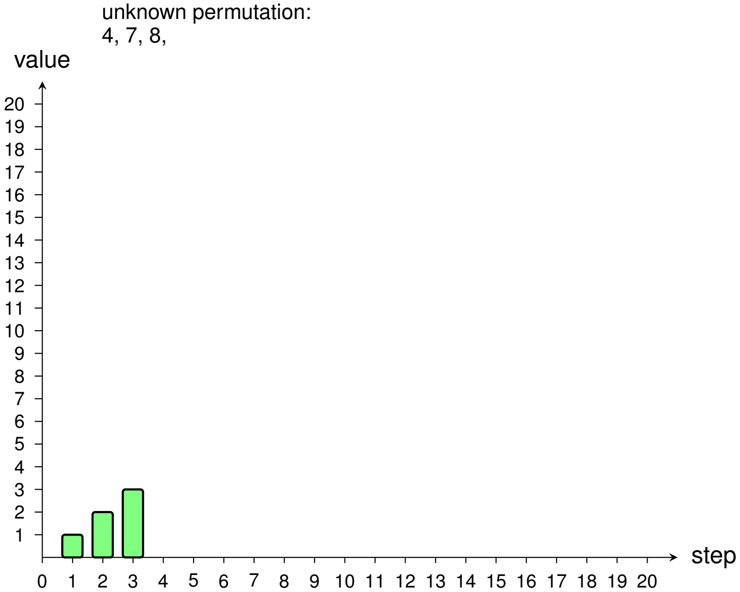

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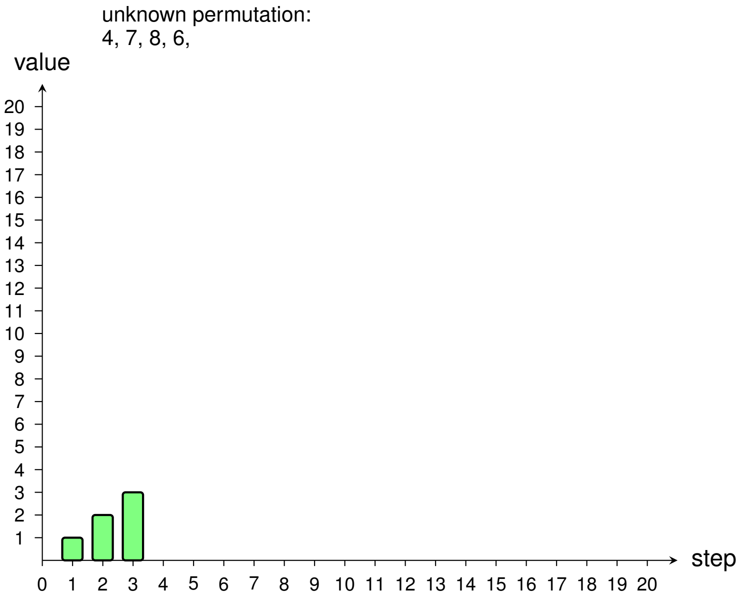

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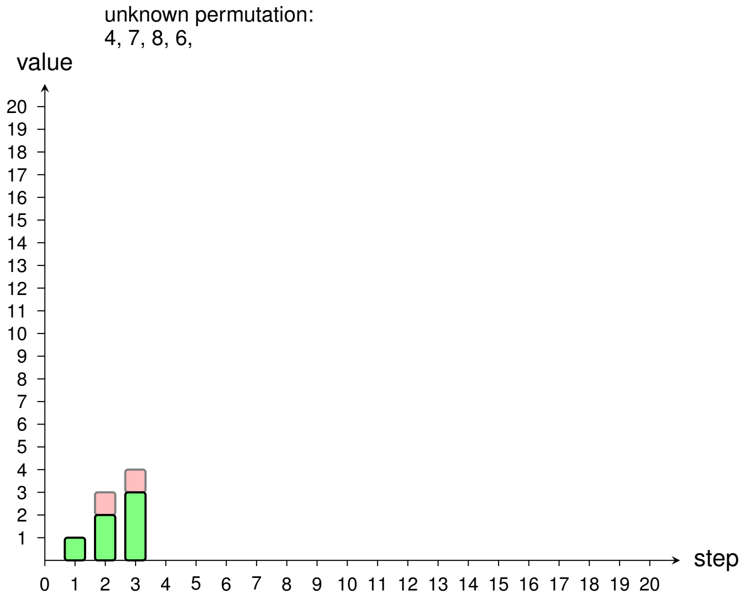

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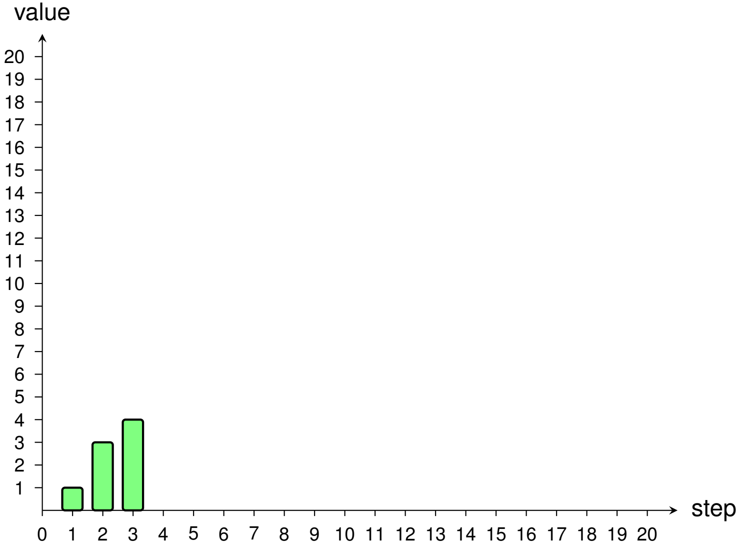

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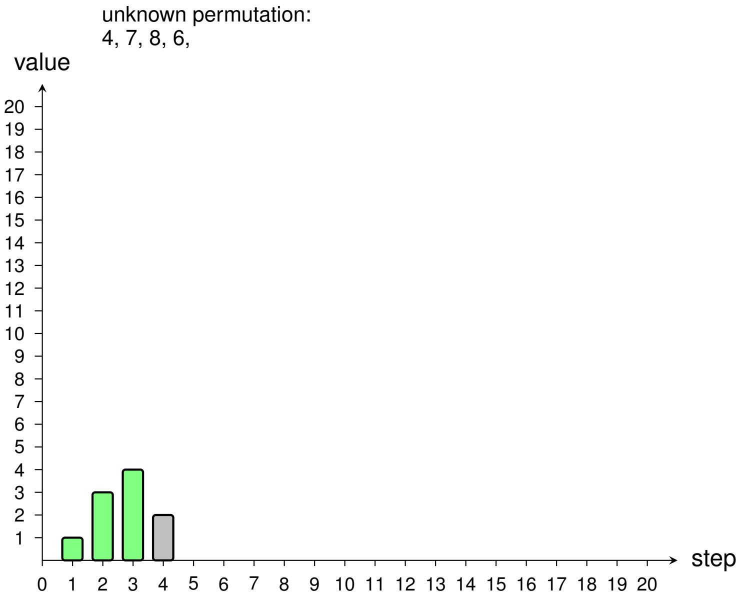
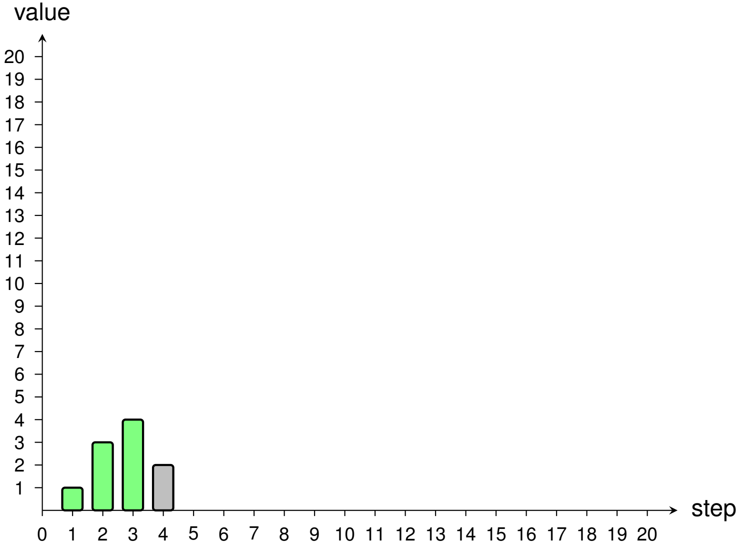


Illustration ($n = 20$)

unknown permutation:
4, 7, 8, 6, 18,



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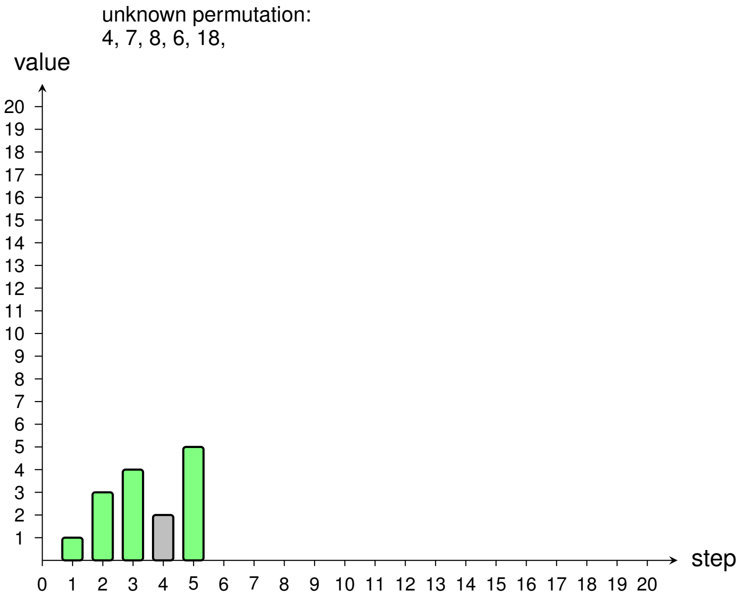

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unknown permutation:
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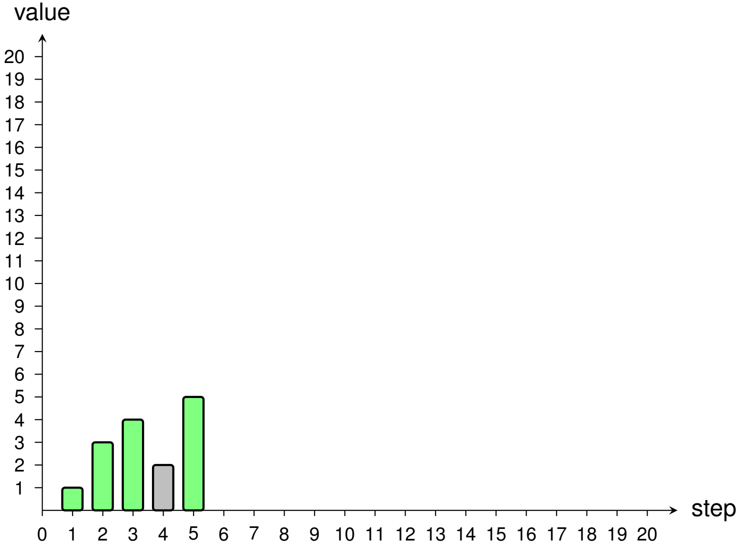

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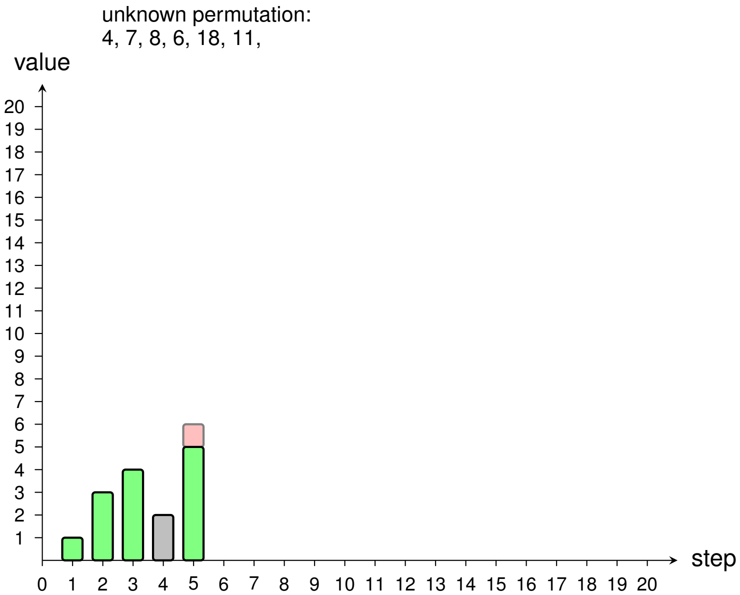

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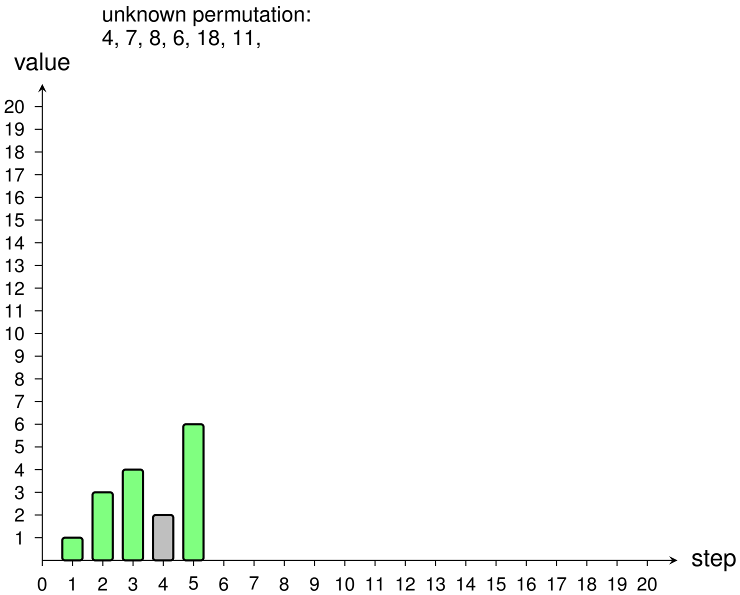

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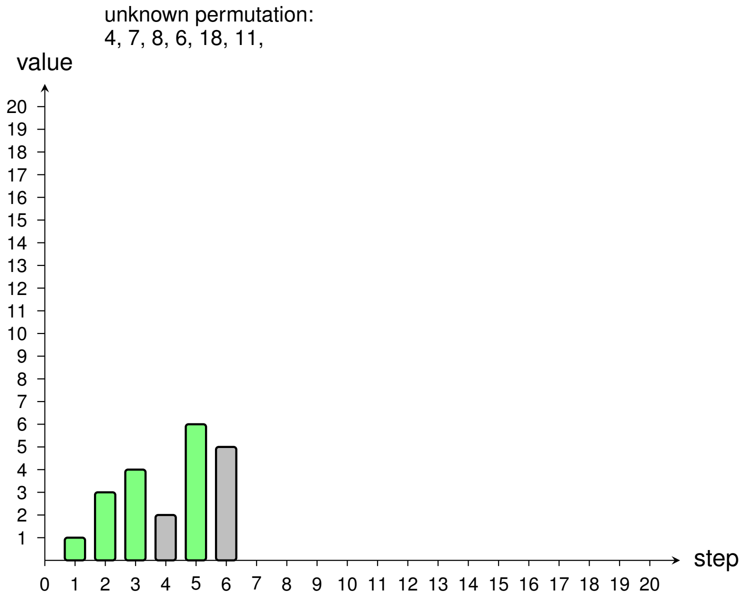

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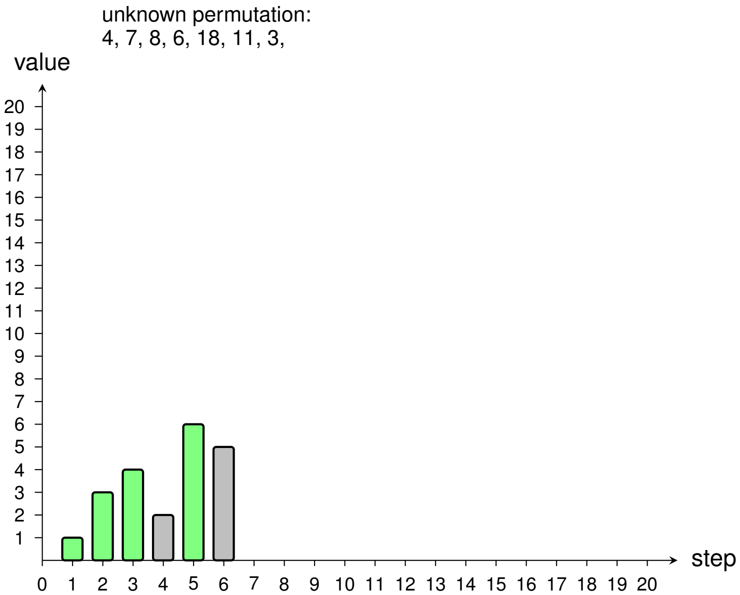

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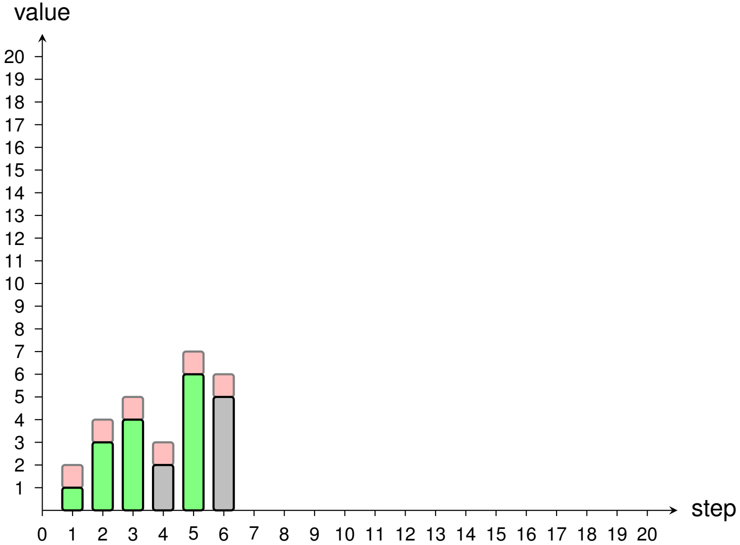

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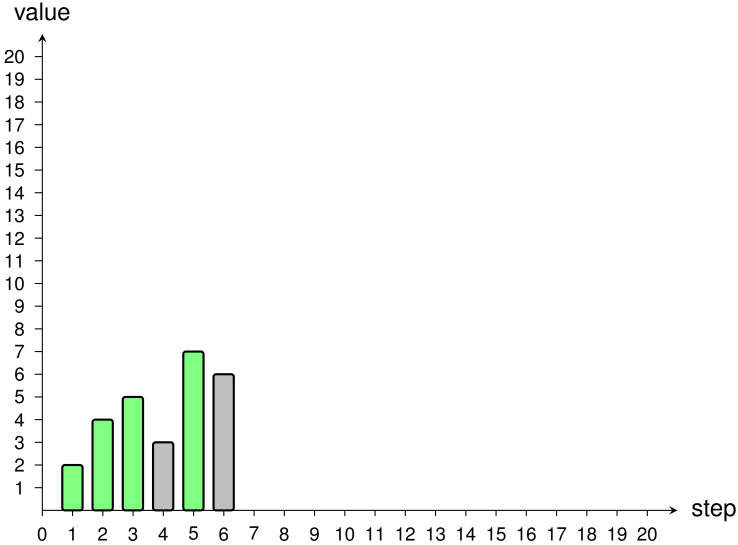

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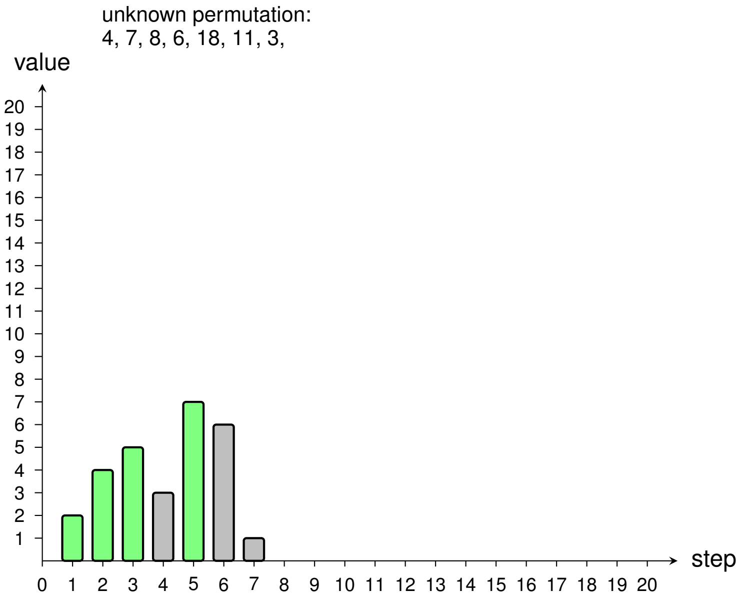

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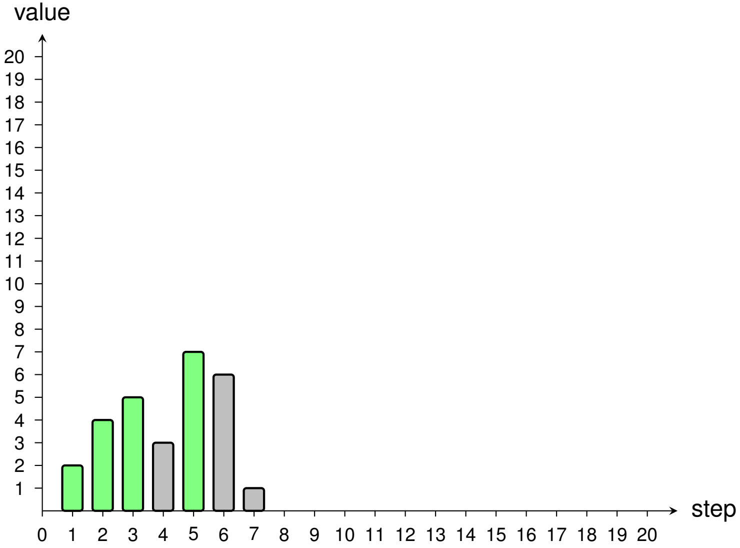

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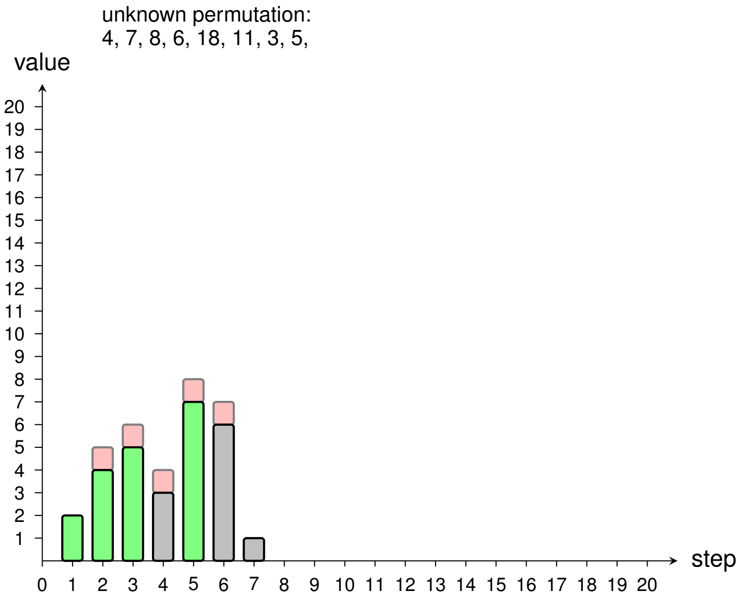

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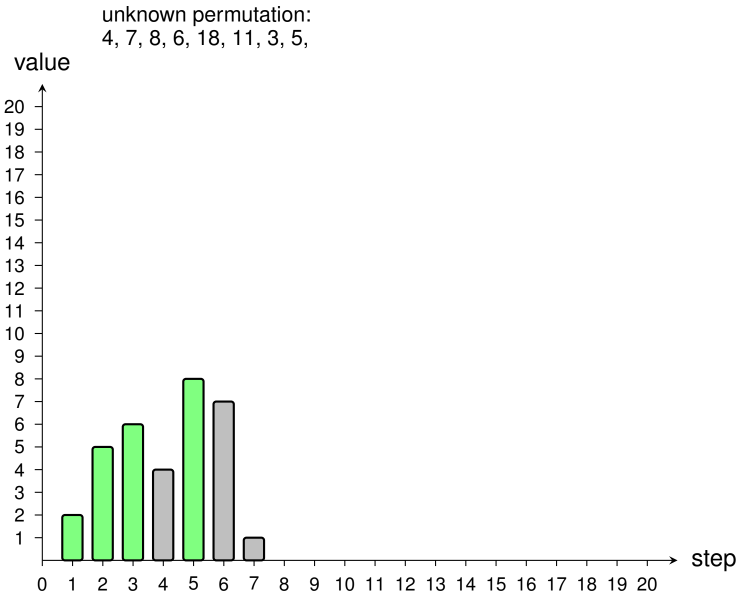

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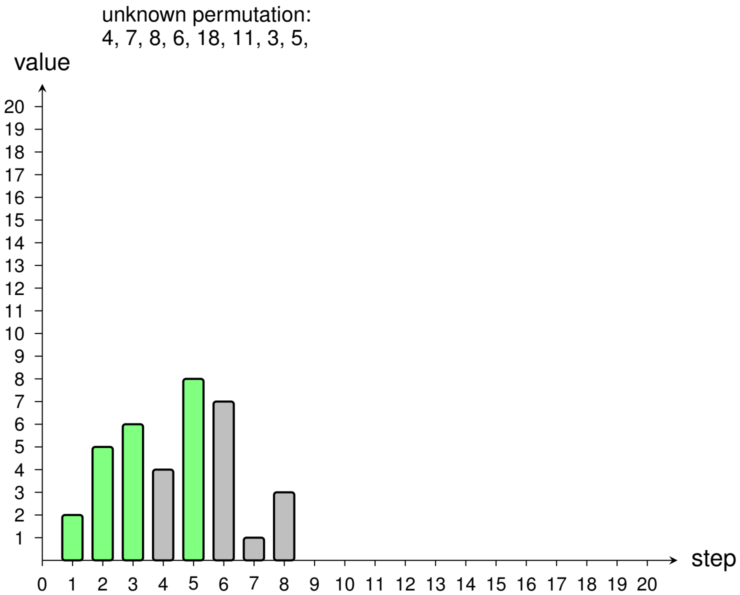

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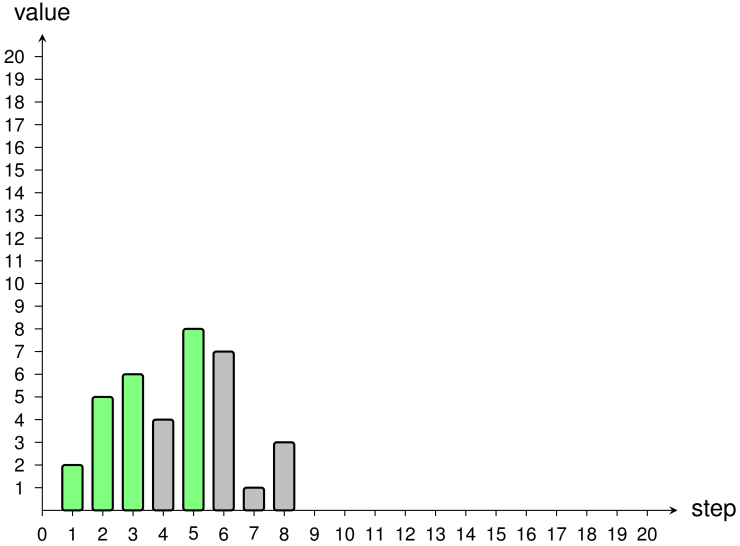

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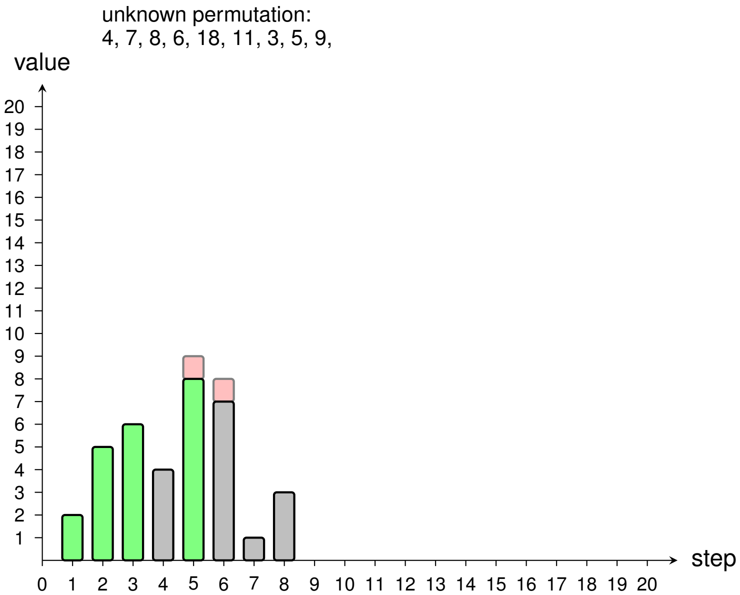

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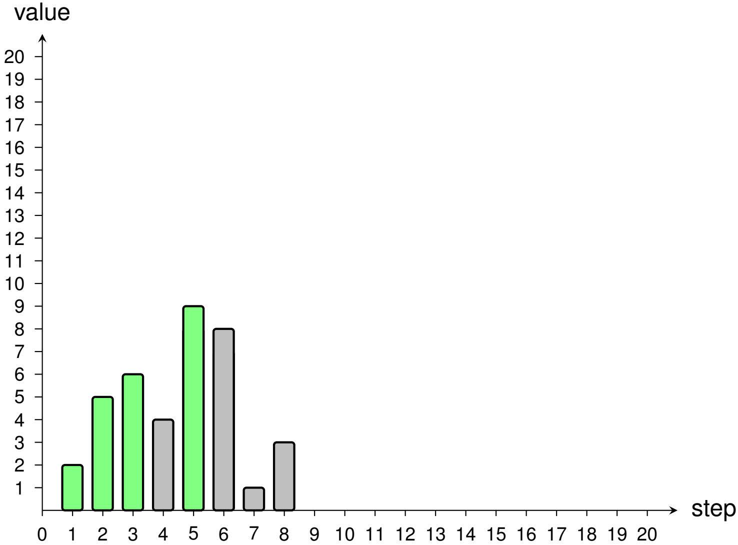

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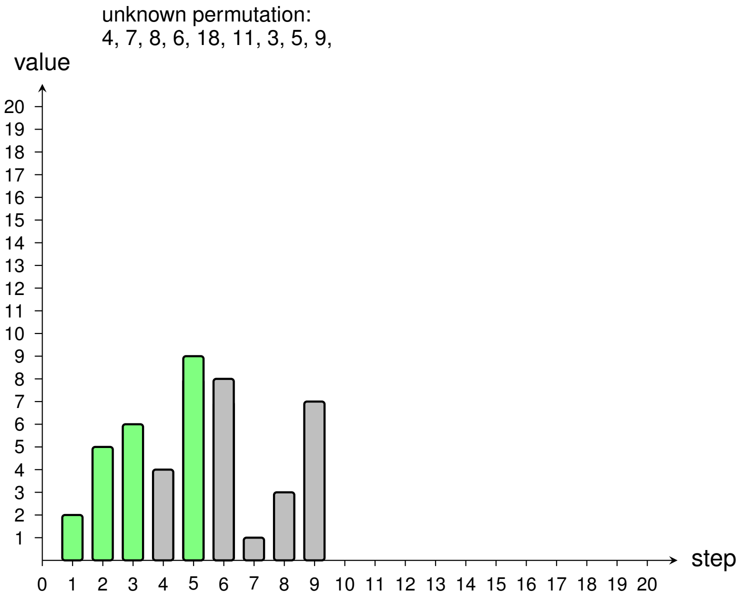

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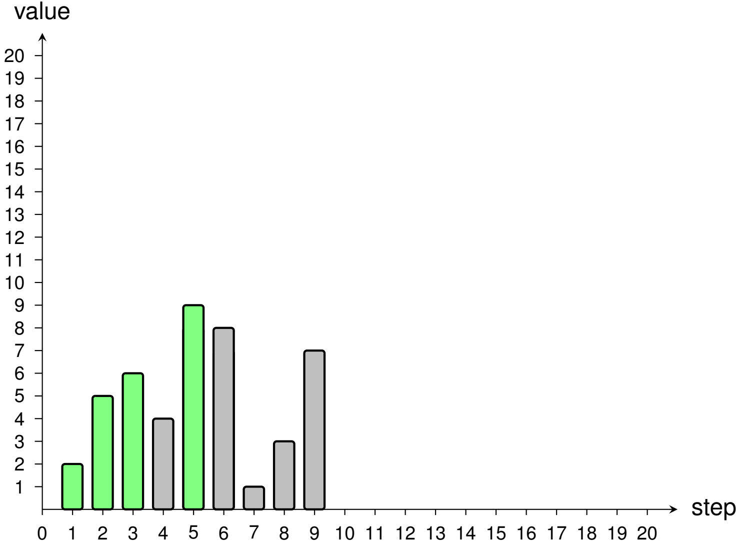

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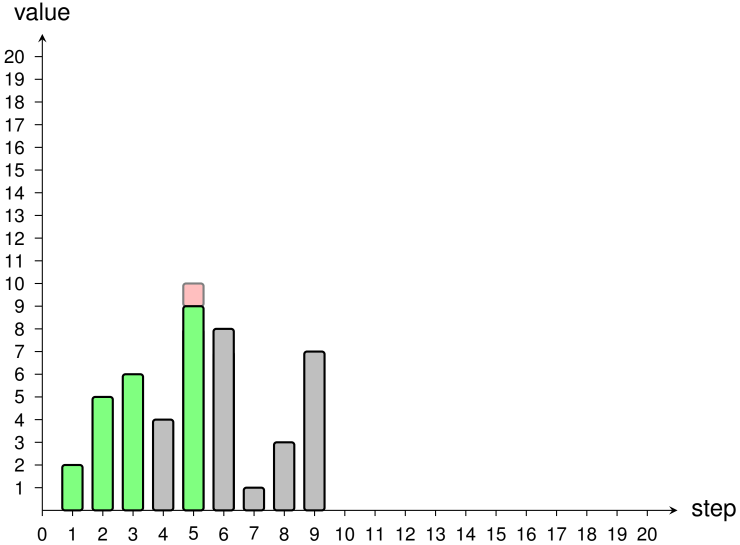

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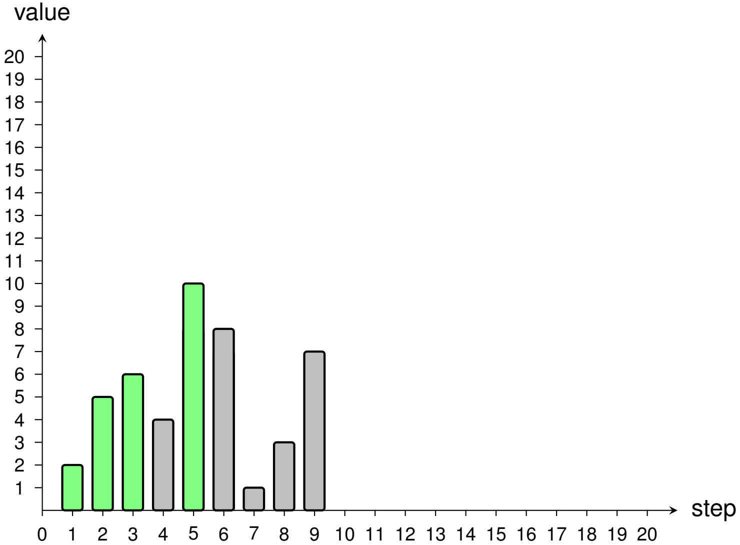

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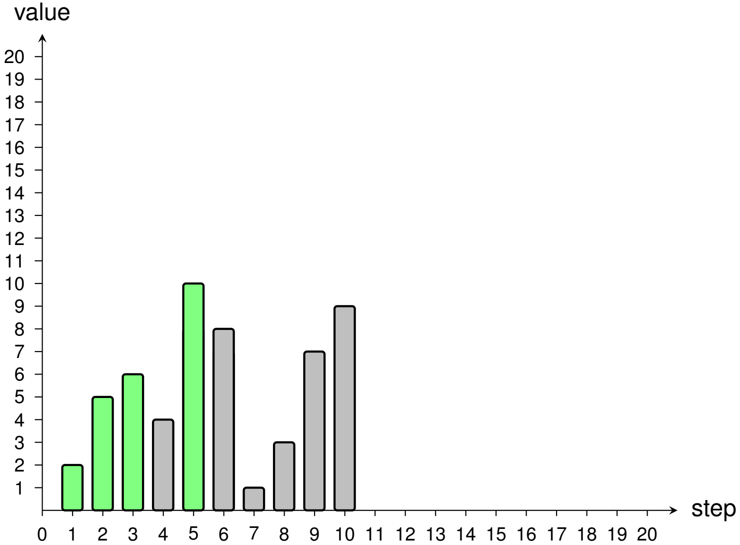

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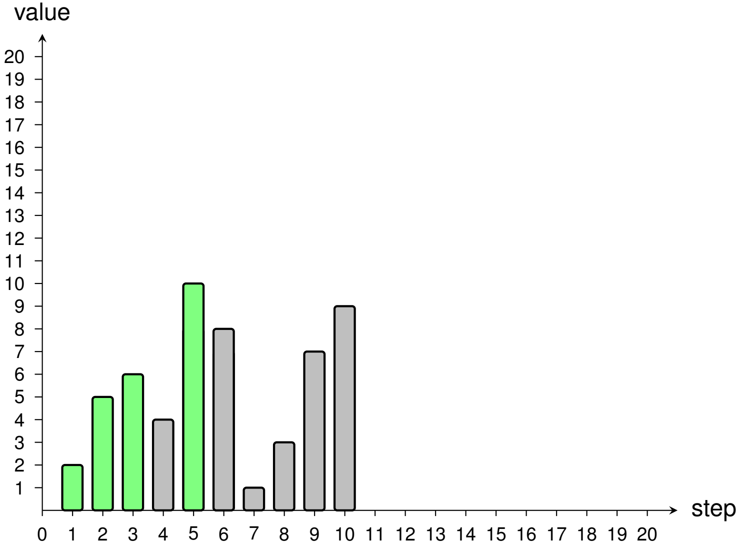

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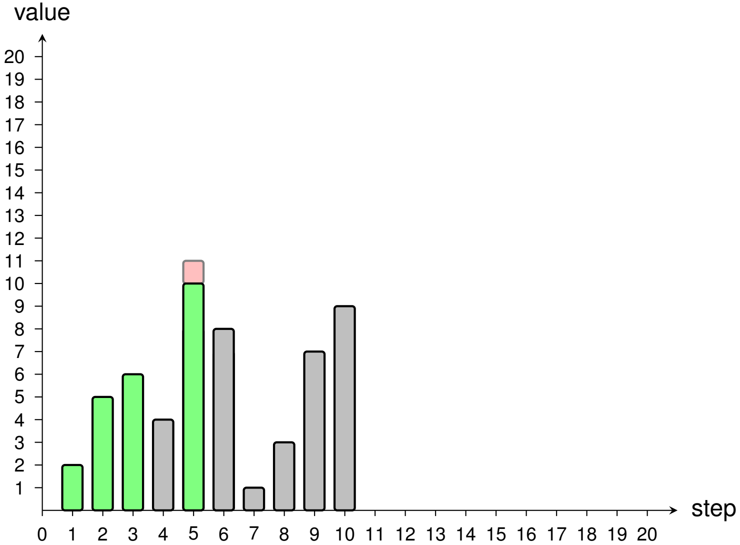
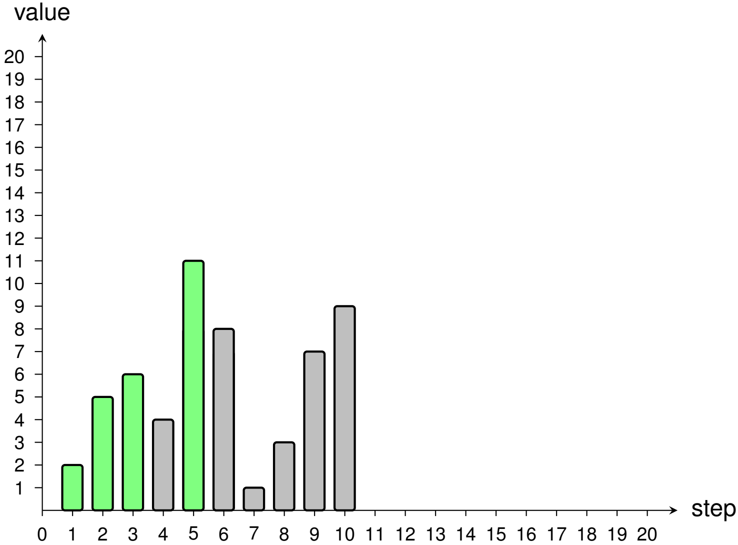


Illustration ($n = 20$)

unknown permutation:
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17,



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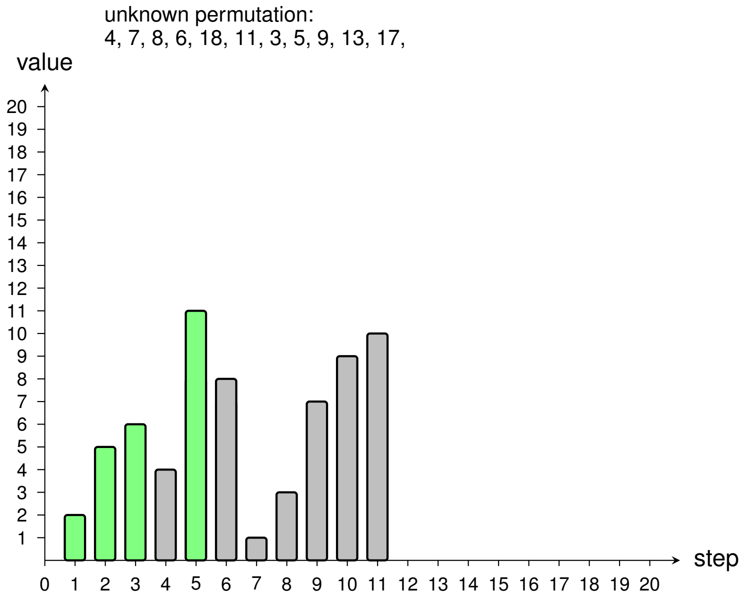

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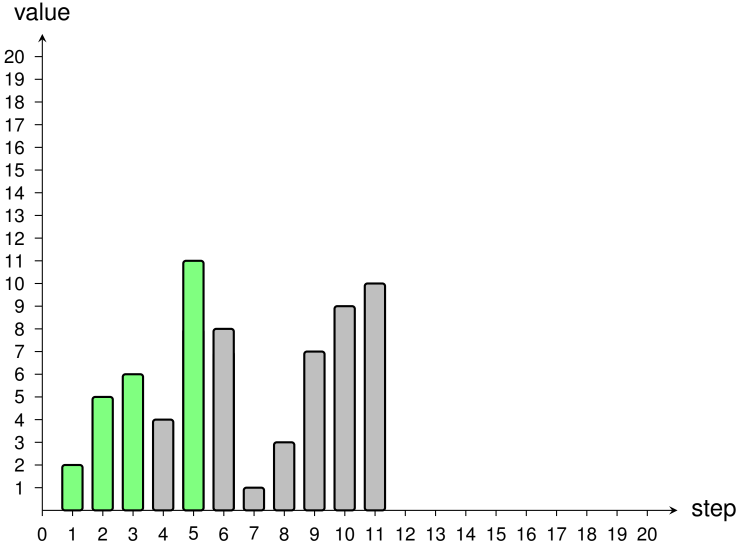
unknown permutation:
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17, 2,



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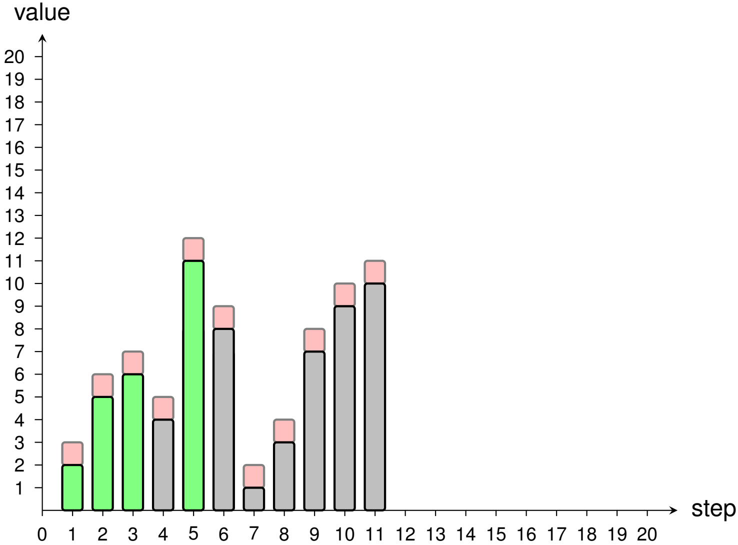
unknown permutation:
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17, 2,



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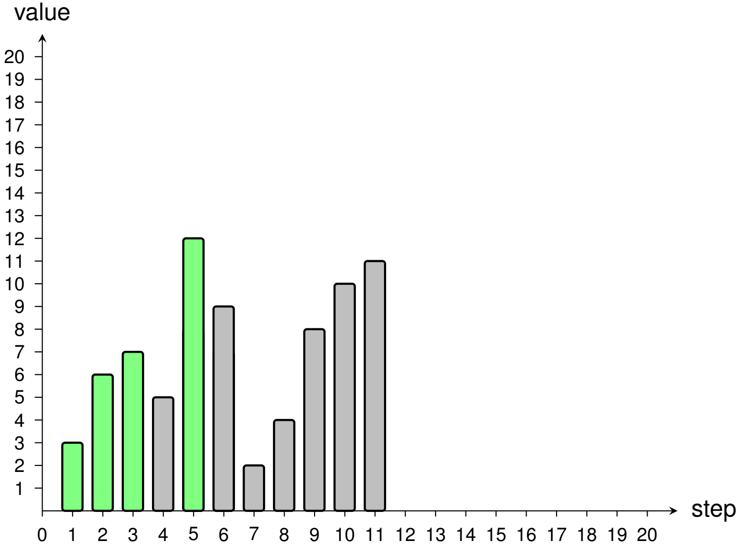

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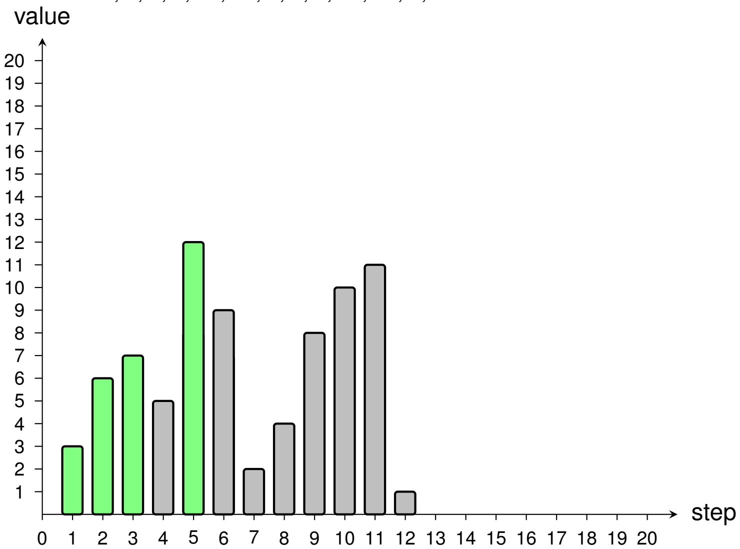

Illustration ($n = 20$)

unknown permutation:
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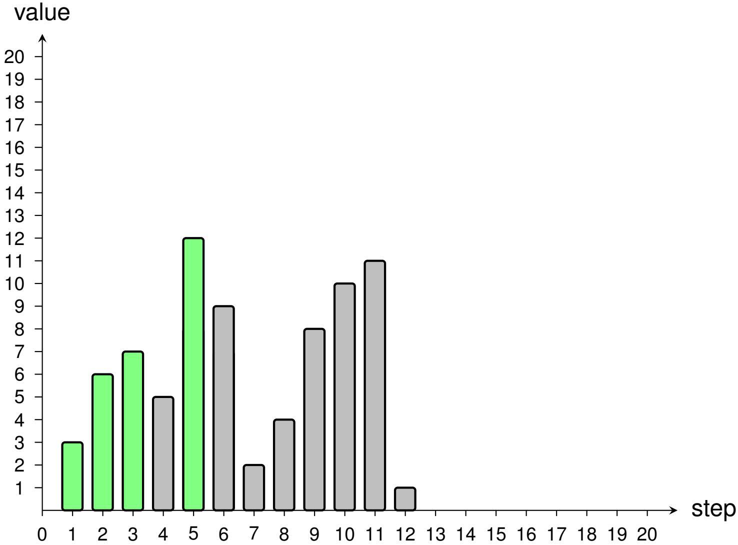

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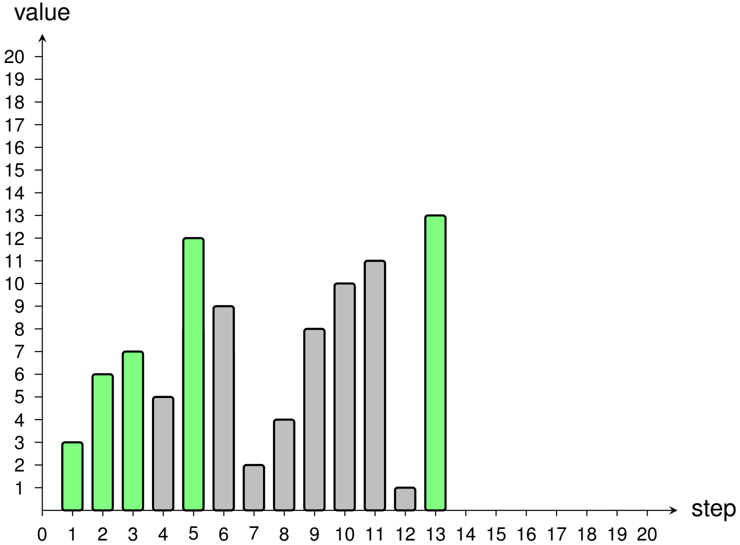

Illustration ($n = 20$)

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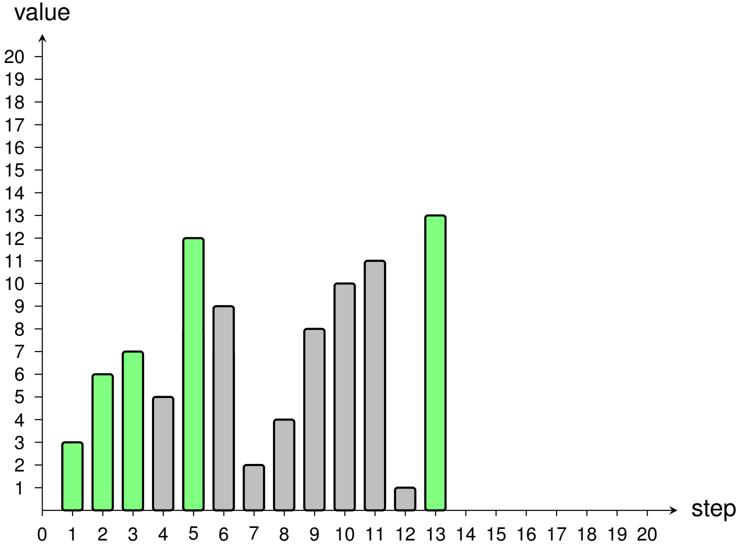

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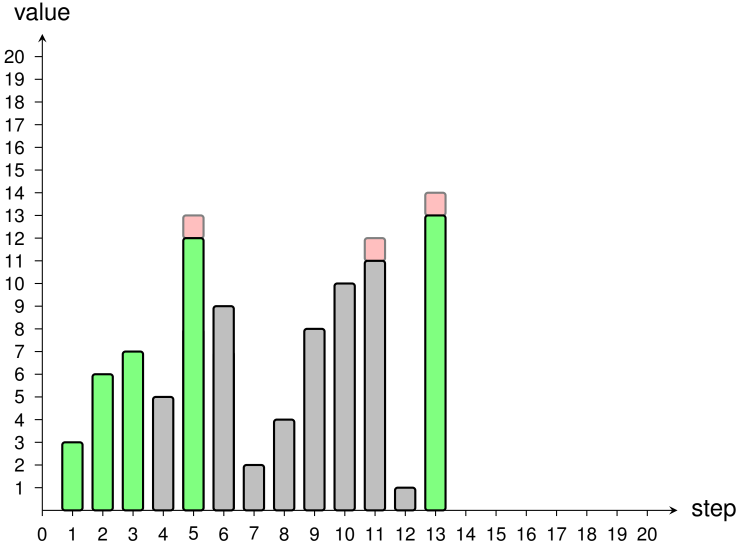

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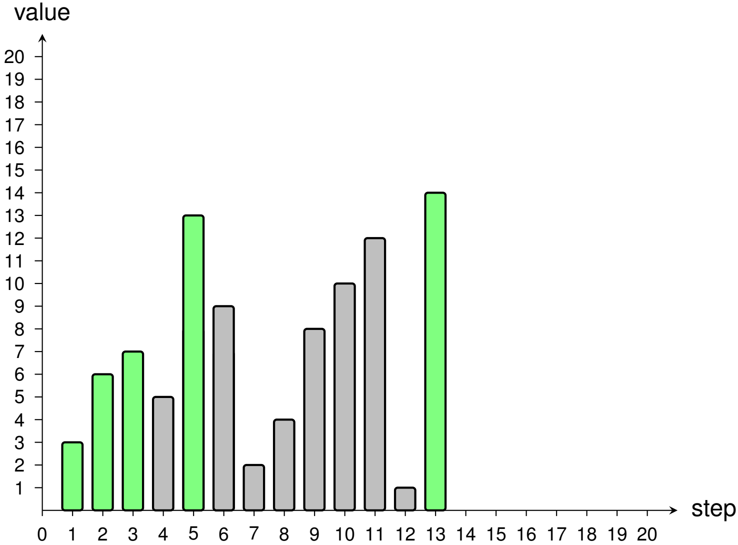

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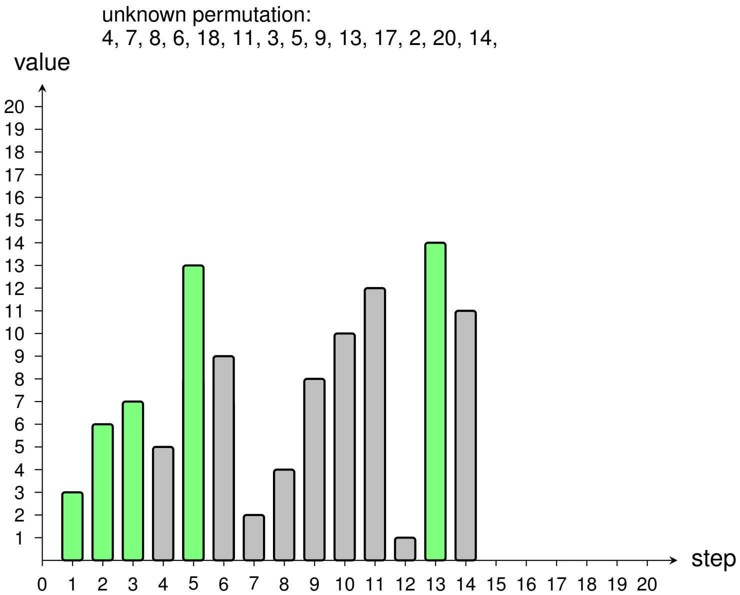
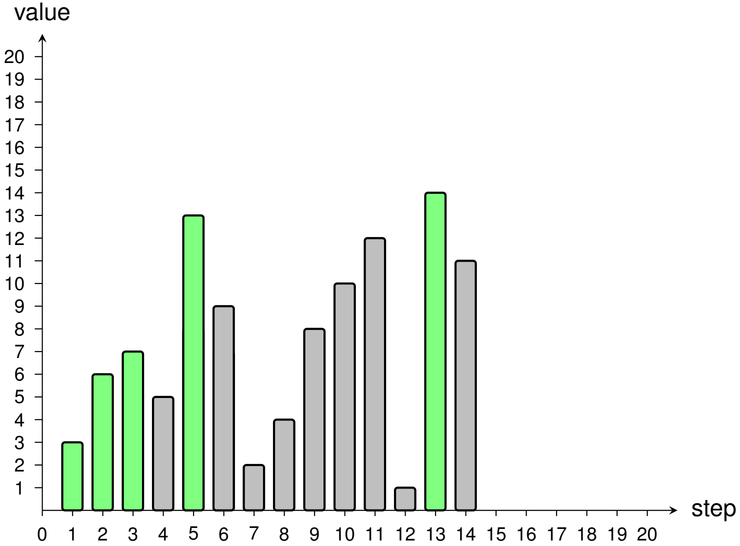


Illustration ($n = 20$)

unknown permutation:
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17, 2, 20, 14, 12,



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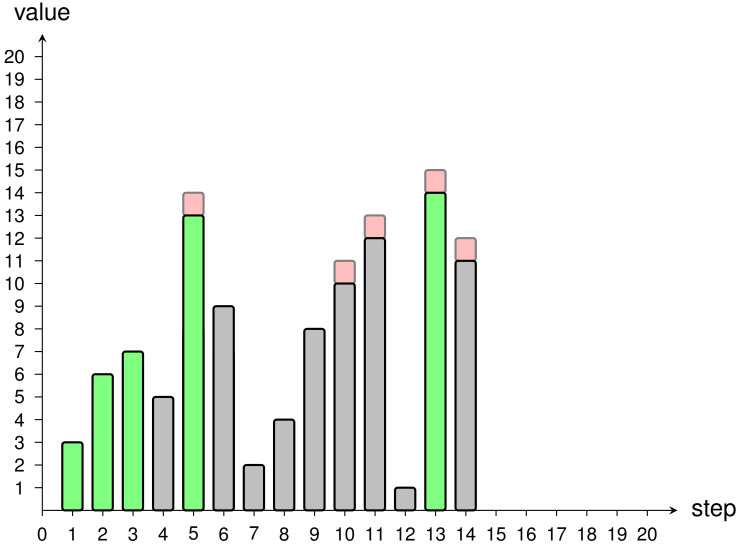

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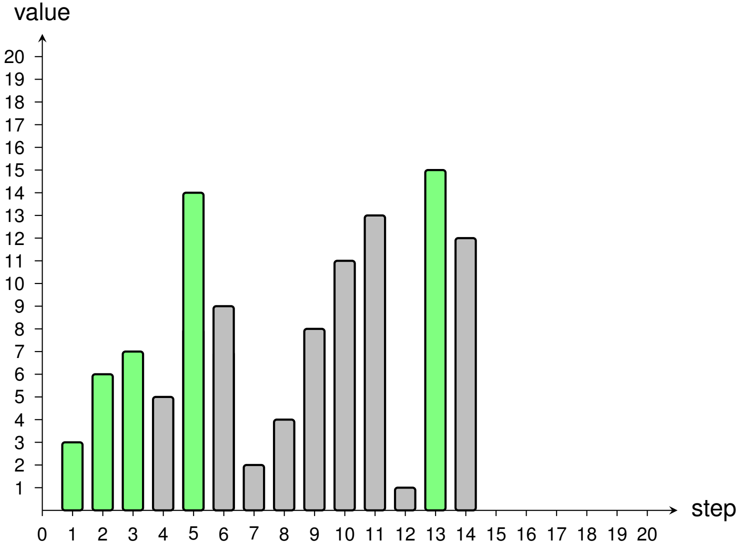

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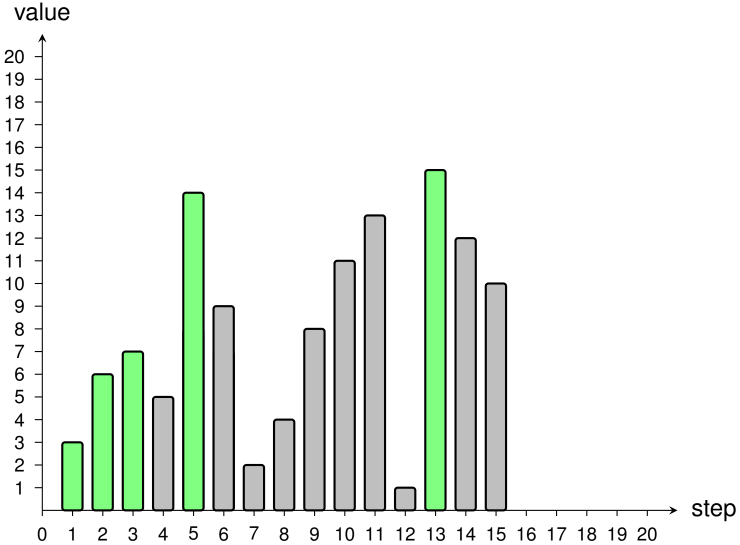

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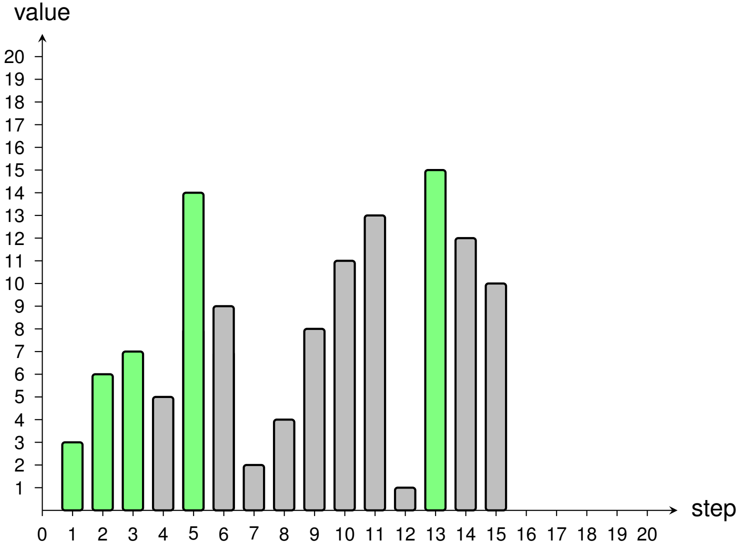

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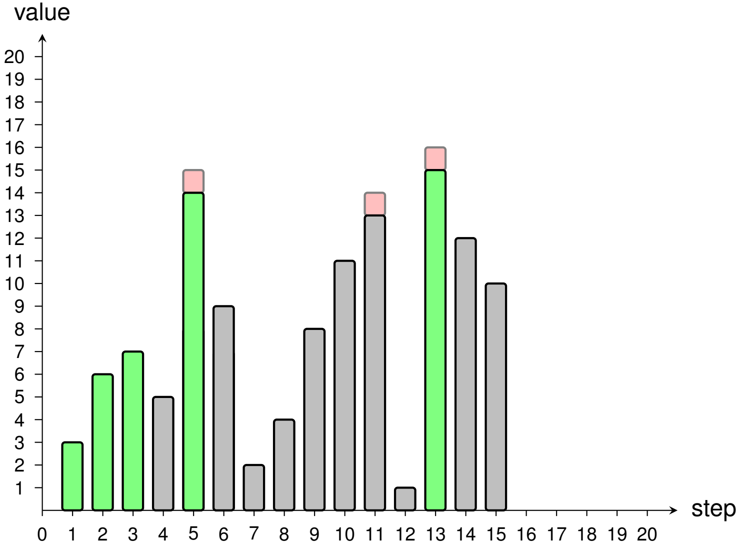

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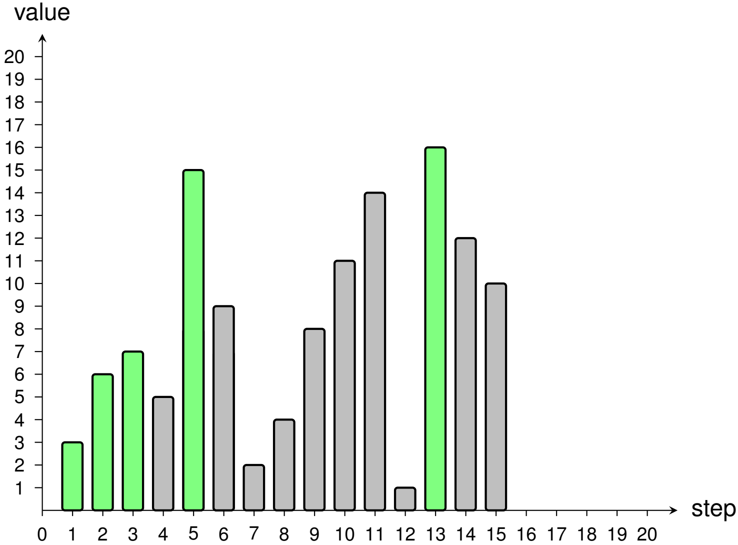

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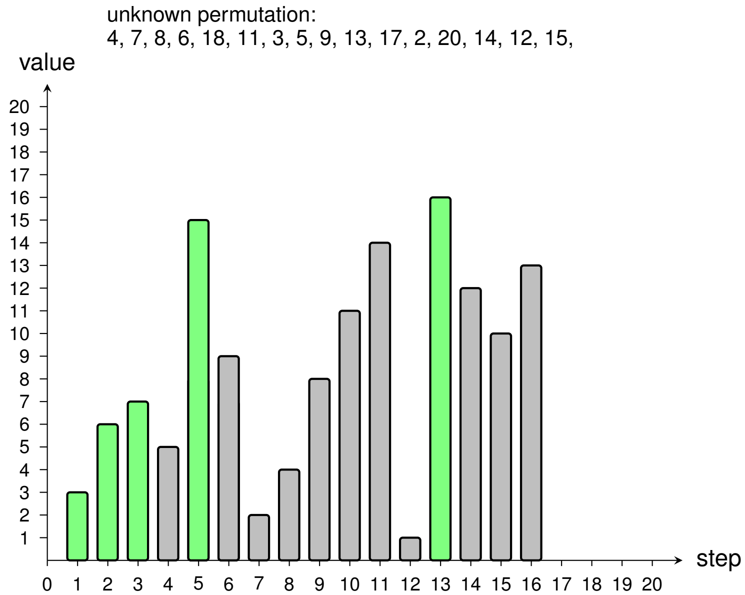

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unknown permutation:
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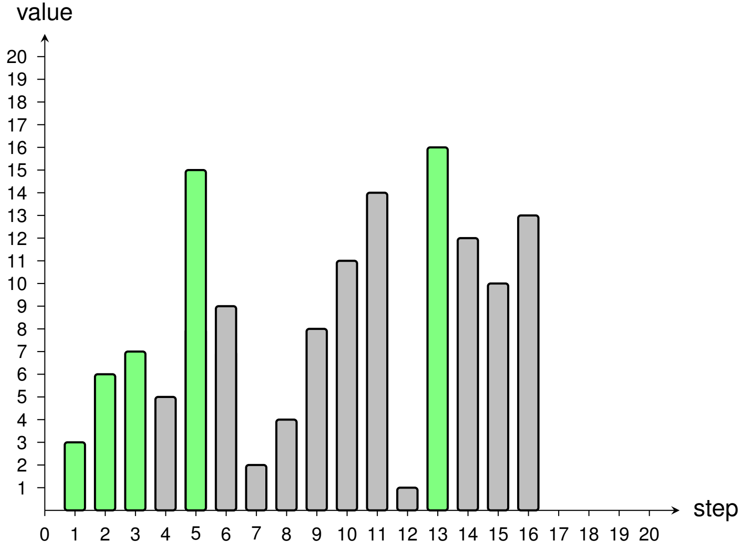

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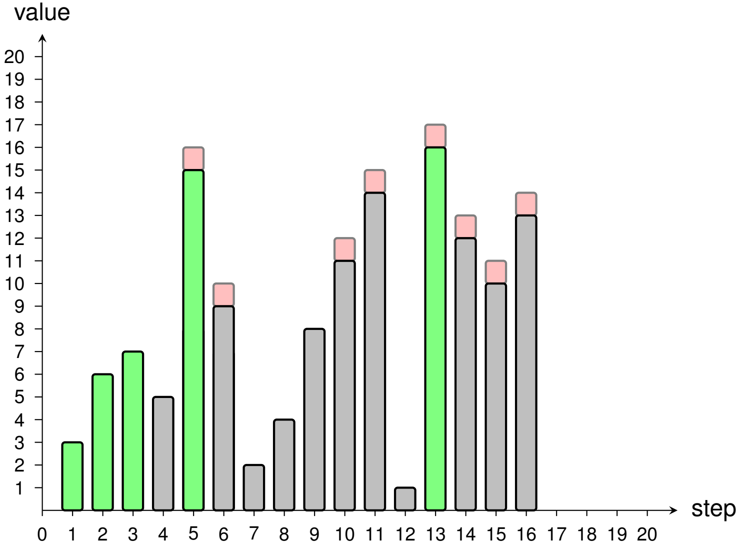

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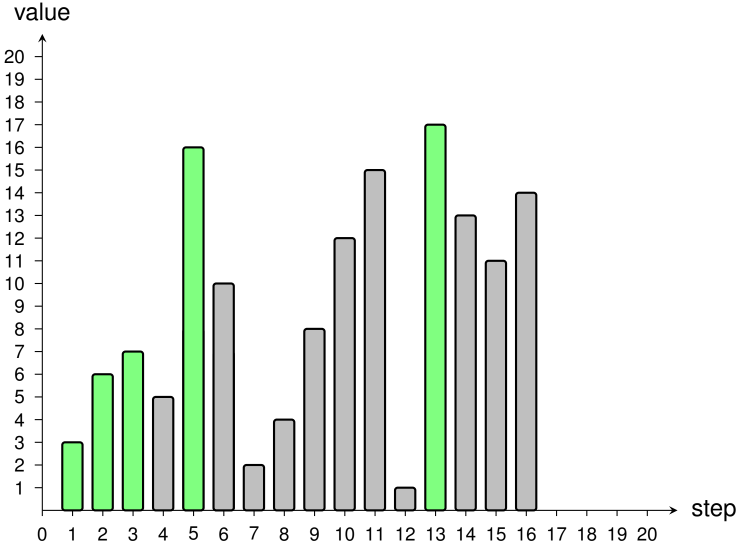

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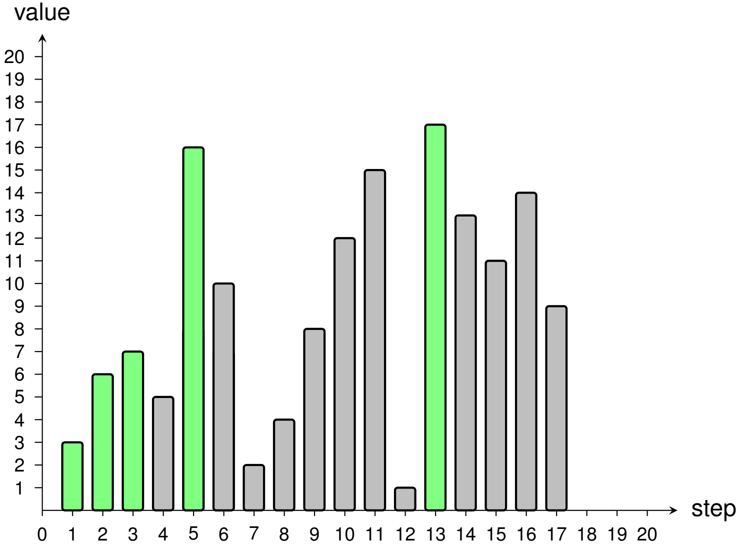

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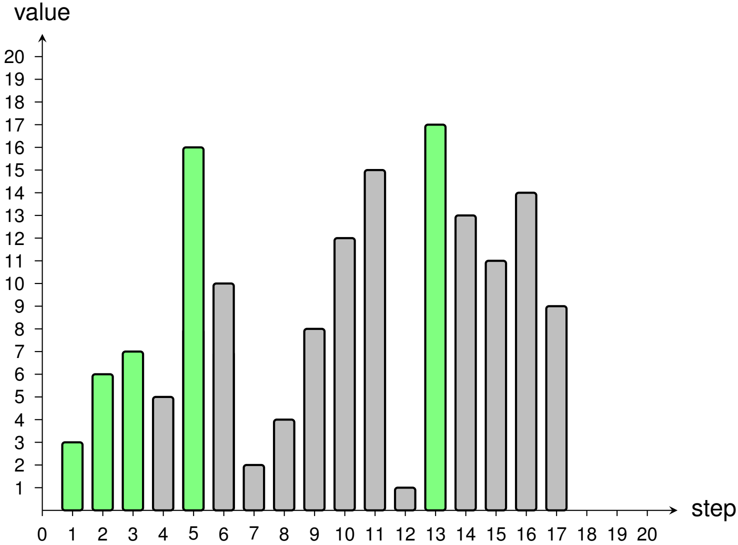
unknown permutation:
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17, 2, 20, 14, 12, 15, 10, 16,



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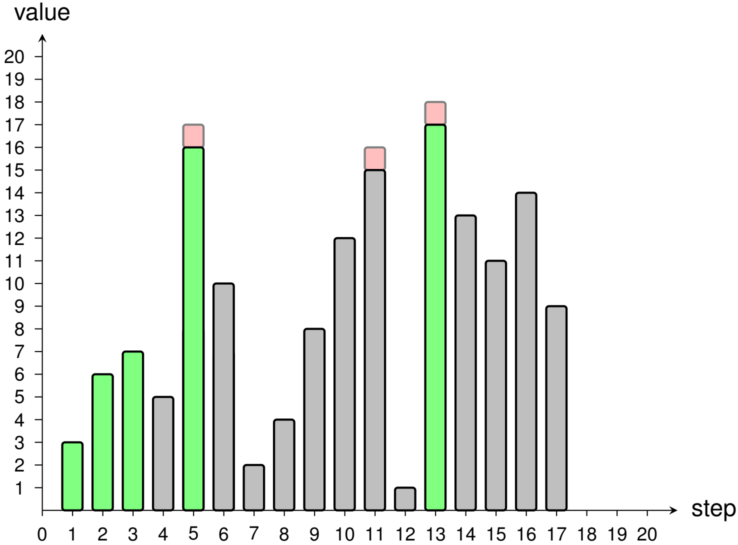

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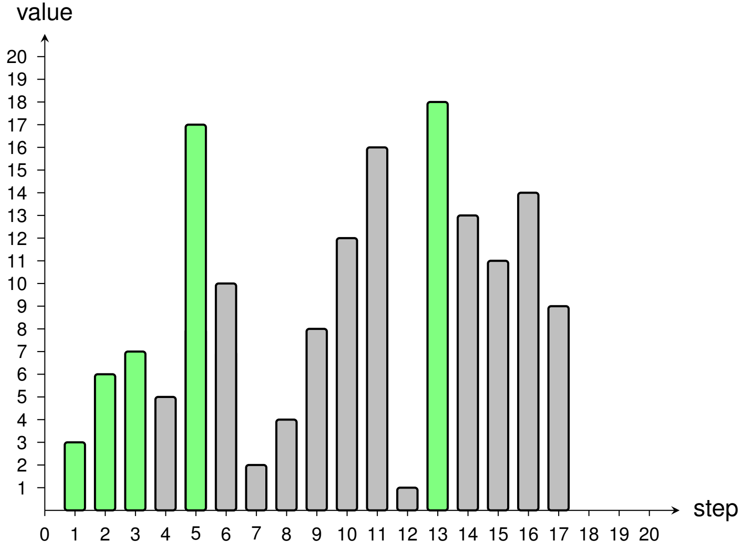

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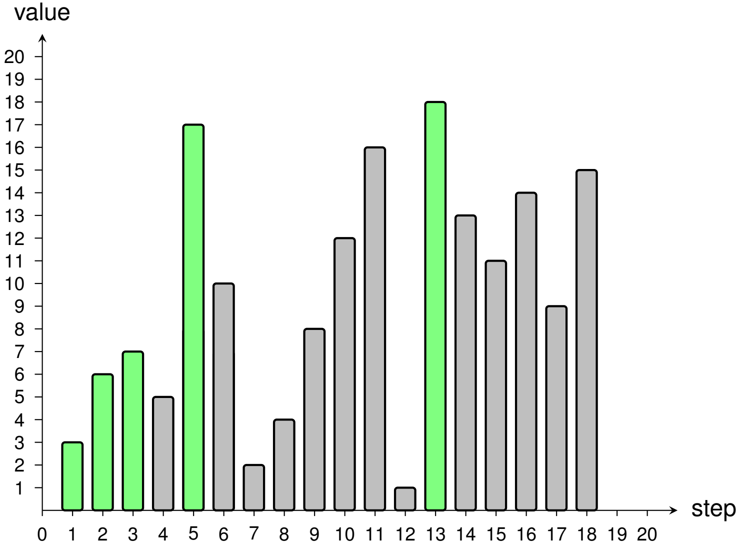

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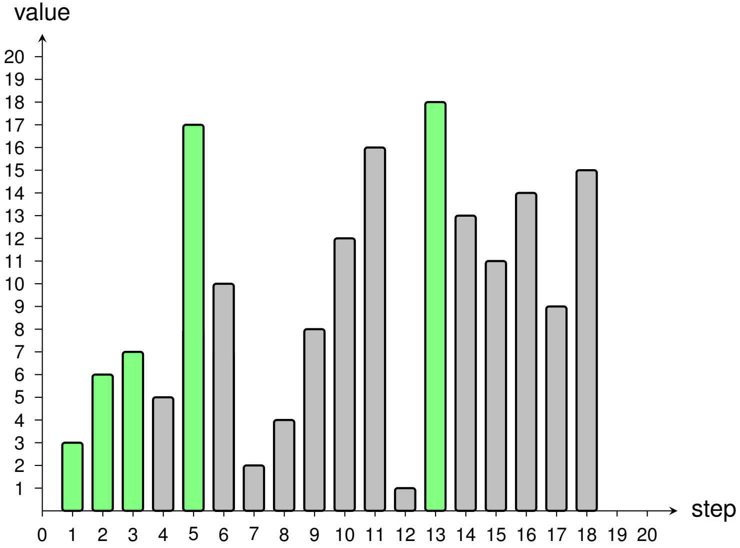

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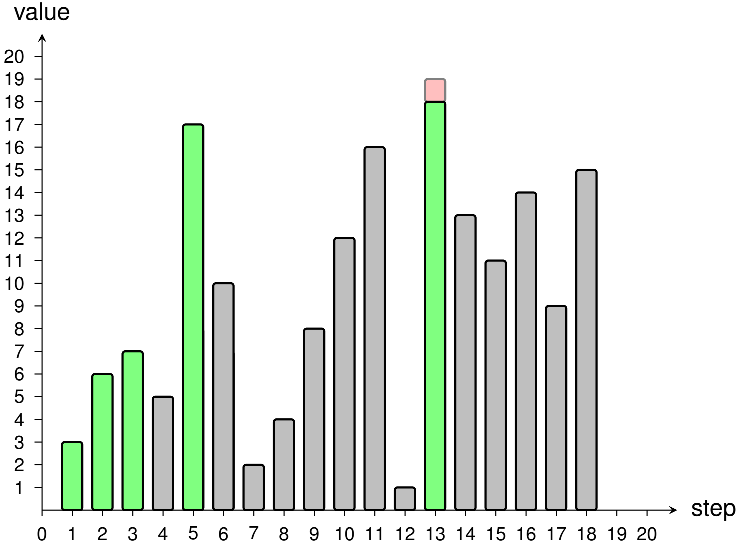

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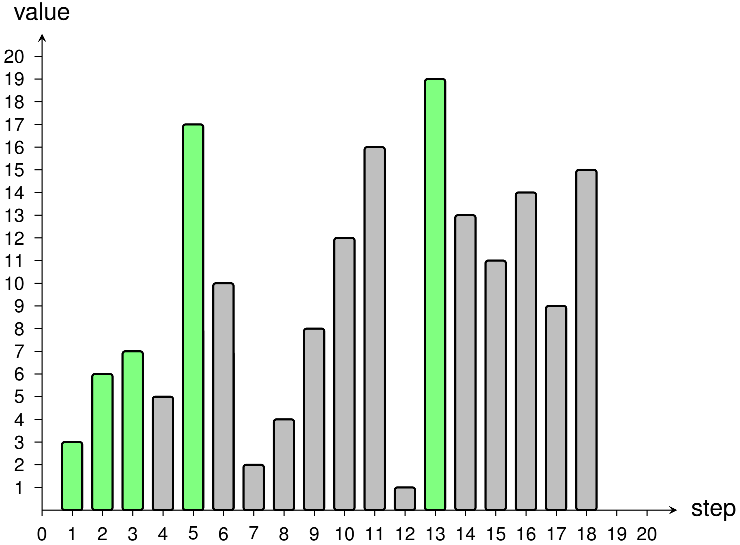

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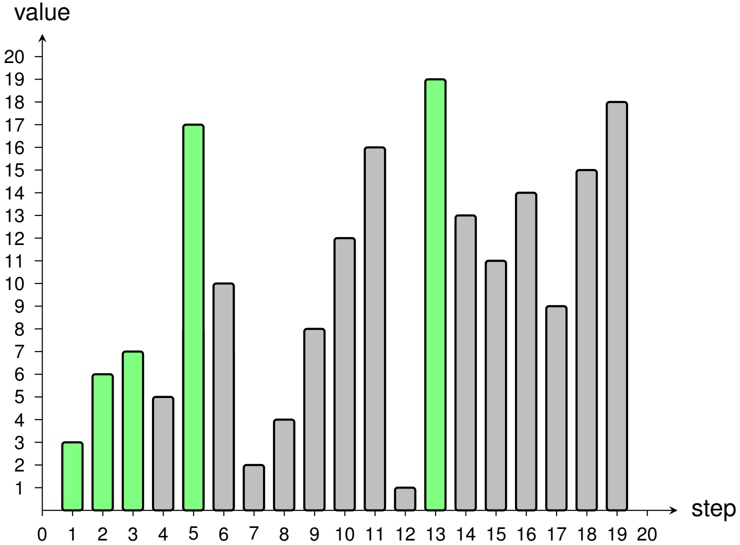

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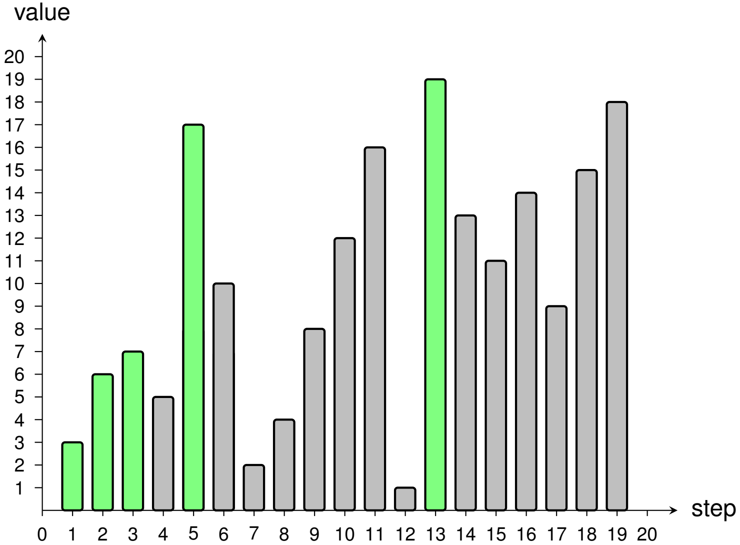

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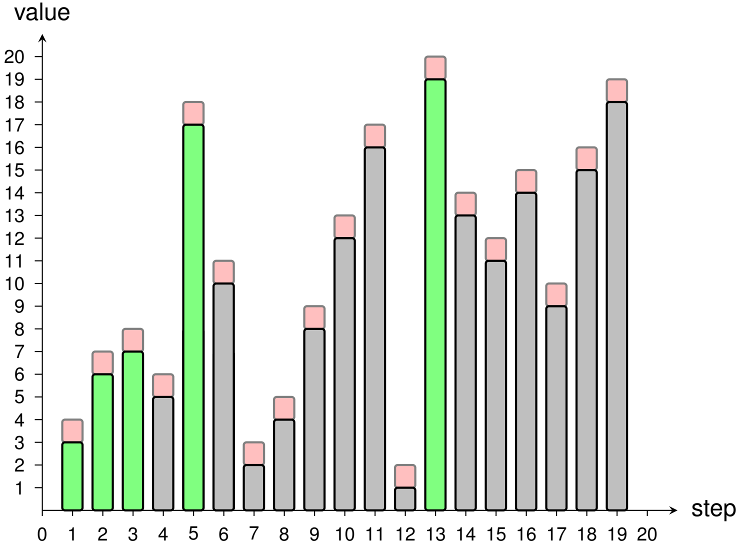

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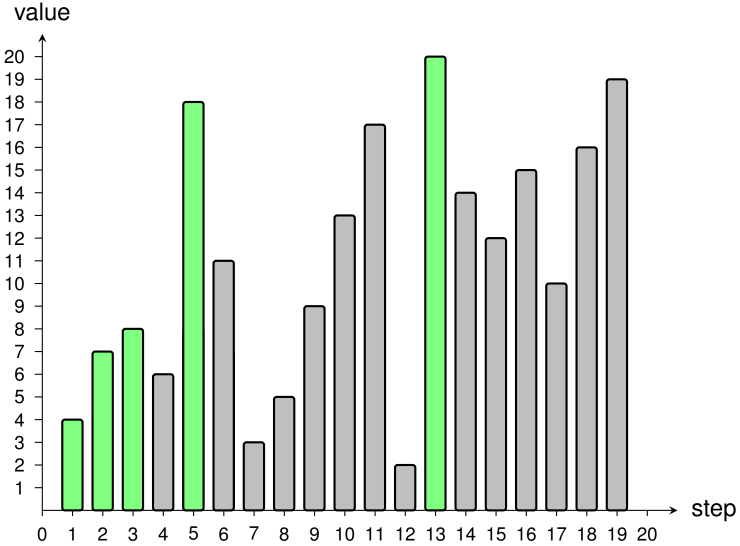

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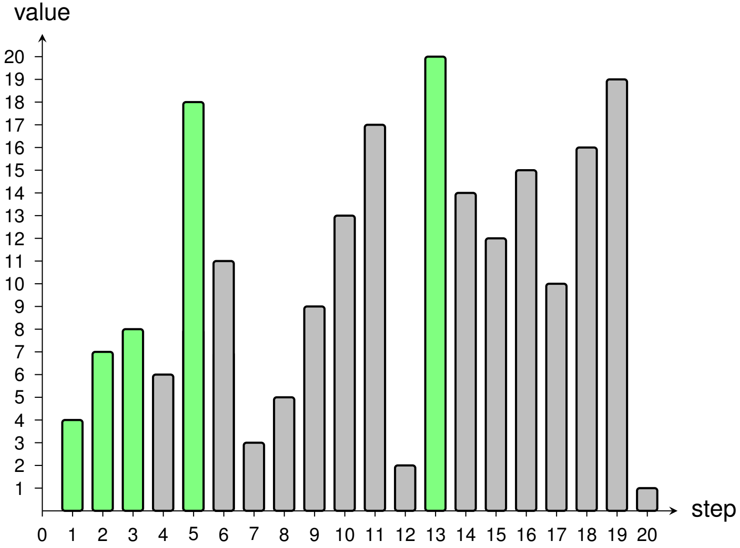

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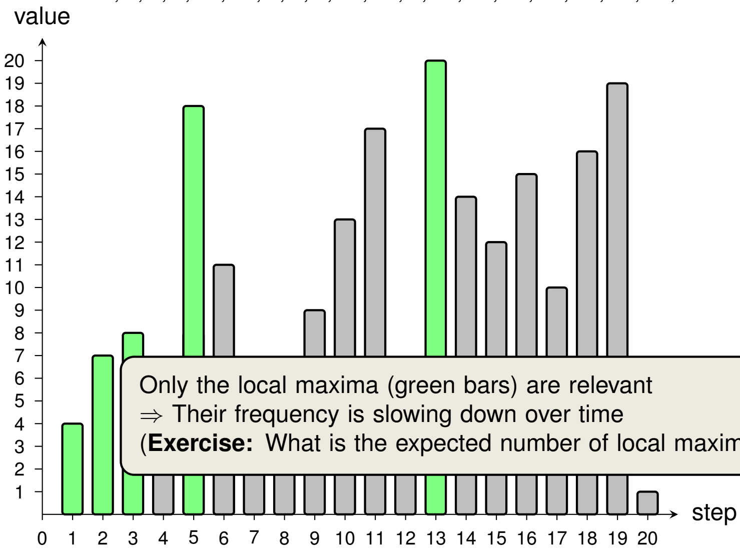

Only the local maxima (green bars) are relevant
$\Rightarrow$ Their frequency is slowing down over time
(**Exercise:** What is the expected number of local maxima?)

Two Basic Strategies

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How good is this approach?

Analysis of the Refined Approach

Example 1

Find a lower bound on the success probability of the refined approach (picking the first candidate better than the first $n/2$).

Answer

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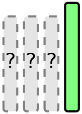

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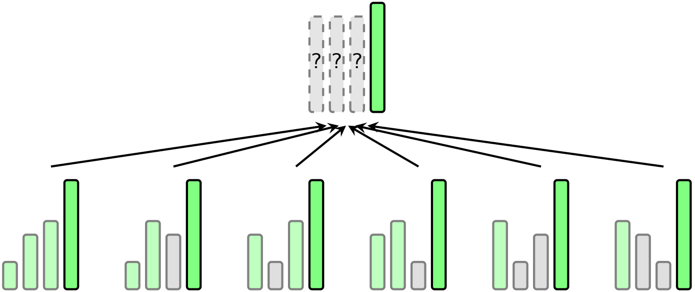

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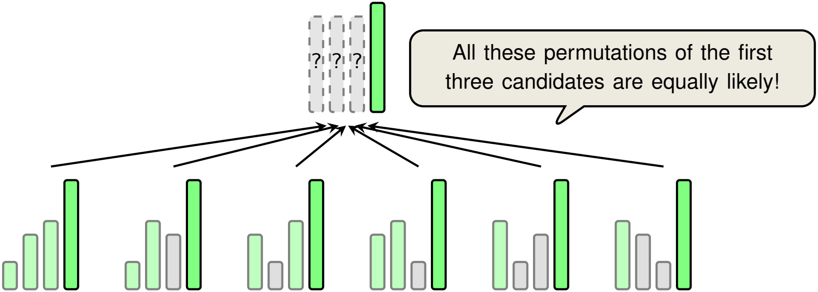

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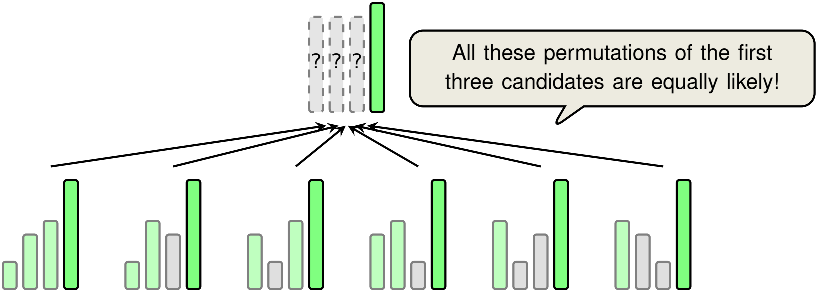

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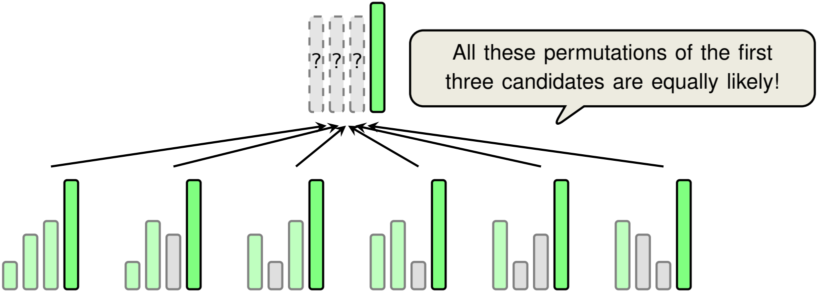

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Optimal Strategy

- Explore but reject the first $x-1$ candidates
- Accept first candidate $i \geq x$ which is better than all candidates before

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Find x which maximises the probability of hiring the best candidate.

Answer

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$$= \sum_{i=1}^{n} \mathbf{P}\left[\,\text{hire candidate } i \cap \text{ candidate } i \text{ is best }\,\right]$$

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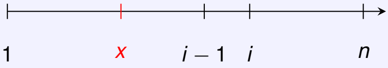

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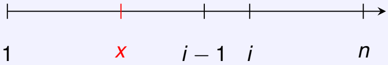

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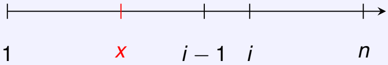

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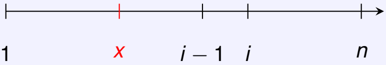

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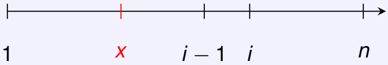

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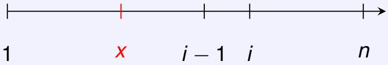

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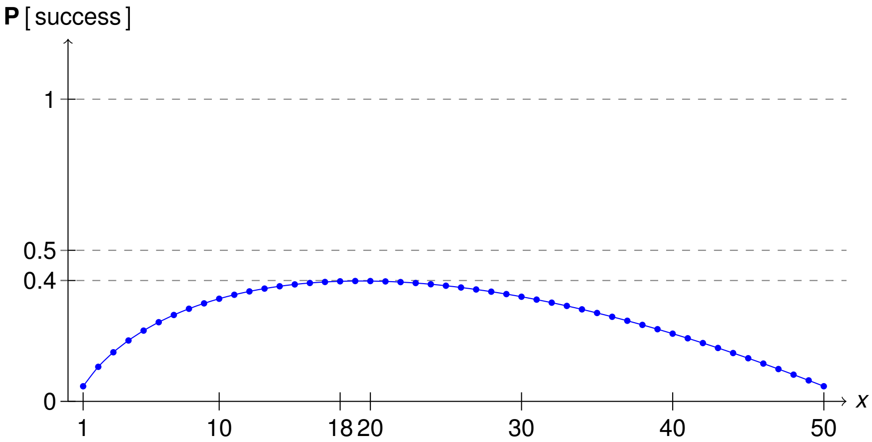
Probability for Success (Illustration)

Suppose $n = 50$:



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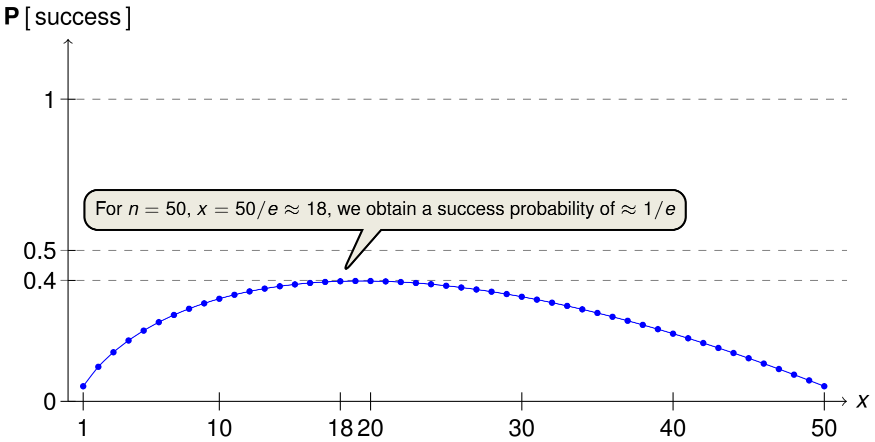

Another Variant of the Secretary Problem

"The Postdoc Variant of the Secretary Problem" (Vanderbei'80)

- same setup as in the secretary problem before
- difference: we want to pick the second-best ("the best [postdoc] is going to Harvard")
- Success probability of the optimal strategy is:

$$\frac{0.25n^2}{n(n-1)} \xrightarrow{n\to\infty} \frac{1}{4}$$

- Thus it is easier to pick the best than the second-best(!)

Outline

Stopping Problem 1: Dice Game

Stopping Problem 2: The Secretary Problem

A Generalisation: The Odds Algorithm (non-examinable)

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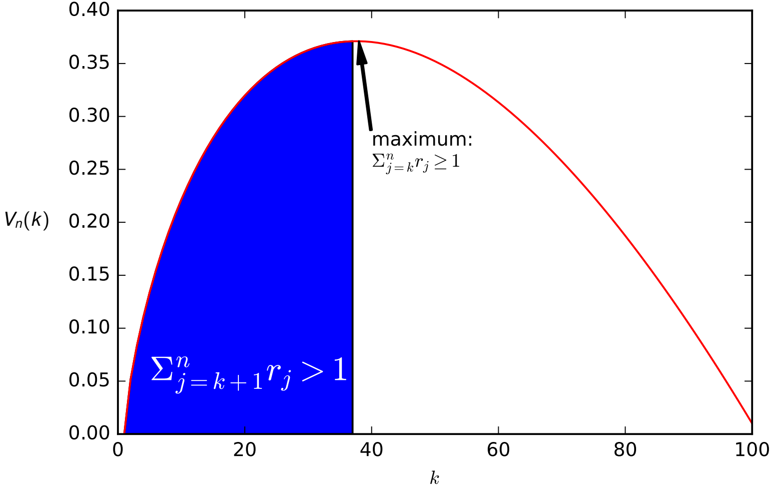
- This algorithm always executes the optimal strategy!

Illustration of the probability of having the last success ($n = 100$)



Source: Group Fibonado

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We re-derived the solution of the secretary problem as a special case!

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Stopping Problem 1: Dice Game

Stopping Problem 2: The Secretary Problem

A Generalisation: The Odds Algorithm (non-examinable)

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List of Lectures

- **Part I: Introduction to Probability**
  - Lecture 1: Conditional probabilities and Bayes' theorem

- **Part II: Random Variables**
  - Lecture 2: Random variables, probability mass function, expectation
  - Lecture 3: Expectation properties, variance, discrete distributions
  - Lecture 4: More discrete distributions: Poisson, Geometric, Negative Binomial
  - Lecture 5: Continuous random variables
  - Lecture 6: Marginals and Joint Distributions
  - Lecture 7: Independence, Covariance and Correlation

- **Part III: Moments and Limit Theorems**
  - Lecture 8: Basic Inequalities and Law of Large Numbers
  - Lecture 9: Central Limit Theorem

- **Part IV: Applications and Statistics**
  - Lecture 10: Estimators (Part I)
  - Lecture 11: Estimators (Part II)
  - Lecture 12: Online Algorithms

List of Distributions

Very Important:

- Bernoulli, Binomial, Poisson
- (Continuous) Uniform, Normal, Exponential

(Somewhat Less) Important:

- Geometric, Negative Binomial, Hypergeometric, Discrete Uniform

Not used or not defined in this course (and thus not examinable):

- Cauchy, Gamma, bivariate Normal
- Beta

Thank you and Best Wishes for the Exam!