Introduction to Probability

Lecture 12: Online Algorithms

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Easter 2025

UNIVERSITY OF CAMBRIDGE

Compiled: May 18, 2025 at 19:51

Outline

Introduction: Dice Game

Dice Game

- We throw a fair, six-sided dice n times
- After each throw, you can either `STOP` or `CONTINUE`
- You win if you `STOP` at the last 6 within the n throws

What is the optimal strategy for maximising the probability of winning?

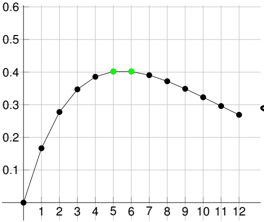
This boils down to finding a threshold from which we `STOP` as soon as a 6 is thrown.

Example ($n = 10$)

- $3, 5, 6, 4, 2, \underbrace{3}_{\texttt{STOP}}, 1, 2, 6, 5 \quad \Rightarrow$ LOSE!
- $3, 5, \underbrace{6}_{\texttt{STOP}}, 4, 2, 3, 1, 2, 6, 5 \quad \Rightarrow$ LOSE!
- $3, 5, 6, 4, 2, 3, 1, 2, \underbrace{6}_{\texttt{STOP}}, 5 \quad \Rightarrow$ WIN!

Dice Game (Solution)

$$\mathbf{P}\left[\text{obtain exactly one 6 in last } k \text{ throws}\right] = \binom{k}{1} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{k-1} = \frac{k}{6} \cdot \left(\frac{5}{6}\right)^{k-1}$$

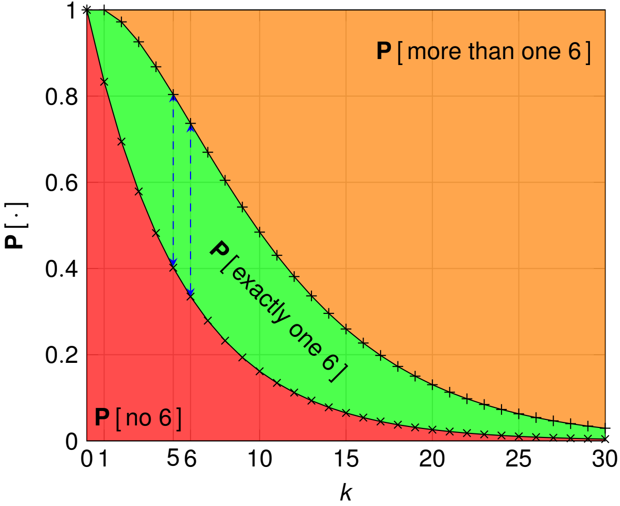


We obtain a unimodal distribution

- This is maximised for $k = 6$ (or $k = 5$) $\Rightarrow$ best strategy: wait until we have 6 (5) throws left, and then STOP at the first 6
- Probability of success is:

$$\left(\frac{5}{6}\right)^{5} \approx 0.40.$$

Illustration of the Three Possibilities



Outline

Stopping Problem 1: Dice Game

Stopping Problem 2: The Secretary Problem

A Generalisation: The Odds Algorithm (non-examinable)

The End...

The Secretary Problem

The Problem

- We are interviewing n candidates for one job in a sequential, random order
- A candidate must be accepted (STOP) or rejected immediately after the interview and cannot be recalled
- **Goal:** maximise the probability of hiring the best candidate

also known as marriage problem (Kepler 1613), hiring problem or best choice problem.

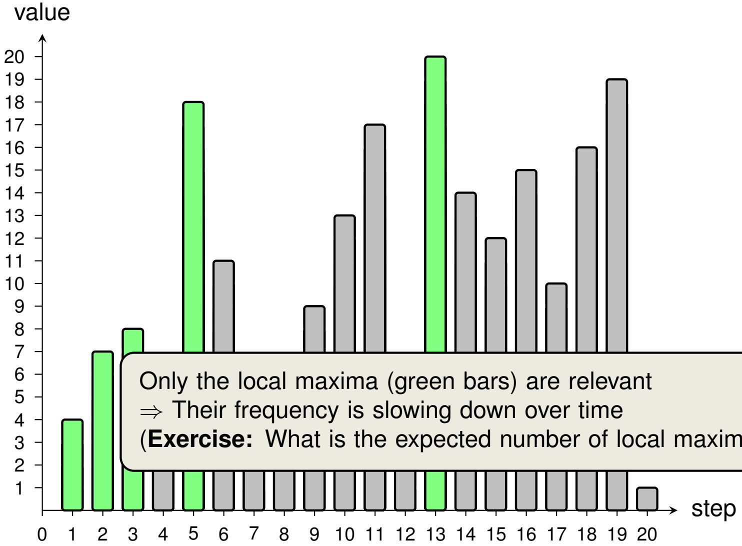
Further Remarks

- After seeing candidate i, we only know the relative order among the first i candidates.

$\Rightarrow$ For our problem we may as well assume that the only information we have when interviewing candidate i is whether that candidate is best among $\{1, \ldots, i\}$ or not.

Illustration ($n = 20$)

unknown permutation:
4, 7, 8, 6, 18, 11, 3, 5, 9, 13, 17, 2, 20, 14, 12, 15, 10, 16, 19, 1.



Only the local maxima (green bars) are relevant
$\Rightarrow$ Their frequency is slowing down over time
(**Exercise:** What is the expected number of local maxima?)

Two Basic Strategies

Naive Approach

- Always pick the first (or any other) candidate
- Probability for success is:

$$\mathbf{P}[\text{ hire best candidate }] = \frac{1}{n}.$$

A typical **exploration-exploitation** based strategy.

Smarter Approach

- Reject the first $n/2$ candidates, then take the first candidate that is better than the first $n/2$ (if none is taken before, take last candidate)

How good is this approach?

Example 1

Find a lower bound on the success probability of the refined approach (picking the first candidate better than the first $n/2$).

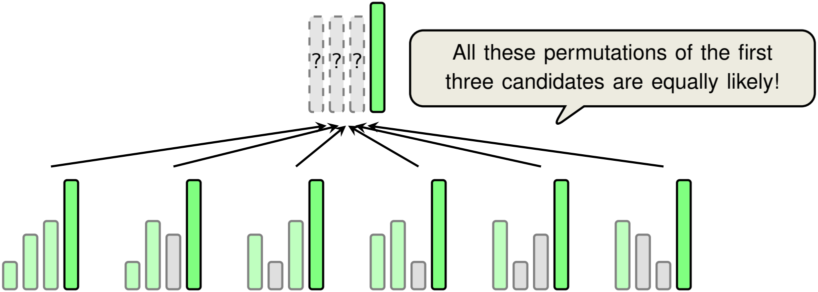
Answer

- Probability for success is:

$$\begin{aligned}
&\mathbf{P}[\text{ hire best candidate }] \\
&\geq \mathbf{P}[\text{ best in 2nd half} \cap \text{ second best in 1st half }] \\
&= \mathbf{P}[\text{ best in 2nd half }] \cdot \mathbf{P}[\text{ second best in 1st half} \mid \text{ best in 2nd half }] \\
&= \frac{n/2}{n} \cdot \frac{n/2}{n-1} > \frac{1}{4}.
\end{aligned}$$

Finding the Optimal Strategy (1/2)

- Observation 1: At interview i, it only matters if current candidate is best so far (i.e., no benefit in counting how many “best-so-far” candidates we had).



- Observation 2: If at interview i, the best strategy is to accept the candidate (if it is “best-so-far”), then the same holds for interview $i + 1$

Optimal Strategy

- Explore but reject the first $x - 1$ candidates
- Accept first candidate $i \geq x$ which is better than all candidates before

Example 2

Find x which maximises the probability of hiring the best candidate.

Answer

- First compute success probability for any $x \in \{1, \ldots, n\}$, and then optimise:

$$\mathbf{P}\left[\text{hire best candidate}\right]$$

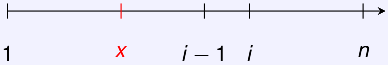
$$= \sum_{i=1}^{n} \mathbf{P}\left[\text{hire candidate } i \cap \text{ candidate } i \text{ is best }\right]$$

$$= \sum_{i=x}^{n} \mathbf{P}\left[\text{hire candidate } i \cap \text{ candidate } i \text{ is best }\right]$$

$$= \sum_{i=x}^{n} \mathbf{P}\left[\text{hire candidate } i \mid \text{ candidate } i \text{ is best }\right] \cdot \mathbf{P}\left[\text{ candidate } i \text{ is best }\right]$$

$$= \frac{1}{n} \cdot \sum_{i=x}^{n} \mathbf{P}\left[\text{second best of first } i \text{ candidates is in the first } x-1 \mid \text{ candidate } i \text{ is best }\right]$$

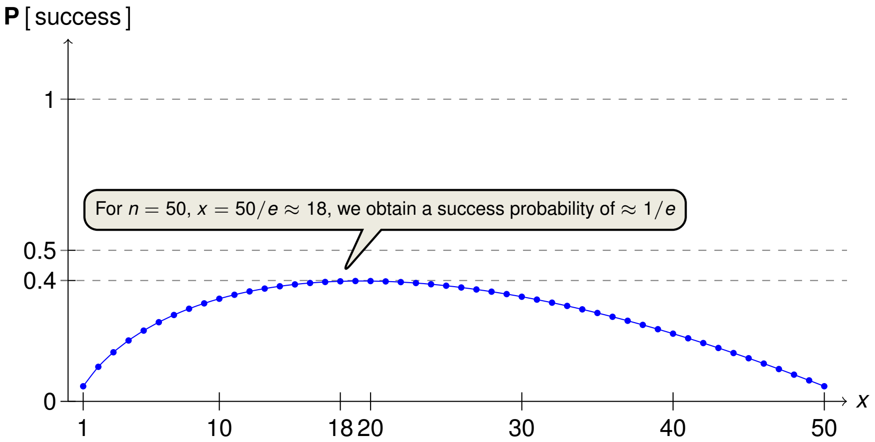
$$= \frac{1}{n} \cdot \sum_{i=x}^{n} \frac{x-1}{i-1} = \frac{x-1}{n} \cdot \sum_{i=x}^{n} \frac{1}{i-1}$$



$$\Rightarrow \quad \sum_{i=x}^{n} \frac{1}{i-1} \approx \ln(n/x) \quad \Rightarrow \quad \text{maximum success probability for } x = \tfrac{1}{e} \cdot n.$$

Probability for Success (Illustration)

Suppose $n = 50$:



Another Variant of the Secretary Problem

"The Postdoc Variant of the Secretary Problem" (Vanderbei'80)

- same setup as in the secretary problem before
- difference: we want to pick the second-best ("the best [postdoc] is going to Harvard")
- Success probability of the optimal strategy is:

$$\frac{0.25n^2}{n(n-1)} \xrightarrow{n\to\infty} \frac{1}{4}$$

- Thus it is easier to pick the best than the second-best(!)

Outline

Details of the Odds Algorithm

- Let $I_1, I_2, \ldots, I_n$ be a sequence of independent indicators and let $p_j = \mathbf{E}[\,I_j\,]$
- Let $r_j := \frac{p_j}{1-p_j}$ (the odds) and $p_j \in (0,1)$ for all $j = 1, 2, \ldots, n$

Example 3

What is the probability that after trial k, there is exactly one success?

Answer

$$\mathbf{P}\left[\sum_{j=k}^{n} I_j = 1\right] = \sum_{j=k}^{n} p_j \cdot \prod_{k \leq j \leq n, j \neq i}^{n} (1-p_i) = \sum_{j=k}^{n} r_j \cdot \left(\prod_{i=k}^{n} (1-p_i)\right)$$

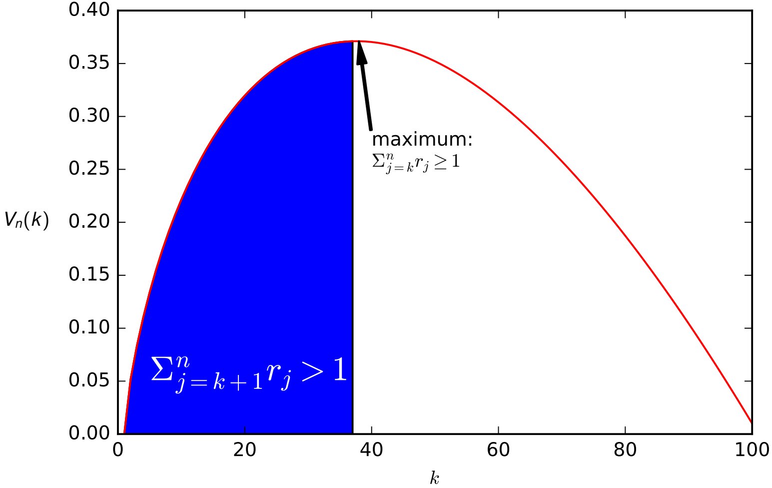
- One can prove that $\mathbf{P}\left[\sum_{j=k}^{n} I_j = 1\right]$ is unimodal in $k \Rightarrow$ there is an ideal point from which on we should STOP at the first success!

Odds Algorithm (“Sum the Odds to One and Stop”, F. Thomas Bruss, 2000)

1. Let k^* be the largest k such that $\sum_{j=k}^{n} r_j \geq 1$
2. Ignore everything before the k^*-th trial, then STOP at the first success.

- The success probability is $\sum_{j=k^*}^{n} r_j \cdot \left(\prod_{i=k^*}^{n} (1-p_i)\right)$.
- This algorithm always executes the optimal strategy!

Illustration of the probability of having the last success ($n = 100$)



Source: Group Fibonado

Example 4

Use the Odds Algorithm to analyse the Secretary Problem.

Answer

- Let $I_j = 1$ if and only if secretary j is the best secretary so far.
- The I_j's are independent (this is an question is on the exercise sheet)
- Then:

$$p_j = \mathbf{P}\left[\, I_j = 1 \,\right] = \frac{1}{j}$$

$$r_j = \frac{p_j}{1 - p_j} = \frac{1/j}{(j-1)/j} = \frac{1}{j-1}$$

- Largest k for which $\sum_{j=k}^{n} \frac{1}{j-1} \geq 1$ is $k = 1/e \cdot n$
- Probability for success:

$$\mathbf{P}\left[\sum_{j=k}^{n} I_j = 1\right] = \sum_{j=k}^{n} r_j \cdot \left(\prod_{i=k}^{n} (1 - p_i)\right)$$

$$= \sum_{j=k}^{n} \frac{1}{j-1} \cdot \left(\prod_{i=k}^{n} \frac{i-1}{i}\right)$$

$$= \sum_{j=k}^{n} \frac{1}{j-1} \cdot \frac{k-1}{n} \approx \frac{1}{e}.$$

We re-derived the solution of the secretary problem as a special case!

Outline

List of Lectures

- **Part I: Introduction to Probability**
  - Lecture 1: Conditional probabilities and Bayes' theorem
- **Part II: Random Variables**
  - Lecture 2: Random variables, probability mass function, expectation
  - Lecture 3: Expectation properties, variance, discrete distributions
  - Lecture 4: More discrete distributions: Poisson, Geometric, Negative Binomial
  - Lecture 5: Continuous random variables
  - Lecture 6: Marginals and Joint Distributions
  - Lecture 7: Independence, Covariance and Correlation
- **Part III: Moments and Limit Theorems**
  - Lecture 8: Basic Inequalities and Law of Large Numbers
  - Lecture 9: Central Limit Theorem
- **Part IV: Applications and Statistics**
  - Lecture 10: Estimators (Part I)
  - Lecture 11: Estimators (Part II)
  - Lecture 12: Online Algorithms

List of Distributions

Very Important:

- Bernoulli, Binomial, Poisson
- (Continuous) Uniform, Normal, Exponential

(Somewhat Less) Important:

- Geometric, Negative Binomial, Hypergeometric, Discrete Uniform

Not used or not defined in this course (and thus not examinable):

- Cauchy, Gamma, bivariate Normal
- Beta

Thank you and Best Wishes for the Exam!